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Dynamical systems in economics

The dynamics of the economic evolution with the capital depreciation

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Abstract. The mathematical model of sector capital distribution dynamics over efficiency levels with the depreciation is proposed. The qualitative analysis of the model is presented. The globally stable set (attractor) is constructed. The equilibria of the proposed dynamical model are determined, in the case of two efficiency levels the global stability of one of them is proved. It is shown that the global stability of the equilibrium means that the depreciation process causes excess production. It is proved that in the case of three efficiency levels there exists a unique equilibrium. In the case without the capital migrating from the second level to the first one the global stability of the equilibrium is proved.

Keywords: dynamical systems, Schumpeterian dynamics, global stability.

1 Introduction

The Austrian economist J. Schumpeter proposed the theory of endogenous economic development [1] based on two interacting processes: innovations (the creation of new technologies) and imitations (their borrowing). This theory is actively developed currently (for example, see [4, 5, 6, 11], etc.).

The mathematical model of the Schumpeterian dynamics with depreciation, under the action of which the firms transfer to the lower efficiency level, was proposed by V. M. Polterovich and G. M. Henkin [13]. Their evolution equation describing the dynamics of the firms distribution over efficiency levels is as follows

$$\dot{F}_i = \frac{dF_i}{dt} = -(\alpha+\beta(1-F_i(t)))(F_i(t)-F_{i-1}(t))+\mu((F_{i+1}(t)-F_i(t)), \quad i = 1, 2, \ldots$$

under the boundary and initial conditions

$$F_0(t) \equiv 0{,}0 < F_{i-1}(0) < F_i(0) < 1 \quad \text{if } 1 < i < N, F_i(0) = 1, \quad \text{if } i \geqslant N,$$

where, $F_i(t)$ is the share of firms corresponding to the levels which numbers are less or equal to i at the moment of time t, N is the initial number of levels for $t = 0$. Note that the addend $-\alpha(F_i - F_{i-1})$ describes the innovation process, the addend $-\beta(1 - F_i)(F_i - F_{i-1})$ — the imitation process and the addend $\mu((F_{i+1}(t) - F_i(t))$ — the depreciation process. Here, $\alpha > 0$ is the innovation rate, $\beta > 0$ is the imitation rate, $\mu > 0$ is the depreciation rate. The numerical analysis of this model was conducted in [14]. However, the qualitative analysis of the models with depreciation was not presented in the investigations and this problem was formulated in [14]. Note that in [13] for the model without the depreciation ($\mu = 0$) the wave process, generated by the above system of ordinary differential equations, were studied.

A. A. Shananin and G. M. Henkin described the mathematical model of the firm capacity dynamics in their article [12]. Let $M_i(t)$ be the integrated firm capacity at the i-th level, λ_i is the profit per capacity unit at the i-th level, $\varphi_i(t)$ is the share of the investments of the firms at the i-th level for the creation of the capacities at the next $i + 1$-th level, $0 \leqslant \varphi_i(t) \leqslant 1$. Then the capacity dynamics equation is as follows

$$\dot{M}_i = (1 - \varphi_i)\lambda_i M_i + \varphi_{i-1}\lambda_{i-1}M_{i-1}, \quad i = 1, 2, \ldots \tag{1}$$

under the boundary and initial conditions

$$M_0(t) \equiv 0, M_i(0) \geqslant 0, \sum_{i=1}^{N} M_i(0) > 0, M_i(0) = 0, \quad \text{if } i > N,$$

where N is the initial number of levels. Here, $\varphi_i = \alpha + \beta(1 - F_i(T))$, $\alpha > 0$, $\beta > 0$ are constants, $F_i(t) = \frac{\sum_{k=0}^{i} M_k}{\sum_{k=0}^{\infty} M_k}$. In [10] it is shown that $M_i(t) \to \infty$ as

$t \to \infty$. It means the unbounded growth of the capacities, which is incorrect from the economic perspective.

