DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 3, 2020
Electronic Journal,
reg. N ФC77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru

Stochastic differential equations
Numerical methods
Computer modeling in dynamical and control systems

Application of Multiple Fourier–Legendre Series to Implementation of Strong Exponential Milstein and Wagner–Platen Methods for Non-Commutative Semilinear Stochastic Partial Differential Equations

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to the application of multiple Fourier–Legendre series for the approximation of iterated stochastic Itô integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. These iterated stochastic integrals are a part of the so-called exponential Milstein and Wagner–Platen numerical methods for semilinear stochastic partial differential equations with nonlinear multiplicative trace class noise. The mentioned numerical methods have strong orders of convergence 1.0− and 1.5− correspondingly with respect to the temporal discretization. The theorem on the mean-square convergence of approximations of iterated stochastic Itô integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process is formulated and proved. The results of this article can be applied to implementation of high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations with nonlinear multiplicative trace class noise.

Key words: Non-commutative semilinear stochastic partial differential equation, infinite-dimensional Q-Wiener process, multiplicative trace class noise, iterated stochastic Itô integral, generalized multiple Fourier series, multiple Fourier–Legendre series, exponential Milstein scheme, exponential Wagner–Platen scheme, Legendre polynomials, mean-square approximation, expansion.

1 Introduction

This paper continues the author's research [1] on methods of the mean-square approximation of iterated stochastic Itô integrals with respect to the infinite dimensional Q-Wiener process.

It is well-known that one of the effective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for semilinear stochastic partial differential equations (SPDEs) is based on the Taylor formula in Banach spaces and the exponential formula for the mild solution of SPDE [2]-[7]. A significant step in this direction was made in [6], [7], where the exponential Milstein and Wagner–Platen methods for semilinear SPDEs were constructed. Under the appropriate conditions [6], [7] these methods have strong orders of convergence $1.0-\varepsilon$ and $1.5-\varepsilon$ correspondingly with respect to the temporal variable (where ε is an arbitrary small posilive real number). It should be noted that in [8] the convergence of the exponential Milstein scheme for semilinear SPDEs with strong order 1.0 has been proved under additional smoothness assumptions.

An important feature of the mentioned numerical methods is the presence in them the so-called iterated stochastic Itô integrals with respect to the infinite-dimensional Q-Wiener process [9]. The problem of numerical modeling of these stochastic integrals with multiplicities 1 to 3 was solved in [6], [7] for the case when special commutativity conditions for SPDE are fulfilled.

If the mentioned commutativity conditions are not satisfied, which often corresponds to SPDEs in numerous applications, the numerical modeling of iterated stochastic Itô integrals with respect to the infinite-dimensional Q-Wiener process becomes much more difficult. Note that the exponential Milstein scheme [6] contains the iterated stochastic Itô integrals of multiplicities 1 and 2 with respect to the infinite-dimensional Q-Wiener process and the exponential Wagner–Platen scheme [7] contains the mentioned stochastic integrals of multiplicities 1 to 3. In [10], [11] two methods of the mean-square approximation of iterated stochastic Itô integrals from the exponential Milstein scheme for semilinear SPDEs without the commutativity conditions have been considered. Note that the mean-square error of approximation of these stochastic integrals consists of two components [10], [11]. The first component is related with the finite-dimentional approximation of the infinite-dimentional Q-Wiener process while the second one is connected with the approximation of iterated stochastic Itô integrals with respect to the scalar standard Brownian motions. In the author's publication [1] the problem of the mean-square approximation of iterated stochastic Itô integrals with respect to the infinite-dimensional Q-Wiener process in the sense of second component of approximation error (see above) has been solved for arbitraty multiplicity k ($k \in \mathbb{N}$) of stochastic integrals and without the assumptions of commutativity for SPDE. More precisely, in [1] the method of generalized multiple Fourier series [12]-[22] for the approximation of iterated stochastic Itô integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic Itô integrals with respect to the infinite-dimensional Q-Wiener process (in the sense of the second component of approximation error).

In this article, we extend the method [10], [11] for estimating the first component of approximation error for iterated stochastic Itô integrals of multiplicities 1 to 3 with respect to the infinite-dimensional Q-Wiener process. In addition, we combine the obtained results with results from [1]. Thus, results of the paper can be applied to the implementation of exponential Milstein and Wagner–Platen schemes for semilinear SPDEs with nonlinear multiplicative trace class noise and without the commutativity conditions.

2 Exponential Milstein and Wagner–Platen Numerical Schemes for Non-Commutative Semilinear SPDEs

Let U, H be separable $\mathbb{R}$-Hilbert spaces and $L_{HS}(U, H)$ be a space of Hilbert–Schmidt operators. Let $(\Omega, \mathbf{F}, \mathsf{P})$ be a probability space with a normal filtration $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$ [9], let $\mathbf{W}_t$ be an U-valued Q-Wiener process with respect to $\{\mathbf{F}_t, t \in [0, \bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here $L(U)$ denotes all bounded linear operators on U. Let U_0 be an $\mathbb{R}$-Hilbert space $U_0 = Q^{1/2}(U)$ with a scalar product [6], [7]

$$\langle u, w \rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w \right\rangle_U$$

for all $u, w \in U_0$.

Consider the semilinear parabolic SPDE with multiplicative trace class noise

$$dX_t = (AX_t + F(X_t))\, dt + B(X_t)d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}], \tag{1}$$

where nonlinear operators F, B ($F : H \to H$, $B : H \to L_{HS}(U_0, H)$), linear operator $A : D(A) \subset H \to H$ as well as the initial value ξ are assumed to satisfy the conditions of existence and uniqueness of the SPDE mild solution (see [7], Assumptions A1–A4).

It is well-known [24] that Assumptions A1–A4 [7] guarantee the existence and uniqueness (up to modifications) of the mild solution $X_t : [0, \bar{T}] \times \Omega \to H$ of SPDE (1)

$$X_t = \exp(At)\xi + \int\limits_0^t \exp(A(t-\tau))F(X_\tau)d\tau + \int\limits_0^t \exp(A(t-\tau))B(X_\tau)d\mathbf{W}_\tau \tag{2}$$

with probability 1 (further w. p. 1) for all $t \in [0, \bar{T}]$, where $\exp(At)$ is the semigroup generated by the operator A.

Consider eigenvalues λ_i and eigenfunctions $e_i(x)$ of the covariance operator Q, where $i = (i_1, \ldots, i_d) \in J$, $J = \{i : \ i \in \mathbb{N}^d, \text{ and } \lambda_i > 0\}$, and $x = (x_1, \ldots, x_d) \in U$.

The series representation of the Q-Wiener process $\mathbf{W}_t$ has the following form [9]

$$\mathbf{W}_t = \sum_{i \in J} e_i \sqrt{\lambda_i} \mathbf{w}_t^{(i)} \quad \text{or} \quad \mathbf{W}_t = \sum_{i \in J_M} e_i \, \langle e_i, \mathbf{W}_t \rangle_U,$$

where $t \in [0, \bar{T}]$, $\mathbf{w}_t^{(i)}$ ($i \in J$) are independent standard Wiener processes, and $\langle \cdot, \cdot \rangle_U$ is a scalar product in U. Note that eigenfunctions $e_i(x)$, $i \in J$ form an orthonormal basis of U [9].