The authors, developing the described models, propose the model of the capital distribution dynamics over efficiency levels with depreciation taking into account the boundedness of the economic growth. It is modeled via the introducing of the notion of the economic niche volume which is analogous to the ecological niche volume. The economic niche volume is a limit integrated capital value, for which the growth rate is so low that there is no capital growth. Thus, in this paper the partial solution of the problem formulated in [14] is proposed under the boundedness of the economic growth condition.

2 The model with the depreciation

Consider the economic system (for example, sector) in which the firms are ordered by efficiency levels. Let i be the number of efficiency level, $i \in \{1, \dots, N\}$. Assume that the greater i corresponds to the higher level. Consider the following system of differential equations

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi_1}{\lambda_1} C_1 (V - \sum\limits_{j=1}^{N} C_j) + \mu_2 C_2 = f_1(C), \\ \dot{C}_i = \frac{1-\varphi_i}{\lambda_i} C_i (V - \sum\limits_{j=1}^{N} C_j) + \varphi_{i-1} C_{i-1} - \mu_i C_i + \mu_{i+1} C_{i+1} = f_i(C), i = 2, \dots, N-1, \\ \dot{C}_N = \frac{1-\varphi_N}{\lambda_N} C_N (V - \sum\limits_{j=1}^{N} C_j) + \varphi_{N-1} C_{N-1} - \mu_N C_N = f_N(C), \end{cases} \tag{2}$$

where, C_i is the integrated capital of all firms at the i-th level (one firm can have the capital at different levels), $C = (C_1, \dots, C_N)$, V is the economic niche volume, φ_i is the share of capital of the firms at the i-th level intended to the developing of the production at the next, $i+1$-th, level, λ_i is the unit prime cost at the i-th level (i.e. the unit goods production cost per unit time), $i = 1, .., N$, μ_i is the share of the capital migrating to the lower level due to the depreciation process, $i = 2, .., N$. Here, $V > 0$, $0 < \varphi_i < 1$, $0 < \mu_i < 1$, $\lambda_i > 0$ are constants. Denote $a_i = \frac{1-\varphi_i}{\lambda_i} > 0$.

In what follows the next conditions are assumed to be fulfilled

$$\frac{\mu_{i+1}}{a_i} < V, \frac{\varphi_{i-1}}{a_i} < V.$$

It means that the contribution of the production at each level is much greater than the depreciation from the next level $i + 1$ and the investments from the previous level $i - 1$, which corresponds the healthy economy.

Denote by $C(t, t_0, C^0)$ the solution of (2) for which $C(t, t_0, C^0) = C^0$. If $t_0 = 0$ the solution will be denoted as $C(t, C^0)$. Denote by $r(C^0)$ the positive half-trajectory $r(C^0) = \{C(t, C^0) : t \geqslant 0\}$. If $t_0 > 0$, then $r(t_0, C^0) = \{C(t, C^0) : t \geqslant t_0\}$.

Proposition 1 *The set* $\mathbb{R}^N_+ = \{(C_1, \dots, C_N) \in \mathbb{R}^N : C_i \geqslant 0, i = 1, \dots, N\}$ *is invariant.*

Proof

From (2) we obtain $\dot{C}_i \geqslant 0$ if $C_i = 0$. It means that the positive half-trajectories do not leave $\mathbb{R}^N_+$ through the boundary hyperplanes $C_i = 0$, therefore, $\mathbb{R}^N_+$ is invariant.

Proposition 2 *Any positive half-trajectory* $r(C^0), C^0 \in \mathbb{R}^N_+$ *enters the set* $W = \{(C_1, \dots, C_N) : V \leqslant \sum\limits_{j=1}^{N} C_j \leqslant V + \Delta V\}$, *where* $\Delta V = \sum\limits_{j=1}^{N-1} \frac{\varphi_j}{a_j}$ *and do not leave it.*

Remark The Proposition 2 implies that W is a global attractor.

Proof

Consider the hyperplanes $\pi(R) : C_1 + \ldots + C_N = R$, where $R > 0$ with the normal vector $n = (1, \dots, 1)$. Denote by (f, n) the inner product of $f = (f_1, \dots, f_N)$ and n. Then, if $R \in (0, V]$ we obtain

$$(f, n) = (a_1 C_1 + \ldots + a_N C_N)(V - R) + \sum_{j=1}^{N-1} \varphi_j C_j > 0.$$

If $R \geqslant V + \Delta V$, then

$$(f, n) \leqslant (a_1 C_1 + \ldots + a_N C_N)(-\Delta V) + \sum_{j=1}^{N-1} \varphi_j C_j$$

$$= (a_2 C_2 + \ldots + a_N C_N)(-\frac{\varphi_1}{a_1}) + \ldots + (a_1 C_1 + \ldots + a_{N-2} C_{N-2} + a_N C_N)(-\frac{\varphi_{N-1}}{a_{N-1}}) < 0.$$

Thus, all positive half-trajectories enter the set W and do not leave it.

3 Two efficiency levels

Consider the model with two efficiency levels

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi}{\lambda_1} C_1(V - C_1 - C_2) + \mu C_2 = f_1(C_1, C_2), \\ \dot{C}_2 = \frac{1-\alpha}{\lambda_2} C_2(V - C_1 - C_2) + \varphi C_1 - \mu C_2 = f_2(C_1, C_2). \end{cases} \tag{3}$$

Now let us investigate the stability of the equilibrium $C^* = (C_1^*, C_2^*)$ of the system (3): $C_1^* = \frac{(2a_1a_2V+A)\mu}{2a_1a_2\mu+a_1A}$, $C_2^* = \frac{(2a_1a_2V+A)\mu}{2a_1a_2\mu+a_1A}\frac{A}{2a_2\mu}$, where $A = -a_1\mu + \sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu}$.

Definition 1 *The equilibrium C^* of the system $\dot{C} = f(C)$ is globally stable in $\mathbb{R}_+^n \setminus \{O\}$ if $C(t, C^0) \to C^*$, as $t \to \infty$, for every $C^0 \in \mathbb{R}_+^n \setminus \{O\}$.*

Theorem 1 *The equilibrium $C^* = (C_1^*, C_2^*)$ of the system (3) is globally stable in $\mathbb{R}_+^2 \setminus \{O\}$.*

Proof

Consider the isoclines of (3)

$$\dot{C}_1 = 0 : C_2(C_1) = \frac{a_1C_1(V - C_1)}{a_1C_1 - \mu}, \tag{4}$$

$$\dot{C}_2 = 0 : C_1(C_2) = \frac{a_2C_2(V - C_2 - \frac{\mu}{a_2})}{a_2C_2 - \varphi}. \tag{5}$$

The isocline $\dot{C}_1 = 0$ has the asymptote $C_1 = \frac{\mu}{a_1} < V$, the isocline $\dot{C}_2 = 0$ has the asymptote $C_2 = \frac{\varphi}{a_2} < V$, herewith, $C_2(C_1) \to +\infty$ as $C_1(C_2) \to \frac{\mu}{a_1}$ and $C_1 \to +\infty$ as $C_2 \to \frac{\varphi}{a_2}$. Rewrite (4),(5) as follows

$$\dot{C}_1 = 0 : C_1 = \tilde{C}_1(C_2) = \frac{(V - C_2) + \sqrt{(V - C_2)^2 + \frac{4\mu C_2}{a_1}}}{2},$$

$$\dot{C}_2 = 0 : C_2 = \tilde{C}_2(C_1) = \frac{V - \mu - C_1 + \sqrt{(V - \mu - C_1)^2 + \frac{4\varphi C_1}{a_2}}}{2}.$$

The interior of the set $\mathbb{R}_+^2 \setminus \{O\}$ is divided isoclines into 4 domains D_i as follows (fig. 1).