Consider the finite-dimensional approximation of $\mathbf{W}_t$ [9]

$$\mathbf{W}_t^M = \sum_{i \in J_M} e_i \sqrt{\lambda_i} \mathbf{w}_t^{(i)}, \quad t \in [0, \bar{T}], \tag{3}$$

where

$$J_M = \{i : \ 1 \le i_1, \ldots, i_d \le M, \text{ and } \lambda_i > 0\}. \tag{4}$$

Remark 1. *Obviously, without the loss of generality we can suppose that* $J_M = \{1, 2, \ldots, M\}$.

Let $\Delta > 0$, $\tau_p = p\Delta$ $(p = 0, 1, \ldots, N)$, and $N\Delta = \bar{T}$. Consider the exponential Milstein numerical scheme [6]

$$
Y_{p+1} = \exp\left(A\Delta\right)\left(Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s + \right.
$$

$$
\left. + \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau\right) d\mathbf{W}_s\right), \qquad (5)
$$

and Wagner–Platen numerical scheme [7]

$$
Y_{p+1} = \exp\left(\frac{A\Delta}{2}\right)\left(\exp\left(\frac{A\Delta}{2}\right) Y_p + \Delta F(Y_p) + \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s + \right.
$$

$$
+ \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau\right) d\mathbf{W}_s +
$$

$$
+ \frac{\Delta^2}{2} F'(Y_p)\Big(AY_p + F(Y_p)\Big) + \int\limits_{\tau_p}^{\tau_{p+1}} F'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau\right) ds +
$$

$$
+ \frac{\Delta^2}{4} \sum_{i \in J} \lambda_i F''(Y_p)\Big(B(Y_p)e_i, B(Y_p)e_i\Big) +
$$

$$
+ A\left(\int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau ds - \frac{\Delta}{2} \int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s\right) +
$$

$$
+ \Delta \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\Big(AY_p + F(Y_p)\Big) d\mathbf{W}_s - \int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{s} B'(Y_p)\Big(AY_p + F(Y_p)\Big) d\mathbf{W}_\tau ds +
$$

$$
+ \frac{1}{2} \int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau, \int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau\right) d\mathbf{W}_s +
$$

$$+\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B'(Y_p)\left(\int\limits_{\tau_p}^{\tau} B(Y_p)d\mathbf{W}_\theta\right)d\mathbf{W}_\tau\right)d\mathbf{W}_s\Biggr) \qquad (6)$$

for SPDE (1), where Y_p is an approximation of X_{τ_p} (mild solution (2) at the time moment τ_p), $p = 0, 1, \ldots, N$, and B', B'', F', F'' are Fréchet derivatives [23]. Note that in addition to the temporal discretization, the implementation of numerical schemes (5) and (6) also requires a discretization of the infinite-dimensional Hilbert space H (approximation with respect to the space domain) and a finite-dimensional approximation of the Q-Wiener process.

Let us focus on the approximation connected with the Q-Wiener process. Consider the following iterated Itô stochastic integrals

$$I_{(1)T,t}^{(r_1)} = \int\limits_t^T d\mathbf{w}_{t_1}^{(r_1)}, \quad I_{(10)T,t}^{(r_1 0)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)} dt_2, \quad I_{(01)T,t}^{(0 r_2)} = \int\limits_t^T \int\limits_t^{t_2} dt_1 d\mathbf{w}_{t_2}^{(r_2)}, \qquad (7)$$

$$I_{(11)T,t}^{(r_1 r_2)} = \int\limits_t^T \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)} d\mathbf{w}_{t_2}^{(r_2)}, \quad I_{(111)T,t}^{(r_1 r_2 r_3)} = \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} d\mathbf{w}_{t_1}^{(r_1)} d\mathbf{w}_{t_2}^{(r_2)} d\mathbf{w}_{t_3}^{(r_3)}, \qquad (8)$$

where $r_1, r_2, r_3 \in J_M$, $0 \le t < T \le \bar{T}$, and J_M is defined by (4).

Let us replace the infinite-dimensional Q-Wiener process in the iterated stochastic Itô integrals from (5), (6) by its finite-dimensional approximation (3). Then w. p. 1 we have

$$\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s^M = \sum_{r_1 \in J_M} B(Y_p) e_{r_1} \sqrt{\lambda_{r_1}} I_{(1)\tau_{p+1},\tau_p}^{(r_1)}, \qquad (9)$$

$$A\left(\int\limits_{\tau_p}^{\tau_{p+1}} \int\limits_{\tau_p}^{s} B(Y_p) d\mathbf{W}_\tau^M ds - \frac{\Delta}{2}\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s^M\right) =$$

$$= A\int\limits_{\tau_p}^{\tau_{p+1}} B(Y_p)\left(\frac{\tau_{p+1}}{2} - s + \frac{\tau_p}{2}\right) d\mathbf{W}_s^M =$$

$$= \sum_{r_1 \in J_M} A B(Y_p) e_{r_1} \sqrt{\lambda_{r_1}} \left(\frac{\Delta}{2} I_{(1)\tau_{p+1},\tau_p}^{(r_1)} - I_{(01)\tau_{p+1},\tau_p}^{(0 r_1)}\right), \qquad (10)$$

$$\Delta \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_s^M - \int\limits_{\tau_p}^{\tau_{p+1}}\int\limits_{\tau_p}^{s} B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)d\mathbf{W}_\tau^M ds =$$

$$= \int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\int\limits_{\tau_p}^{s}\Bigl(AY_p+F(Y_p)\Bigr)d\tau d\mathbf{W}_s^M =$$

$$= \sum_{r_1\in J_M} B'(Y_p)\Bigl(AY_p+F(Y_p)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}I_{(01)\tau_{p+1},\tau_p}^{(0r_1)}, \qquad (11)$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} F'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)ds =$$

$$= \sum_{r_1\in J_M} F'(Y_p)B(Y_p)e_{r_1}\sqrt{\lambda_{r_1}}\left(\Delta I_{(1)\tau_{p+1},\tau_p}^{(r_1)} - I_{(01)\tau_{p+1},\tau_p}^{(0r_1)}\right), \qquad (12)$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2\in J_M} B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(11)\tau_{p+1},\tau_p}^{(r_1r_2)},$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B'(Y_p)\left(\int\limits_{\tau_p}^{s} B'(Y_p)\left(\int\limits_{\tau_p}^{\tau} B(Y_p)d\mathbf{W}_\theta^M\right)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$= \sum_{r_1,r_2,r_3\in J_M} B'(Y_p)\left(B'(Y_p)\left(B(Y_p)e_{r_1}\right)e_{r_2}\right)e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}I_{(111)\tau_{p+1},\tau_p}^{(r_1r_2r_3)},$$

$$\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_\tau^M\right)d\mathbf{W}_s^M =$$

$$
= \sum_{r_1,r_2,r_3\in J_M} B''(Y_p)\left(B(Y_p)e_{r_1}, B(Y_p)e_{r_2}\right) e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times
$$

$$
\times \int\limits_{\tau_p}^{\tau_{p+1}} \left(\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)} \int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)}\right) d\mathbf{w}_{s}^{(r_3)}. \tag{13}
$$

Note that in (10)–(12) we used the Itô formula. Moreover, using the Itô formula we obtain

$$
\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)} \int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)} = I_{(11)s,\tau_p}^{(r_1 r_2)} + I_{(11)s,\tau_p}^{(r_2 r_1)} + \mathbf{1}_{\{r_1=r_2\}}(s-\tau_p) \quad \text{w. p. 1}, \tag{14}
$$

where $\mathbf{1}_A$ is the indicator of the set A.