- $D_1 = \{(C_1, C_2) : C_1 < \tilde{C}_1, C_2 < \tilde{C}_2\}$, in which, $f_1 > 0, f_2 > 0$;
- $D_2 = \{(C_1, C_2) : C_1 > \tilde{C}_1, C_2 < \tilde{C}_2\}$, in which, $f_1 < 0, f_2 > 0$;
- $D_3 = \{(C_1, C_2) : C_1 > \tilde{C}_1, C_2 > \tilde{C}_2\}$, in which, $f_1 < 0, f_2 < 0$;
- $D_4 = \{(C_1, C_2) : C_1 < \tilde{C}_1, C_2 > \tilde{C}_2\}$, in which, $f_1 > 0, f_2 < 0$.

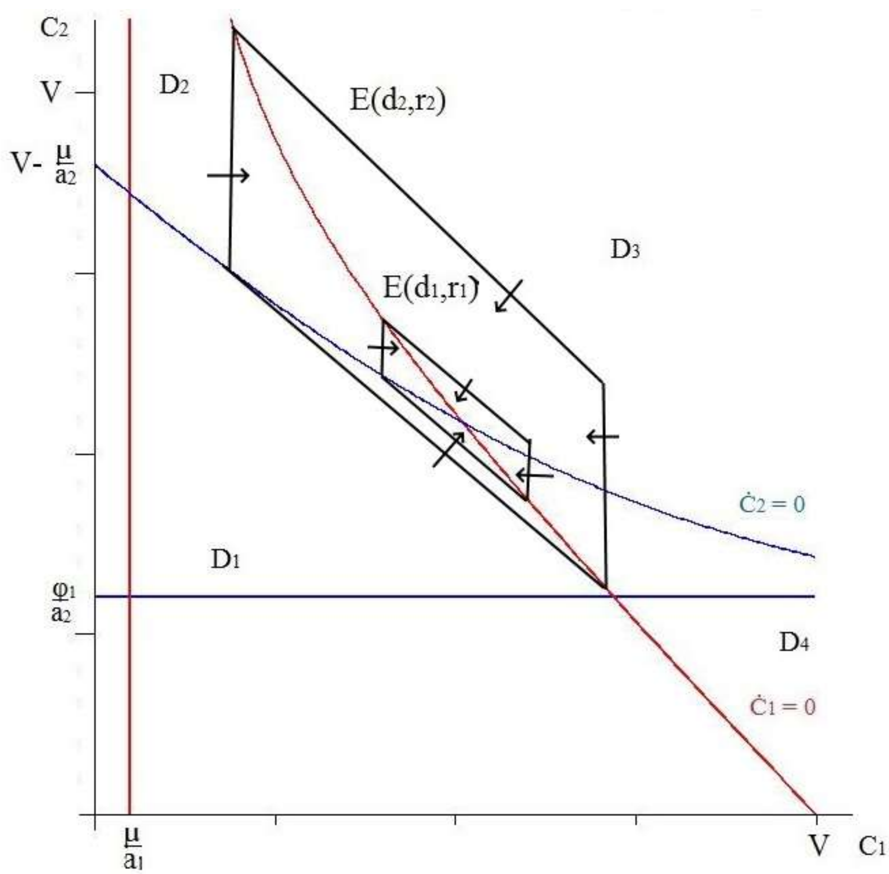

Figure 1: Nested sets $E(d, r)$.