From (14) w. p. 1 we have

$$
\int\limits_{\tau_p}^{\tau_{p+1}} \left(\int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_1)} \int\limits_{\tau_p}^{s} d\mathbf{w}_{\tau}^{(r_2)}\right) d\mathbf{w}_{s}^{(r_3)} = I_{(111)\tau_{p+1},\tau_p}^{(r_1 r_2 r_3)} + I_{(111)\tau_{p+1},\tau_p}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1=r_2\}} I_{(01)\tau_{p+1},\tau_p}^{(0 r_3)}. \tag{15}
$$

After substituting (15) into (13) w. p. 1 we obtain

$$
\int\limits_{\tau_p}^{\tau_{p+1}} B''(Y_p)\left(\int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_{\tau}^{M}, \int\limits_{\tau_p}^{s} B(Y_p)d\mathbf{W}_{\tau}^{M}\right) d\mathbf{W}_{s}^{M} =
$$

$$
= \sum_{r_1,r_2,r_3\in J_M} B''(Z)\left(B(Z)e_{r_1}, B(Z)e_{r_2}\right) e_{r_3}\sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}}\times
$$

$$
\times \left(I_{(111)\tau_{p+1},\tau_p}^{(r_1 r_2 r_3)} + I_{(111)\tau_{p+1},\tau_p}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1=r_2\}} I_{(01)\tau_{p+1},\tau_p}^{(0 r_3)}\right). \tag{16}
$$

Thus, for the implementation of numerical schemes (5) and (6) we need to approximate the following iterated stochastic Itô integrals

$$
I_{(1)T,t}^{(r_1)},\quad I_{(01)T,t}^{(0 r_1)},\quad I_{(11)T,t}^{(r_1 r_2)},\quad I_{(111)T,t}^{(r_1 r_2 r_3)},
$$

where $r_1, r_2, r_3 \in J_M,\ \ 0 \le t < T \le \bar{T}$.

3 Approximation of Iterated Stochastic Itô Integrals of Multiplicity k with Respect to the Q-Wiener Process

At first, consider an efficient method [12] (also see [1], [13]-[22]) of the mean-square approximation of iterated stochastic Itô integrals of the form

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (17)$$

where $0 \le t < T \le \bar{T}$, $\psi_l(\tau)$ $(l = 1,\ldots,k)$ are continuous non-random functions on $[t, T]$, $\mathbf{w}_\tau^{(i)}$ $(i = 1,\ldots,m)$ are independent standard Wiener processes, $\mathbf{w}_\tau^{(0)} = \tau$, $i_1,\ldots,i_k = 0, 1,\ldots,m$.

Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t, T])$ and define the following function on the hypercube $[t, T]^k$

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), & t_1 < \ldots < t_k \\ 0, & \text{otherwise} \end{cases}, \qquad (18)$$

where $t_1,\ldots,t_k \in [t, T]$ $(k \ge 2)$, and $K(t_1) \equiv \psi_1(t_1)$, $t_1 \in [t, T]$.

The function $K(t_1,\ldots,t_k)$ is piecewise continuous on the hypercube $[t, T]^k$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k) \in L_2([t, T]^k)$ converges to $K(t_1,\ldots,t_k)$ on the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \left\| K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|_{L_2([t,T]^k)} = 0, \qquad (19)$$

where

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \qquad (20)$$

is the Fourier coefficient and

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right)^{1/2}.$$

Consider the discretization $\{\tau_j\}_{j=0}^N$ of $[t,T]$ such that

$$t=\tau_0<\ldots<\tau_N=T,\qquad \Delta_N=\max_{0\le j\le N-1}\Delta\tau_j\to 0\ \ \text{if}\ \ N\to\infty, \tag{21}$$

where $\Delta\tau_j=\tau_{j+1}-\tau_j$.

Theorem 1 [12] (also see [1], [13]-[22]). *Suppose that* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *are continuous non-random functions on the interval* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuous functions in* $L_2([t,T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{22}$$

where

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)} \tag{23}$$

are independent standard Gaussian random variables for various i *or* j (*if* $i\ne 0$), $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (20), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is the discretization* (21).

Note that in [12]-[15], [18], [19] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Another version of Theorem 1 related to the application of complete orthonormal systems of functions with weight $r(t_1)\ldots r(t_k)\ge 0$ in $L_2([t,T]^k)$ was considered in [18].

Obtain transformed particular cases of Theorem 1 for $k=1,\ldots,4$ [1], [12]-[22]

$$J[\psi^{(1)}]_{T,t}^{(i_1)}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{24}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} C_{j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right), \qquad (25)$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \right.$$

$$\left. -\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\right), \qquad (26)$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\left(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)} - \right.$$
$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)} -$$
$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} +$$
$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} +$$
$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}} +$$
$$\left. +\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\right), \qquad (27)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Let us consider the generalization of the formulas (24) – (27) for the case of arbitrary k ($k \in \mathbb{N}$).

Theorem 2 [13] (also see [1], [14], [15], [18], [19]). *In conditions of Theorem* 1 *the following mean-square converging expansion is valid*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r \times \right.$$

$$\left. \times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\right), \qquad (28)$$

where $[\cdot]$ *is an integer part of a real number,*

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}$$

means the sum according to all possible permutations of the set

$$(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}),$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$*, braces mean an disordered set, and parentheses mean an ordered set.*

Assume that $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1\ldots p_k}$ is an approximation of stochastic integral (17), which is the prelimit expression in (22) or (28).

Let us denote

$$E^{(i_1\ldots i_k)p_1,\ldots,p_k}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)p_1,\ldots,p_k}\right)^2\right\},$$

$$I_k=\|K\|^2_{L_2([t,T]^k)}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{29}$$

In [15], [18], [19] it was shown that

$$E_k^{(i_1\ldots i_k)p_1,\ldots,p_k}\le k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C^2_{j_k\ldots j_1}\right) \tag{30}$$

for $i_1,\ldots,i_k=1,\ldots,m$ and $T-t\in(0,+\infty)$ or $i_1,\ldots,i_k=0,1,\ldots,m$ and $T-t\in(0,1)$.

Using Theorem 1 and complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$ we obtain the following approximations of iterated stochastic Itô integrals (7), (8) [1], [12]-[22] (also see early publication [25])

$$I_{(1)T,t}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{31}$$

$$I_{(01)T,t}^{(0i_1)}=\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{32}$$

$$I_{(10)T,t}^{(i_10)} = \frac{(T-t)^{3/2}}{2}\biggl(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\biggr), \tag{33}$$

$$I_{(11)T,t}^{(i_1i_2)q} = \frac{T-t}{2}\left(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right) - \mathbf{1}_{\{i_1=i_2\}}\right), \tag{34}$$

$$I_{(111)T,t}^{(i_1i_2i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{35}$$

$$I_{(111)T,t}^{(i_1i_1i_1)} = \frac{1}{6}(T-t)^{3/2}\left(\left(\zeta_0^{(i_1)}\right)^3 - 3\zeta_0^{(i_1)}\right),$$

$$C_{j_3j_2j_1} = \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(T-t)^{3/2}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1} = \int\limits_{-1}^{1} P_{j_3}(z)\int\limits_{-1}^{z} P_{j_2}(y)\int\limits_{-1}^{y} P_{j_1}(x)dxdydz,$$

where the Gaussian random variable $\zeta_j^{(i)}$ (if $i \ne 0$) is defined by (23) and $P_j(x)$ $(j = 0, 1, 2, \ldots)$ is a complete orthonormal system of Legendre polynomials in the space $L_2([-1, 1])$ [26].