To prove the global stability of the equilibrium $C^* = (C_1^*, C_2^*)$ we construct the family of nested sets $\{E(d, r), d > 0, r > 0\}$ containing C^*, diameters of which tend to 0 as $d, r \to 0$, such that all positive half-trajectories enter and do not leave each of these sets. Consider the family of sets (fig. 1):

$$E(d,r) = \{(C_1, C_2) : C_1^* - d \leqslant C_1 \leqslant C_1^* + d,$$
$$C_1^* + C_2^* - r \leqslant C_1 + C_2 \leqslant C_1^* + C_2^* + r\}.$$

Obviously, $E(d_1, r_1) \subset E(d_2, r_2)$ if $d_2 < d_1$, $r_2 < r_1$ and $(C_1^*, C_2^*) \in E(d,r)$ for any $d \geqslant 0$, $r \geqslant 0$. The boundary $\partial E(d,r)$ consists of the segments of the straight lines l_i, where

- in D_1: $l_1 : C_1 + C_2 - C_1^* - C_2^* + d = 0$ with normal vector $n_1 = (1,1)$,
- in D_2: $l_2 : C_1 - C_1^* + r = 0$ with normal vector $n_2 = (1,0)$,
- in D_4: $l_4 : C_1 - C_1^* - r = 0$ with normal vector $n_4 = (-1,0)$,
- in D_3: l_4 and $l_3 : C_1 + C_2 - C_1^* - C_2^* - d = 0$ with normal vectors $n_4 = (-1,0)$ and $n_3 = (-1,-1)$ respectively.

The inner product of the vector $f = (f_1, f_2)$ and the normal vector n_i to each straight line l_i is

- in D_1: $(f, n_1) = f_1 + f_2 > 0$,
- in D_2: $(f, n_2) = f_1 > 0$,
- in D_4: $(f, n_4) = -f_1 > 0$,
- in D_3: $(f, n_3) = -f_1 - f_2 > 0$, $(f, n_4) = -f_1 > 0$.

So, inner product $(f, n_i), i = 1, \ldots, 4$ is positive for all points belonging to $\partial E(d,r)$. Hence, the positive half-trajectories intersect $\partial E(d,r)$ from outside to inside. Therefore, all positive half-trajectories enter each $E(d,r)$ and do not leave it. Taking into account that d, r may be arbitrary small, we obtain the result: $C(t, C^0) \to P = (C_1^*, C_2^*)$ as $t \to \infty$.

Since $\frac{\mu}{a_1} < V$, $\frac{\varphi_1}{a_2} < V$, then, obviously, $C_i^* < V, i = 1,2$. Denote by $d(\mu) = C_1^* + C_2^* - V$. Then,

$$d(\mu) = \frac{(2a_1a_2V + A)\mu}{2a_1a_2\mu + a_1A}\frac{2a_2\mu + A}{2a_2\mu} - V = \frac{-a_1\mu + \sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu}}{2a_1a_2} > 0. \quad (6)$$

Therefore, $C_1^* + C_2^* > V$, which can be interpreted as follows: the depreciation causes the excess production. Moreover, now we show that the excess production increases by the increasing of the depreciation. Differentiate (6) with respect to μ

$$d'(\mu) = -\frac{1}{2a_2}\frac{a_1\mu + a_2\varphi}{a_2\sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu}}. \tag{7}$$

Equating (7) to zero we obtain

$$2a_1a_2\mu + 2a_2^2\varphi = a_2a_1^2\mu^2 + 4a_1a_2\varphi\mu.$$

After the squaring we have the quadratic polynomial of μ

$$3a_1^2a_2^2\mu^2 + 4a_1a_2^3\varphi\mu + 4a_2^4\varphi^2. \tag{8}$$

The discriminant of (8) is

$$D = 16a_1^2a_2^6\varphi^2 - 48a_1^2a_2^6\varphi^2 < 0,$$

and since $3a_1^2a_2^2 > 0$, then $d'(\mu) > 0$. Therefore, $d(\mu)$ is increasing.

Now we show that $C_1^* \to 0$, $C_2^* \to V$ as $\mu \to 0$, which means that there is no excess production without the depreciation ($\mu = 0$). Obviously,

$$A = -a_1\mu + \sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu} \to 0 \text{ as } \mu \to 0. \tag{9}$$

Let us study the behavior of the expressions $\frac{A}{\mu}$ and $\frac{\mu}{A}$ as $\mu \to 0$.

$$\frac{A}{\mu} = \frac{-a_1\mu + \sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu}}{\mu} = \frac{4a_1a_2\varphi}{\sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu} + a_1\mu} \to \infty \text{ as } \mu \to 0. \tag{10}$$

$$\frac{A}{\mu} = \frac{\mu}{-a_1\mu + \sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu}} = \frac{\mu(\sqrt{a_1^2\mu^2 + 4a_1a_2\varphi\mu} + a_1\mu)}{4a_1a_2\varphi\mu} = \frac{A}{4a_1a_2\varphi} \to 0 \text{ as } \mu \to 0. \tag{11}$$

From (9), (10), (11)

$$C_1^* = \frac{2a_1a_2V + A}{2a_1a_2 + a_1\frac{A}{\mu}} \to 0 \text{ as } \mu \to 0, \tag{12}$$

$$C_2^* = \frac{2a_1a_2V + A}{2a_1a_2 + a_1\frac{A}{\mu}}\frac{A}{2a_2\mu} = \frac{2a_1a_2V + A}{4a_1a_2^2\frac{\mu}{A} + 2a_1a_2} \to V \text{ as } \mu \to 0. \tag{13}$$

Thus, the maximal excess production occures if $\mu = 1$, and if $\mu = 0$, there is no excess production in the long run. Thus, large depreciation influences the economy negatively by causing the excess production.

4 Three efficiency levels

Now let us consider the model with three efficiency levels.

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi_1}{\lambda_1}C_1(V - C_1 - C_2 - C_3) + \mu_2 C_2 = f_1(C), \\ \dot{C}_2 = \frac{1-\varphi_2}{\lambda_2}C_2(V - C_1 - C_2 - C_3) + \varphi_1 C_1 - \mu_2 C_2 + \mu_3 C_3 = f_2(C), \\ \dot{C}_N = \frac{1-\alpha}{\lambda_3}C_3(V - C_1 - C_2 - C_3) + \varphi_2 C_2 - \mu_3 C_3 = f_3(C). \end{cases} \quad (14)$$

Proposition 3 *The system (14) has a unique equilibrium C^* such that $C_i^* > 0, i = 1, 2, 3$.*

Proof Denote $x = V - C_1 - C_2 - C_3$, $y = \frac{C_2}{C_1} > 0$, $z = \frac{C_3}{C_1} > 0$. Then the system $f_i(C) = 0, i = 1, 2, 3$ has the following form

$$\begin{cases} a_1 x + \mu_2 y = 0, \\ a_2 xy - \mu_2 y + \mu_3 z + \varphi_1 = 0, \\ a_3 xz - \mu_3 z + \varphi_2 y = 0. \end{cases} \quad (15)$$

From (15) we obtain the following polynomial

$$H(y) = \frac{a_3\mu_2 a_2}{a_1^2}y^3 + \left(\frac{a_3\mu_3}{a_1} + \frac{a_2\mu_2}{a_1}\right)y^2 + \left(-\frac{a_3\varphi_1}{a_1} + \mu_3 - \varphi_2\right)y - \varphi_1 = 0 \quad (16)$$

From the Vieta's formulas for the roots y_i^* of (16)

$$\begin{cases} y_1^* + y_2^* + y_3^* = -\left(\frac{a_3\mu_3}{a_1} + \frac{a_2\mu_2}{a_1}\right) < 0, \\ y_1^* y_2^* y_3^* = \varphi_1 > 0. \end{cases}$$

It is possible if and only if the polynomial (16) has the unique real root $y_1^* > 0$ and the other roots, y_2^*, y_3^*, are either negative real or complex-conjugate.

Since in our framework $y = \frac{C_2}{C_1}$ must be positive real then there exists only the root $y_1^* > 0$ of (16) is feasible. Therefore, the system (15) has a unique solution, which means that the system (14) has a unique equilibrium C^* such that $C_i^* > 0, i = 1, 2, 3$.