Note that for pairwise different $i_1, i_2, i_3 = 1, \ldots, m$ we have [1], [12]-[22], [25]

$$E^{(i_1i_2)q,q} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{q}\frac{1}{4i^2-1}\right), \tag{36}$$

$$E^{(i_1i_2i_3)p,p,p} = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}^2. \tag{37}$$

Consider the iterated stochastic Itô integral with respect to the Q-Wiener process in the following form

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}\right)\psi_2(t_2)d\mathbf{W}_{t_2}\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}, \qquad (38)$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, every non-random function $\psi_l(\tau)$ $(l=1,\ldots,k)$ is continuous on the interval $[t,T]$, and $\Phi_k(v)(\ldots(\Phi_2(v)(\Phi_1(v)))\ldots)$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0\times\ldots\times U_0}_{k\ \text{times}}$ to H for all $v\in H$.

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M$ be an approximation of the stochastic integral (38)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^M=\int\limits_t^T\Phi_k(Z)\left(\ldots\left(\int\limits_t^{t_3}\Phi_2(Z)\times\right.\right.$$

$$\left.\left.\times\left(\int\limits_t^{t_2}\Phi_1(Z)\psi_1(t_1)d\mathbf{W}_{t_1}^M\right)\psi_2(t_2)d\mathbf{W}_{t_2}^M\right)\ldots\right)\psi_k(t_k)d\mathbf{W}_{t_k}^M=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^k\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}, \qquad (39)$$

where $0\le t<T\le\bar{T}$, and

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_3}\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(r_1)}d\mathbf{w}_{t_2}^{(r_2)}\ldots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated stochastic Itô integral (17).

Let $I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}$ be an approximation of the iterated stochastic integral (39)

$$I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t}^{M,p_1\ldots,p_k}=$$

$$=\sum_{r_1,r_2,\ldots,r_k\in J_M}\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\left(\prod_{l=1}^{k}\lambda_{r_l}\right)^{1/2}\times$$

$$\times J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}, \tag{40}$$

where $J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}$ is defined as a prelimit expression in (22)

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(r_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(r_k)}\Biggr) \tag{41}$$

or as a prelimit expression in (28)

$$J[\psi^{(k)}]_{T,t}^{(r_1r_2\ldots r_k)p_1\ldots p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(r_l)}+\sum_{m=1}^{[k/2]}(-1)^m\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2m-1},g_{2m}\}\},\{q_1,\ldots,q_{k-2m}\})\\ \{g_1,g_2,\ldots,g_{2m-1},g_{2m},q_1,\ldots,q_{k-2m}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{m}\mathbf{1}_{\{r_{g_{2s-1}}=\ r_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2m}\zeta_{j_{q_l}}^{(r_{q_l})}\Biggr). \tag{42}$$

Let U, H be separable $\mathbb{R}$-Hilbert spaces, $U_0=Q^{1/2}(U)$, and $L(U,H)$ be the space of linear and bounded operators mapping from U to H. Let

$$L(U,H)_0=\{T|_{U_0}:\ T\in L(U,H)\},$$

where $T|_{U_0}$ is the restriction of operator T to the space U_0. It is known [9] that $L(U,H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0,H)$.

Theorem 3 [1], [22]. *Let the conditions of Theorem* 1 *be fulfilled as well as the following conditions*:

1. *$Q \in L(U)$ is a nonnegative and symmetric trace class operator (λ_i and e_i $(i \in J)$ are its eigenvalues and eigenfunctions correspondingly), and $\mathbf{W}_\tau$, $\tau \in [0, \bar{T}]$ is an U-valued Q-Wiener process.*

2. *$Z : \Omega \to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping.*

3. *$\Phi_1 \in L(U, H)_0$, $\Phi_2 \in L(H, L(U, H)_0)$, and $\Phi_k(v)(\ \ldots(\Phi_2(v)(\Phi_1(v)))\ldots\ )$ is a k-linear Hilbert–Schmidt operator mapping from $\underbrace{U_0 \times \ldots \times U_0}_{k\ times}$ to H for all $v \in H$ such that*

$$
\left\|\Phi_k(Z)\left(\ldots\left(\Phi_2(Z)\left(\Phi_1(Z)e_{r_1}\right)e_{r_2}\right)\ldots\right)e_{r_k}\right\|_H^2 \le L_k < \infty
$$

w. p. 1 for all $r_1, r_2, \ldots, r_k \in J_M$, $M \in \mathbb{N}$. Then

$$
\mathsf{M}\left\{\left\|I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1\ldots p_k}\right\|_H^2\right\} \le
$$

$$
\le L_k (k!)^2 \left(\mathrm{tr}\ Q\right)^k \left(I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2\right), \tag{43}
$$

where I_k is defined by (29), and

$$
\mathrm{tr}\ Q = \sum_{i \in J} \lambda_i.
$$

Note that the right-hand side of the inequality (43) is independent of M and tends to zero if $p_1, \ldots, p_k \to \infty$ due to the Parseval's equality.

4 Approximation of Iterated Stochastic Integrals From the Exponential Milstein and Wagner–Platen Schemes for Non-Commutative Semilinear SPDEs

This section is devoted to the approximation of iterated stochastic integrals from the Milstein scheme (5) and Wagner–Platen scheme (6) for non-commutative semilinear SPDEs. These integrals have the following form

$$J_1[B(Z)]_{T,t} = \int\limits_t^T B(Z)d\mathbf{W}_{t_1}, \tag{44}$$

$$J_2[B(Z)]_{T,t} = A\left(\int\limits_t^T\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}dt_2 - \frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_{t_1}\right), \tag{45}$$

$$J_3[B(Z),F(Z)]_{T,t} = (T-t)\int\limits_t^T B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}-$$

$$-\int\limits_t^T\int\limits_t^{t_2} B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}dt_2, \tag{46}$$

$$J_4[B(Z),F(Z)]_{T,t} = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)dt_2, \tag{47}$$

$$I_1[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{48}$$

$$I_2[B(Z)]_{T,t} = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}, \tag{49}$$

$$I_3[B(Z)]_{T,t} = \int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right)d\mathbf{W}_{t_2}, \tag{50}$$

where $Z:\Omega\to H$ is an $\mathbf{F}_t/\mathcal{B}(H)$-measurable mapping, $0\le t<T\le\bar{T}$.

Note that according to (9)–(12), (31), and (32) we can write the following relatively simple formulas for numerical modeling

$$J_1[B(Z)]_{T,t}^M = \int\limits_t^T B(Z)d\mathbf{W}_s^M =$$

$$= (T-t)^{1/2}\sum_{r_1\in J_M} B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_0^{(r_1)},$$

$$J_2[B(Z)]_{T,t}^M = A\left(\int\limits_t^T\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M dt_2 - \frac{(T-t)}{2}\int\limits_t^T B(Z)d\mathbf{W}_{t_1}^M\right) =$$

$$= -\frac{(T-t)^{3/2}}{2\sqrt{3}}\sum_{r_1\in J_M} AB(Z)e_{r_1}\sqrt{\lambda_{r_1}}\zeta_1^{(r_1)}, \qquad (51)$$

$$J_3[B(Z),F(Z)]_{T,t}^M = (T-t)\int\limits_t^T B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}^M -$$

$$-\int\limits_t^T\int\limits_t^{t_2} B'(Z)\Bigl(AZ+F(Z)\Bigr)d\mathbf{W}_{t_1}^M dt_2 =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} B'(Z)\Bigl(AZ+F(Z)\Bigr)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \qquad (52)$$

$$J_4[B(Z),F(Z)]_{T,t}^M = \int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)dt_2 =$$

$$= \frac{(T-t)^{3/2}}{2}\sum_{r_1\in J_M} F'(Z)B(Z)e_{r_1}\sqrt{\lambda_{r_1}}\left(\zeta_0^{(r_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(r_1)}\right), \qquad (53)$$

where $\zeta_0^{(r_1)}$, $\zeta_1^{(r_1)}$ $(r_1 \in J_M)$ are independent standard Gaussian random variables.

Further consider the more complicate for approximation stochastic integrals (48)–(50) in detail.

Let $I_1[B(Z)]_{T,t}^M$, $I_2[B(Z)]_{T,t}^M$, $I_3[B(Z)]_{T,t}^M$ be approximations of stochastic integrals (48)–(50), which have the following form

$$I_1[B(Z)]_{T,t}^M = \int\limits_t^T B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M =$$

$$= \sum_{r_1,r_2\in J_M} B'(Z)\left(B(Z)e_{r_1}\right)e_{r_2}\sqrt{\lambda_{r_1}\lambda_{r_2}}I_{(11)T,t}^{(r_1r_2)}, \qquad (54)$$

$$I_2[B(Z)]_{T,t}^M = \int_t^T B'(Z) \left(\int_t^{t_3} B'(Z) \left(\int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M \right) d\mathbf{W}_{t_2}^M \right) d\mathbf{W}_{t_3}^M =$$

$$= \sum_{r_1,r_2,r_3 \in J_M} B'(Z) \left(B'(Z) \left(B(Z) e_{r_1} \right) e_{r_2} \right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} I_{(111)T,t}^{(r_1 r_2 r_3)}, \tag{55}$$

$$I_3[B(Z)]_{T,t}^M = \int_t^T B''(Z) \left(\int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M, \int_t^{t_2} B(Z) d\mathbf{W}_{t_1}^M \right) d\mathbf{W}_{t_2}^M =$$

$$= \sum_{r_1,r_2,r_3 \in J_M} B''(Z) \left(B(Z) e_{r_1}, B(Z) e_{r_2} \right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(I_{(111)T,t}^{(r_1 r_2 r_3)} + I_{(111)T,t}^{(r_2 r_1 r_3)} + \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(0 r_3)} \right). \tag{56}$$

Let $I_1[B(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M.q}$, $I_3[B(Z)]_{T,t}^{M,q}$ be approximations of stochastic integrals (54)–(56), which are represented as follows

$$I_1[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2 \in J_M} B'(Z) \left(B(Z) e_{r_1} \right) e_{r_2} \sqrt{\lambda_{r_1}\lambda_{r_2}} I_{(11)T,t}^{(r_1 r_2)q},$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3 \in J_M} B'(Z) \left(B'(Z) \left(B(Z) e_{r_1} \right) e_{r_2} \right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} I_{(111)T,t}^{(r_1 r_2 r_3)q}, \tag{57}$$

$$I_3[B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3 \in J_M} B''(Z) \left(B(Z) e_{r_1}, B(Z) e_{r_2} \right) e_{r_3} \sqrt{\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}} \times$$

$$\times \left(I_{(111)T,t}^{(r_1 r_2 r_3)q} + I_{(111)T,t}^{(r_2 r_1 r_3)q} + \mathbf{1}_{\{r_1 = r_2\}} I_{(01)T,t}^{(0 r_3)} \right), \tag{58}$$

where $q \geq 1$, approximations $I_{(11)T,t}^{(r_1 r_2)q}$, $I_{(111)T,t}^{(r_1 r_2 r_3)q}$, $I_{(111)T,t}^{(r_2 r_1 r_3)q}$ are defined by (34), (35), and $I_{(01)T,t}^{(0 r_3)}$ has the form (32).

Recall that $L_{HS}(U_0, H)$ is a space of Hilbert–Schmidt operators from U_0 to H. Moreover, let $L_{HS}^{(2)}(U_0, H)$ and $L_{HS}^{(3)}(U_0, H)$ be spaces of bilinear and 3-linear

Hilbert–Schmidt operators from $U_0 \times U_0$ to H and from $U_0 \times U_0 \times U_0$ to H correspondingly. Furthermore, let $\|\cdot\|_{L_{HS}(U_0,H)}$, $\|\cdot\|_{L_{HS}^{(2)}(U_0,H)}$, and $\|\cdot\|_{L_{HS}^{(3)}(U_0,H)}$ be operator norms in these spaces.

Theorem 4. *Let the conditions* 1, 2 *of Theorem* 3 *as well as the conditions of Theorem* 1 *be fulfilled. Futhermore, let*

$$B(v) \in L_{HS}(U_0, H), \quad B'(v)(B(v)) \in L_{HS}^{(2)}(U_0, H),$$

$$B'(v)(B'(v)(B(v))), \quad B''(v)(B(v), B(v)) \in L_{HS}^{(3)}(U_0, H)$$

for all $v \in H$ (*we suppose that Frêchet derivatives* B', B'' *exist*; *see Sect.* 2). *Moreover, let there exists a constant* C *such that w. p.* 1

$$\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)} < C, \quad \left\|B'(Z)(B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(2)}(U_0,H)} < C,$$

$$\left\|B'(Z)(B'(Z)(B(Z)))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)} < C,$$

$$\left\|B''(Z)(B(Z), B(Z))Q^{-\alpha}\right\|_{L_{HS}^{(3)}(U_0,H)} < C$$

for some $\alpha > 0$. *Then*

$$\mathsf{M}\left\{\left\|I_1[B(Z)]_{T,t} - I_1[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le (T-t)^2 \left(C_0 \left(\text{tr } Q\right)^2 \left(\frac{1}{2} - \sum_{j=1}^{q} \frac{1}{4j^2-1}\right) + K_Q \left(\sup_{i \in J \backslash J_M} \lambda_i\right)^{2\alpha}\right), \qquad (59)$$

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\} \le$$

$$\le (T-t)^3 \left(C_1 \left(\text{tr } Q\right)^3 \left(\frac{1}{6} - \sum_{j_1,j_2,j_3=0}^{q} \hat{C}_{j_3 j_2 j_1}^2\right) + L_Q \left(\sup_{i \in J \backslash J_M} \lambda_i\right)^{2\alpha}\right), \qquad (60)$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\le(T-t)^3\left(C_2\,(\mathrm{tr}\ Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}_{j_3j_2j_1}^2\right)+M_Q\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\right),\qquad(61)$$

where $q\in\mathbb{N}$, $C_0,C_1,C_2,K_Q,L_Q,M_Q<\infty$, *and*

$$\hat{C}_{j_3j_2j_1}=\frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}}{8}\bar{C}_{j_3j_2j_1},$$

$$\bar{C}_{j_3j_2j_1}=\int\limits_{-1}^{1}P_{j_3}(z)\int\limits_{-1}^{z}P_{j_2}(y)\int\limits_{-1}^{y}P_{j_1}(x)dxdydz,$$

where $P_j(x)$ $(j=0,1,2,\ldots)$ *is the Legendre polynomial.*

Remark 2. *Note that the estimate similar to* (59) *has been derived in* [10], [11] (*also see* [6]) *with the difference connected with the first term on the right-hand side of* (59). *In* [10] *the authors used the Karhunen–Loeve expansion of the Brownian bridge process for the approximation of iterated stochastic Itô integrals with respect to the finite-dimensional Wiener process. In this article we apply Theorem* 1 *and the system of Legendre polynomials to obtain the first term on the right-hand side of* (59).