However, we can only prove the existence and the uniqueness of the equilibrium C^*, but we cannot find this equilibrium in the explicit form in general. Let us consider the case $\mu_2 = 0$. It means that the first efficiency level is so inefficient or obsolete that the firms at the second level under the action of the depreciation process do not transfer to the first level. The system (14) in this case takes the form

$$\begin{cases} \dot{C}_1 = \frac{1-\varphi_1}{\lambda_1} C_1(V - C_1 - C_2 - C_3) = f_1(C), \\ \dot{C}_2 = \frac{1-\varphi_2}{\lambda_2} C_2(V - C_1 - C_2 - C_3) + \varphi_1 C_1 + \mu_3 C_3 = f_2(C), \\ \dot{C}_N = \frac{1-\alpha}{\lambda_3} C_3(V - C_1 - C_2 - C_3) + \varphi_2 C_2 - \mu_3 C_3 = f_3(C). \end{cases} \tag{17}$$

The system (17) has the equilibria $O = (0,0,0)$ and $P = (0, C_2^*, C_3^*)$, where $C_2^* = \frac{(2a_2a_3V+A)\mu_3}{2a_2a_3\mu_3+a_2A}$, $C_3^* = \frac{(2a_2a_3V+A)\mu_3}{2a_2a_3\mu_3+a_2A}\frac{A}{2a_3\mu_3}$.

Theorem 2 *The equilibrium $C^* = (0, C_2^*, C_3^*)$ of the system (17) is globally stable in $\mathbb{R}^3_+ \setminus \{O\}$.*

Proof

According to Proposition 2 all positive half-trajectories enter the set $W = \{(C_1, \dots, C_N) : V \leqslant \sum\limits_{j=1}^{N} C_j \leqslant V + \Delta V\}$, where $\Delta V = \sum\limits_{j=1}^{N-1} C_j \frac{\varphi_j}{a_j}$, and do not leave it. Therefore, all positive half-trajectories are positively Lagrange stable. Hence, the ω-limit set $\Omega_C \neq \varnothing$ for any positive half-trajectory $r(C^0)$ [2].

From (17), it is easily to understand, that in W: $C_1(t) \to 0$ as $t \to \infty$ for any positive half-trajectory. Hence, $C_1 = 0$ for any point belonging to Ω_C, or $\Omega_C \subset \mathbb{R}^2_+ = \{(C_1, C_2, C_3) \in \mathbb{R}^3_+ : C_1 = 0\}$. Let us show that $\Omega_C = P = (0, C_2^*, C_3^*) \in \mathbb{R}^3_+$ for any positive half-trajectory $r(t) \in \mathbb{R}^3_+ \setminus \{O\}$. Assume, on the contrary, that there exists a positive half-trajectory $r^*(C^0)$ for which there exists an ω-limit point $Q \in \Omega_C \subset \mathbb{R}^2_+$ and $Q \neq P$.

Since $\mathbb{R}^2_+$ is the invariant set of (17), then (17) in $\mathbb{R}^2_+$ has the following form

$$\begin{cases} \dot{C}_2 = \frac{1-\varphi_2}{\lambda_2} C_2(V - C_2 - C_3) + \mu_3 C_3 = f_2(C), \\ \dot{C}_3 = \frac{1-\alpha}{\lambda_3} C_3(V - C_2 - C_3) + \varphi_2 C_2 - \mu_3 C_3 = f_3(C), \end{cases} \tag{18}$$

Thus, we can assume that the system (17) has the form (18). One can see, that the system (18) and the system (3) have the same form. Hence, all conclusions obtained in the Section 3 are valid for the system (18). From the Theorem 1 the equilibrium $C^* = (C_2^*, C_3^*) \in \mathbb{R}^2_+$ of this system is globally stable in $\mathbb{R}^2_+ \setminus \{O\}$.