Remark 3. *If we assume that* $\lambda_i\le C'i^{-\gamma}$ $(\gamma>1,C'<\infty)$ *for* $i\in J$, *then the parameter* $\alpha>0$ *obviously increases with decreasing* γ [11].

Proof. The estimate (59) follows directly from (43) for $k=2$ (the first term on the right-hand side of (59)) and Theorem 1 from [10] (the second term on the right-hand side of (59)). Further C_3, C_4, ... denote various constants.

Let us prove the estimates (60), (61). Using Theorem 3 we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]_{T,t}^{M}\right\|_H^2\right\}+$$

$$+2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}^{M}-I_2[B(Z)]_{T,t}^{M,q}\right\|_H^2\right\}\le$$

$$\leq 2\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t}-I_2[B(Z)]^{M}_{T,t}\right\|^2_H\right\}+$$

$$+C_3(T-t)^3\,(\text{tr }Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}^2_{j_3j_2j_1}\right),\qquad(62)$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]^{M,q}_{T,t}\right\|^2_H\right\}\leq 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]^{M}_{T,t}\right\|^2_H\right\}+$$

$$+2\mathsf{M}\left\{\left\|I_3[B(Z)]^{M}_{T,t}-I_3[B(Z)]^{M,q}_{T,t}\right\|^2_H\right\}.\qquad(63)$$

Repeating with insignificant modification the proof of Theorem 3 for the case $k=3$ (see for details [1], pages 39–44) we have

$$\mathsf{M}\left\{\left\|I_3[B(Z)]^{M}_{T,t}-I_3[B(Z)]^{M,q}_{T,t}\right\|^2_H\right\}\leq$$

$$\leq 4\tilde{C}(3!)^2\,(\text{tr }Q)^3\,(T-t)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}^2_{j_3j_2j_1}\right),\qquad(64)$$

where the constant $\tilde{C}$ has the same meaning as the constant L_k in Theorem 3 (k is the multiplicity of the iterated stochastic integral).

Combining (63) and (64) we obtain

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]^{M,q}_{T,t}\right\|^2_H\right\}\leq 2\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t}-I_3[B(Z)]^{M}_{T,t}\right\|^2_H\right\}+$$

$$+C_4(T-t)^3\,(\text{tr }Q)^3\left(\frac{1}{6}-\sum_{j_1,j_2,j_3=0}^{q}\hat{C}^2_{j_3j_2j_1}\right).\qquad(65)$$

Let us evaluate the right-hand sides of (62) and (65).

Using the elementary inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we obtain

$$\mathsf{M}\left\{\left\|I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^M\right\|_H^2\right\} \le 3\left(E_{T,t}^{1,M} + E_{T,t}^{2,M} + E_{T,t}^{3,M}\right), \qquad (66)$$

$$\mathsf{M}\left\{\left\|I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^M\right\|_H^2\right\} \le 3\left(G_{T,t}^{1,M} + G_{T,t}^{2,M} + G_{T,t}^{3,M}\right), \qquad (67)$$

where

$$E_{T,t}^{1,M} =$$

$$= \mathsf{M}\left\{\left\|\int_t^T B'(Z)\left(\int_t^{t_3} B'(Z)\left(\int_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{2,M} =$$

$$= \mathsf{M}\left\{\left\|\int_t^T B'(Z)\left(\int_t^{t_3} B'(Z)\left(\int_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2} - \mathbf{W}_{t_2}^M\right)\right)d\mathbf{W}_{t_3}\right\|_H^2\right\},$$

$$E_{T,t}^{3,M} =$$

$$= \mathsf{M}\left\{\left\|\int_t^T B'(Z)\left(\int_t^{t_3} B'(Z)\left(\int_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)d\left(\mathbf{W}_{t_3} - \mathbf{W}_{t_3}^M\right)\right\|_H^2\right\},$$

$$G_{T,t}^{1,M} = \mathsf{M}\left\{\left\|\int_t^T B''(Z)\left(\int_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{2,M} = \mathsf{M}\left\{\left\|\int_t^T B''(Z)\left(\int_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right), \int_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}\right\|_H^2\right\},$$

$$G_{T,t}^{3,M} = \mathsf{M}\left\{\left\|\int\limits_t^T B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) d\left(\mathbf{W}_{t_2} - \mathbf{W}_{t_2}^M\right)\right\|_H^2\right\}.$$

We have

$$E_{T,t}^{1,M} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3 \le$$

$$\le C_5 \int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)d\mathbf{W}_{t_2}\right\|_H^2\right\}dt_3 =$$

$$= C_5 \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2 dt_3 \le$$

$$\le C_6 \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2 dt_3 \le \tag{68}$$

$$\le C_6 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1 dt_2 dt_3 \le \tag{69}$$

$$\le C_7 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \tag{70}$$

Note that the transition from (68) to (69) was made by analogy with the proof of Theorem 1 in [10] (also see [6]). More precisely, taking into account the relation $Q^{\alpha}e_i = \lambda_i^{\alpha}e_i$ we have (see [10], Sect. 3.1)

$$\mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}_{t_1}^M\right)\right\|_H^2\right\} =$$

$$= \mathsf{M}\left\{\left\|\sum_{i\in J\backslash J_M}\sqrt{\lambda_i}\int\limits_t^{t_2} B(Z)e_i d\mathbf{w}_{t_1}^{(i)}\right\|_H^2\right\} =$$

$$= \sum_{i\in J\backslash J_M}\lambda_i\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}Q^{\alpha}e_i\right\|_H^2\right\}dt_1 =$$

$$= \sum_{i\in J\backslash J_M}\lambda_i^{1+2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 =$$

$$= \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\sum_{i\in J\backslash J_M}\lambda_i\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 \le$$

$$\le \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\sum_{i\in J}\lambda_i\left\|B(Z)Q^{-\alpha}e_i\right\|_H^2\right\}dt_1 =$$

$$= \left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^{t_2}\mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1. \qquad (71)$$

Further, we will also use the estimate similar to (71).

We have

$$E_{T,t}^{2,M} =$$

$$= \int\limits_t^T\mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3}B'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right)\right\|_{L_{HS}(U_0,H)}^2\right\}dt_3 \le$$

$$\le C_8\int\limits_t^T\mathsf{M}\left\{\left\|\int\limits_t^{t_3}B'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)d\left(\mathbf{W}_{t_2}-\mathbf{W}_{t_2}^M\right)\right\|_H^2\right\}dt_3 \le$$

$$\le C_8\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_3}\mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_2}B(Z)d\mathbf{W}_{t_1}^M\right)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_2dt_3 \le$$

$$\leq C_8 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \int\limits_t^T \int\limits_t^{t_3} \mathsf{M}\left\{\left\|B'(Z)\left(B(Z)\right)Q^{-\alpha}\right\|^2_{L_{HS}^{(2)}(U_0,H)}\right\}(t_2-t)dt_2dt_3 \leq$$

$$\leq C_9 \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (72)$$

Moreover,

$$E_{T,t}^{3,M} \leq \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times$$

$$\times \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(\int\limits_t^{t_3} B'(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right)d\mathbf{W}_{t_2}^M\right)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\}dt_3 \leq$$

$$\leq C_{10} \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times$$

$$\times \int\limits_t^T \mathsf{M}\left\{\left\|B'(Z)\left(B'(Z)\left(B(Z)\right)\right)Q^{-\alpha}\right\|^2_{L_{HS}^{(3)}(U_0,H)}\right\}\frac{(t_3-t)^2}{2}dt_3 \leq$$

$$\leq C_{11} \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (73)$$

Combining (62), (66), (70), (72), (73) we obtain (60).

We have

$$G_{T,t}^{1,M} =$$

$$= \int\limits_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}, \int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|^2_{L_{HS}(U_0,H)}\right\}dt_3 \leq$$

$$\leq C_{12} \int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\mathbf{W}_{t_1}\right\|^2_H \left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|^2_H\right\}dt_3 \leq$$

$$
\leq C_{12}\int\limits_{t}^{T}\left(\mathsf{M}\left\{\left\|\int\limits_{t}^{t_2}B(Z)d\mathbf{W}_{t_1}\right\|_H^4\right\}\right)^{1/2}\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_{t}^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^{M}\right)\right\|_H^4\right\}\right)^{1/2}dt_3\leq
$$

$$
\leq C_{13}\int\limits_{t}^{T}\int\limits_{t}^{t_2}\left(\mathsf{M}\left\{\left\|B(Z)\right\|_{L_{HS}(U_0,H)}^4\right\}\right)^{1/2}dt_1\times
$$

$$
\times\left(\mathsf{M}\left\{\left\|\int\limits_{t}^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^{M}\right)\right\|_H^4\right\}\right)^{1/2}dt_3\leq
$$

$$
\leq C_{14}\int\limits_{t}^{T}(t_2-t)\left(\mathsf{M}\left\{\left\|\int\limits_{t}^{t_2}B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^{M}\right)\right\|_H^4\right\}\right)^{1/2}dt_3. \qquad (74)
$$

Let us estimate the right-hand side of (74). If $s>t$, then for fixed $M\in\mathbb{N}$ and for some $N>M$ $(N\in\mathbb{N})$ we have

$$
\mathsf{M}\left\{\left\|\int\limits_{t}^{s}B(Z)d\left(\mathbf{W}_{t_1}^{N}-\mathbf{W}_{t_1}^{M}\right)\right\|_H^4\right\}=
$$

$$
=\mathsf{M}\left\{\left\langle\sum_{j\in J_N\backslash J_M}\sqrt{\lambda_j}B(Z)e_j\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right),\right.\right.
$$

$$
\left.\left.\sum_{j'\in J_N\backslash J_M}\sqrt{\lambda_{j'}}B(Z)e_{j'}\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\right\rangle_H^2\right\}=
$$

$$
=\sum_{j,j',l,l'\in J_N\backslash J_M}\sqrt{\lambda_j\lambda_{j'}\lambda_l\lambda_{l'}}\ \ \mathsf{M}\Biggl\{\left\langle B(Z)e_j,B(Z)e_{j'}\right\rangle_H\left\langle B(Z)e_l,B(Z)e_{l'}\right\rangle_H\times
$$

$$
\times\mathsf{M}\left\{\left(\mathbf{w}_s^{(j)}-\mathbf{w}_t^{(j)}\right)\left(\mathbf{w}_s^{(j')}-\mathbf{w}_t^{(j')}\right)\left(\mathbf{w}_s^{(l)}-\mathbf{w}_t^{(l)}\right)\left(\mathbf{w}_s^{(l')}-\mathbf{w}_t^{(l')}\right)\Bigg|\mathbf{F}_t\right\}\Biggr\}=
$$

$$= 3(s-t)^2 \sum_{j \in J_N \setminus J_M} \lambda_j^2 \mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^4\right\} +$$

$$+(s-t)^2 \sum_{j,j' \in J_N \setminus J_M (j \neq j')} \lambda_j \lambda_{j'} \Biggl(\mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^2 \left\|B(Z)e_{j'}\right\|_H^2\right\} +$$

$$+2\left\langle B(Z)e_j, B(Z)e_{j'}\right\rangle_H^2\Biggr) \le$$

$$\le 3(s-t)^2 \Biggl(\sum_{j \in J_N \setminus J_M} \lambda_j^2 \mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^4\right\} +$$

$$+ \sum_{j,j' \in J_N \setminus J_M (j \neq j')} \lambda_j \lambda_{j'} \mathsf{M}\left\{\left\|B(Z)e_j\right\|_H^2 \left\|B(Z)e_{j'}\right\|_H^2\right\}\Biggr) =$$

$$= 3(s-t)^2 \mathsf{M}\left\{\left(\sum_{j \in J_N \setminus J_M} \lambda_j \left\|B(Z)e_j\right\|_H^2\right)^2\right\} \le$$

$$\le 3(s-t)^2 \left(\sup_{i \in J_N \setminus J_M} \lambda_i\right)^{4\alpha} \mathsf{M}\left\{\left(\sum_{j \in J_N \setminus J_M} \lambda_j \left\|B(Z)Q^{-\alpha}e_j\right\|_H^2\right)^2\right\} \le$$

$$\le C_{15}(s-t)^2 \left(\sup_{i \in J_N \setminus J_M} \lambda_i\right)^{4\alpha} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^4\right\}. \qquad (75)$$

Carrying out the passage to the limit $\lim\limits_{N\to\infty}$ in (75) and using (74) we obtain

$$G_{T,t}^{1,M} \le C_{16} \left(\sup_{i \in J \setminus J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (76)$$

Absolutely analogously we get

$$G_{T,t}^{2,M} \le C_{17} \left(\sup_{i \in J \setminus J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (77)$$

Let us estimate $G_{T,t}^{3,M}$. We have

$$G_{T,t}^{3,M} \le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \times$$

$$\times \int_t^T \mathsf{M}\left\{\left\|B''(Z)\left(\int_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M, \int_t^{t_2} B(Z)d\mathbf{W}_{t_1}^M\right) Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\} dt_2 \le$$

$$\le \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} \sum_{i\in J} \sum_{j,l\in J_M} \lambda_i\lambda_j\lambda_l \times$$

$$\times \int_t^T (t_2-t)^2 \left(\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H^2\right\} + \right.$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_j)Q^{-\alpha}e_i\right\|_H \left\|B''(Z)(B(Z)e_l, B(Z)e_l)Q^{-\alpha}e_i\right\|_H\right\} +$$

$$+\mathsf{M}\left\{\left\|B''(Z)(B(Z)e_j, B(Z)e_l)Q^{-\alpha}e_i\right\|_H \times\right.$$

$$\left.\left.\times\left\|B''(Z)(B(Z)e_l, B(Z)e_j)Q^{-\alpha}e_i\right\|_H\right\}\right) dt_2 \le$$

$$\le C_{18} \left(\sup_{i\in J\backslash J_M} \lambda_i\right)^{2\alpha} (T-t)^3. \qquad (78)$$

Combaining (65), (67), and (76)–(78) we obtain (61). Theorem 4 is proved.