Consider the positive half-trajectory $r(t_Q, Q)$. Consider a point A of the intersection of the positive half-trajectory $r(t_Q, Q)$ with the boundary of the set W. Let us show that this point exists. Consider in $\mathbb{R}^2_+$ the set $W^2 = W \cap \{(C_1, C_2, C_3) : C_1 = 0\}$ and the system $\dot{C} = -f(C)$. Obviously, the boundary

$\partial W^2 = \partial W \cap \{(C_1, C_2, C_3) : C_1 = 0\} \subset \mathbb{R}^2_+$ and in $\mathbb{R}^2_+$ boundaries of W and W^2 coincide. The inner product $(f, n) = (-a_1C_1 - a_2C_2)(-R) + \varphi_1 C_1 > 0$, where $n = (1, 1)$ is the normal of the straight line $C_1 + C_2 = V + R, R \geqslant 0$. Therefore, this positive half-trajectory exits W^2 through ∂W^2 and there exists the point $A = \partial W^2 \cap \{r(t_Q, Q)\} = \partial W \cap \{r(t_Q, Q)\}$. Denote by $\tilde{Q}$ the point such that $C(\tilde{t}, t_Q, Q) = \tilde{Q}$ (i.e. the time of movement along the positive half-trajectory $r(t_Q, Q)$ from Q to $\tilde{Q}$ equals $\tilde{t}$). Let $\pi(A), \pi(\tilde{Q})$ be planes transversal to the positive half-trajectory $r(t_Q, Q)$ at the points A, $\tilde{Q}$ respectively. From the Theorem (17.4) [3] there exists a closed trajectory cylinder $Z \subset \mathbb{R}^3_+$ with the trajectory $r(t_Q, Q)$ as its axis and $U_Z(A) \subset \pi(A)$, $U_Z(\tilde{Q}) \subset \pi(\tilde{Q})$ as its bases. Though $U_Z(A), U_Z(\tilde{Q}) \not\subset \mathbb{R}^3_+$, but because of invariance of $\mathbb{R}^2_+$ the positive half-trajectories, intersecting the sets $U^+_Z(A) = U_Z(A) \cap \mathbb{R}^3_+$, $U^+_Z(\tilde{Q}) = U_Z(\tilde{Q}) \cap \mathbb{R}^3_+$, belong to $\mathbb{R}^3_+$. By construction, $Z \subset W$. Since Q is ω-limit point for $r^*(C^0)$, then this positive half-trajectory has to enter the cylinder $Z \subset W$ infinite number of times. The entrance to the cylinder Z occurs through $U^+_Z(A)$ from $\mathbb{R}^3 \setminus W$. Thus, the positive half-trajectory $r^*(C^0)$ has to leave W before entering the cylinder Z. But the positive half-trajectories cannot leave W. We have reached a contradiction. Thus, Q is not a ω-limit point of this positive half-trajectory. Thus we have, $\Omega_C = \{C^* = (0, C^*_2, C^*_3)\}$ for any trajectory $r(C^0)$. Since any positive half-trajectory $r(C^0)$ is positively Lagrange stable then, according to the Theorem (3.07) [2], $\rho(C(t), \Omega_C) \to 0$ for $t \to \infty$, where $\Omega_C = P$, for any $C(t)$. Thus, $C^* = (0, C^*_2, C^*_3)$ is globally stable in $\mathbb{R}^3_+ \setminus \{O\}$.

Conclusions

In this article, the mathematical model of sector capital distribution dynamics with the depreciation are proposed. The qualitative analysis of the model is presented. The global attractor for the case of the arbitrary number of efficiency levels is established. The equilibria of the constructed dynamical model are determined in the case of two efficiency levels, their global stability is proved. The global stability of the equilibrium $C^* = (C^*_1, C^*_2)$ such that $C^*_1 + C^*_2 > V$ means that the depreciation process causes excess production. It is proved that in the case of three efficiency levels there exists a unique equilibrium. In the case without the capital transferring from the second level to the first level the global stability of the equilibrium is proved.

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