Let us consider the convergence analysis for the stochastic integrals (45)–(47) (convergence of the stochastic integral (44) follows from (71) (see Theorem 1 in [10] or [6])).

Using the Itô formula w. p. 1 we obtain [7]

$$J_2[B(Z)]_{T,t} = \int_t^T \left(\frac{T}{2} - s + \frac{t}{2}\right) AB(Z)d\mathbf{W}_s,$$

$$J_3[B(Z), F(Z)]_{T,t} = \int\limits_t^T (s-t)B'(Z)\Bigl(AZ + F(Z)\Bigr)d\mathbf{W}_s.$$

Suppose that

$$\mathsf{M}\left\{\left\|B'(Z)\Bigl(AZ + F(Z)\Bigr)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\} < \infty,$$

$$\mathsf{M}\left\{\left\|AB(Z)Q^{-\alpha}\right\|^2_{L_{HS}(U_0,H)}\right\} < \infty$$

for some $\alpha > 0$.

Then by analogy with (71) we get

$$\mathsf{M}\left\{\left\|J_2[B(Z)]_{T,t} - J_2[B(Z)]^M_{T,t}\right\|^2_H\right\} \le$$

$$\le C_{19}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},$$

$$\mathsf{M}\left\{\left\|J_3[B(Z), F(Z)]_{T,t} - J_3[B(Z), F(Z)]^M_{T,t}\right\|^2_H\right\} \le$$

$$\le C_{20}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha},$$

where $J_2[B(Z)]^M_{T,t}$, $J_3[B(Z), F(Z)]^M_{T,t}$ are defined by (51), (52).

Moreover, in conditions given in Sect. 2 we obtain for some $\alpha > 0$

$$\mathsf{M}\left\{\left\|J_4[B(Z), F(Z)]_{T,t} - J_4[B(Z), F(Z)]^M_{T,t}\right\|^2_H\right\} =$$

$$= \mathsf{M}\left\{\left\|\int\limits_t^T F'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1} - \mathbf{W}^M_{t_1}\right)\right)dt_2\right\|^2_H\right\} \le$$

$$\leq (T-t)\int\limits_t^T \mathsf{M}\left\{\left\|F'(Z)\left(\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right)\right\|_H^2\right\}dt_2 \leq$$

$$\leq C_{21}(T-t)\int\limits_t^T \mathsf{M}\left\{\left\|\int\limits_t^{t_2} B(Z)d\left(\mathbf{W}_{t_1}-\mathbf{W}_{t_1}^M\right)\right\|_H^2\right\}dt_2 \leq$$

$$\leq C_{21}(T-t)\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}\int\limits_t^T\int\limits_t^{t_2} \mathsf{M}\left\{\left\|B(Z)Q^{-\alpha}\right\|_{L_{HS}(U_0,H)}^2\right\}dt_1dt_2 \leq$$

$$\leq C_{22}(T-t)^3\left(\sup_{i\in J\backslash J_M}\lambda_i\right)^{2\alpha}.$$

where $J_4[B(Z),F(Z)]_{T,t}^M$ is defined by (53).

Bibliography

[1] *Kuznetsov, D.F.* Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2019), 18-62. Available at:
http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

[2] *Jentzen, A.* Taylor expansions of solutions of stochastic partial differential equations. Discrete Contin. Dyn. Systems - B. 14, 2 (2010), 515-557.

[3] *Jentzen, A., Kloeden, P.E.* Taylor expansions of solutions of stochastic partial differential equations with additive noise. Ann. Probab. 38, 2 (2010), 532-569.

[4] *Jentzen, A., Kloeden, P.E.* Taylor approximations for stochastic partial differential equations. SIAM, Philadelphia, 2011, 224 pp.

[5] *Jentzen, A., Röckner, M.* Regularity analysis of stochastic partial differential equations with nonlinear multiplicative trace class noise. J. Differ. Eq. 252, 1 (2012), 114-136.

[6] *Jentzen, A., Röckner, M.* A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.

[7] *Becker, S., Jentzen, A., Kloeden, P.E.* An exponential Wagner-Platen type scheme for SPDEs. SIAM J. Numer. Anal. 54, 4 (2016), 2389-2426.

[8] *Mishura, Y.S., Shevchenko, G.M.* Approximation schemes for stochastic differential equations in a Hilbert space. Theor. Prob. Appl. 51, 3 (2007), 442-458.

[9] *Prévôt, C., Röckner, M.* A concise course on stochastic partial differential equations, V. 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007, 148 pp.

[10] *Leonhard, C., Rößler, A.* Iterated stochastic integrals in infinite dimensions: approximation and error estimates. Stoch. PDE: Anal. Comp. 7, 2 (2018), 209-239.

[11] *Leonhard, C.* Derivative-free numerical schemes for stochastic partial differential equations. Ph. D., Institute of Mathematics of the University of Lübeck, 2017. 131 pp.

[12] *Kuznetsov, D.F.* Numerical integration of stochastic differential equations. 2. [In Russian]. *Polytechn. Univ. Publ.*, St.-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227

[13] *Kuznetsov, D.F.* Stochastic differential equations: theory and practice of numerical solution. With MatLab programs. 4th Ed. [In Russian]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2010, 816 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-231

[14] *Kuznetsov, D.F.* Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. *Polytechn. Univ. Publ. House*, St.-Petersburg, 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234

[15] *Kuznetsov, D.F.* Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In

English]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (2017), A.1-A.385. Available at:
http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[16] *Kuznetsov, D.F.* Development and application of the Fourier method for the numerical solution of Itô stochastic differential equations. Comp. Math. Math. Phys. 58, 7 (2018), 1058-1070.
DOI: http://doi.org/10.1134/S0965542518070096

[17] *Kuznetsov, D.F.* On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. Autom. Remote Control. 79, 7 (2018), 1240-1254.
DOI: http://doi.org/10.1134/S0005117918070056

[18] *Kuznetsov, D.F.* Stochastic differential equations: theory and practice of numerical solution. With MATLAB programs, 6th Ed. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[19] *Kuznetsov, D.F.* Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2017, 65 pp.

[20] *Kuznetsov, D.F.* On numerical modeling of the multidimentional dynamic systems under random perturbations with the 2.5 order of strong convergence. Autom. Remote Control. 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[21] *Kuznetsov, D.F.* Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. Comp. Math. Math. Phys. 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[22] *Kuznetsov, D.F.* Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2020, 585 pp.

[23] *Kolmogorov, A.N., Fomin, S.V.* Elements of the Theory of Functions and Functional Analysis. Nauka Publ., Moscow, 1976, 546 pp.

[24] *Da Prato, G., Zabczyk, J.* Stochastic equations in infinite dimensions. 2nd Ed. Cambridge Univ. Press, Cambridge, 2014, 493 pp.

[25] *Kuznetsov, D.F.* A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[26] *Suetin, P.K.* Classical orthogonal polynomials. 3rd Ed. Fizmatlit, Moscow, 2005. 480 pp.