*DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N. 4, 2022
Electronic Journal,
reg. N ΦC77-39410 at 15.04.2010
ISSN 1817-2172*

*http://diffjournal.spbu.ru/
e-mail: jodiff@mail.ru*

Stochastic differential equations
Numerical methods
Computer modeling in dynamical and control systems

A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to the development of a new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of the multidimensional Wiener process. This approach is based on multiple Fourier series in complete orthonormal systems of Legendre polynomials and trigonometric functions in Hilbert space. The theorem on the mean-square convergent expansion for the iterated Stratonovich stochastic integrals of multiplicity 6 is formulated and proved. In the first part of the paper, expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 were obtained. These results allow us to construct efficient approximation procedures for iterated Stratonovich stochastic integrals that are necessary for the implementation of strong numerical methods with orders 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô stochastic differential equations with non-commutative noise (in the framework of the approach based on the Taylor–Stratonovich expansion).
Key words: iterated Stratonovich stochastic integral, iterated Itô stochastic integral, Itô stochastic differential equation, generalized multiple Fourier series,

multiple Fourier–Legendre series, multiple trigonometric Fourier series, mean-square convergence, expansion.

Contents

1 Introduction **136**

2 Preliminary Results **138**

2.1 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$) Based on Generalized Multiple Fourier Series Converging in the Mean 138

2.2 Stratonovich Stochastic Integral 141

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$) Under the Condition of Convergence of Trace Series 142

2.4 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 5 . 149

3 Main Result **151**

3.1 Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6 151

3.2 Proof of Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6 152

4 Conclusion **186**

References **186**

1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t \in [0, T]\}$ be a non-decreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process, which is F_t-measurable for any $t \in [0, T]$. We assume that the components $\mathbf{w}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int\limits_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)d\tau + \int\limits_0^t B(\mathbf{x}_\tau, \tau)d\mathbf{w}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \tag{1}$$

Here $\mathbf{x}_t$ is some n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a}: \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $B: \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [1]. The second integral on the right-hand side of (1) is interpreted as the Itô stochastic integral. Further, $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent when $t > 0$.

Consider the following iterated Itô and Stratonovich stochastic integrals

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k)\ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t,T]$, $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0,\ 1,\ \ldots,$ m,

$$\int \ \text{and} \ \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively.

It is well known that the above stochastic integrals, with a special choice of weight functions $\psi_1(\tau), \ldots, \psi_k(\tau)$, play an important role when solving the problem of numerical integration of Itô SDEs using an approach based on the Taylor–Itô and Taylor–Stratonovich expansions [2]-[5]. The importance of the noted problem is explained by a wide range of applications of Itô SDEs [2]-[5].

More pecisely, $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$, $i_1, \ldots, i_k = 0, 1, \ldots, m$ in the classical Taylor–Itô and Taylor–Stratonovich expansions [2]-[8] and $\psi_l(\tau) \equiv (t-\tau)^{q_l}$, $q_l = 0, 1, \ldots$ $(l = 1, \ldots, k)$, $i_1, \ldots, i_k = 1, \ldots, m$ in the unified Taylor–Ito and Taylor–Stratonovich expansions [9]-[13].

The development of a new effective approach [14] to the series expansion and mean-square approximation of the iterated Stratonovich stochastic integrals (3) composes the subject of the article.

We also note other approaches to the mean-square approximation of the iterated Itô and Stratonovich stochastic integrals (2) and (3) [2]-[5], [15]-[32].

2 Preliminary Results

2.1 Expansion of Iterated Itô Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$) Based on Generalized Multiple Fourier Series Converging in the Mean

The results of this section are auxiliary to the proof of the main result of this article (see Theorem 8 below).

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \mathbf{1}_{\{t_1 < \ldots < t_k\}} \psi_1(t_1) \ldots \psi_k(t_k), \tag{4}$$

where $t_1, \ldots, t_k \in [t, T]$ $(k \geq 2)$, $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$, and $\mathbf{1}_A$ is the indicator of the set A.

Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t, T])$. It is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K - K_{p_1 \ldots p_k} \right\|_{L_2([t,T]^k)} = 0,$$

where

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2},$$

$$K_{p_1 \ldots p_k}(t_1, \ldots, t_k) = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l),$$

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \tag{5}$$

is the Fourier coefficient.

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t, T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta\tau_j \to 0 \ \text{ if } \ N \to \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \tag{6}$$

Theorem 1 [11] (2006), [12], [13], [33]-[53]. *Suppose that every* $\psi_l(\tau)$ $(l=1,\ldots,k)$ *is a continuous nonrandom function on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of continuous functions in the space* $L_2([t,T])$. *Then*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} -$$

$$- \underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{7}$$

$i_1,\ldots,i_k=0,1,\ldots,m$, l.i.m. *is a limit in the mean-square sense,* $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (2),

$$\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$$

$$\mathrm{L}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\},$$

$$\zeta_j^{(i)}=\int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (5), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,\ 1,\ \ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of the interval* $[t,T]$, *which satisfies the condition* (6).

A number of generalizations and modifications of Theorem 1 can be found in [13], Chapter 1 (see also bibliography therein).

Let us consider corollaries from Theorem 1 (see (7)) for $k=1,\ldots,5$ [11]

$$J[\psi^{(1)}]_{T,t}^{(i_1)}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{8}$$

$$J[\psi^{(2)}]_{T,t}^{(i_1 i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{9}$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}=\underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad (10)
$$

$$
J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}=\underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \quad (11)
$$

$$
J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}=\underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-
$$

$$
-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+
$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{12}$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal system of functions in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ [54], [13], [45]. At that we will base on our notations [12] (2009), [13], [45].

Theorem 2 [13] (Sect. 1.11), [45] (Sect. 15). *Suppose that $\psi_1(\tau),\ldots,\psi_k(\tau)$ $\in L_2([t,T])$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}+\sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}}\prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\,i_{g_{2s}}\ne 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\,j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \tag{13}$$

that converges in the mean-square sense is valid, where $[x]$ is an integer part of a real number x, the sum in the second line of the formula (13) *means the sum with respect to all possible permutations of the set*

$$(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\}), \tag{14}$$

braces mean an unordered set, and parentheses mean an ordered set, $\{g_1,g_2,\ldots,$ $g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$; another notations are the same as in Theorem 1.

2.2 Stratonovich Stochastic Integral

Let $\mathrm{M}_2([t,T])$ $(0\le t<T<\infty)$ be the class of random functions $\xi(\tau,\omega)\stackrel{\text{def}}{=}\xi_\tau:$ $[t,T]\times\Omega\to\mathbb{R}$, which satisfy the following conditions: $\xi(\tau,\omega)$ is measurable with respect to the pair of variables (τ,ω), ξ_τ is F_τ-measurable for all $\tau\in[t,T]$, ξ_τ is independent with increments $\mathbf{w}_{s+\Delta}-\mathbf{w}_s$ for $s\ge\tau$, $\Delta>0$, and

$$\int\limits_t^T \mathsf{M}\left\{(\xi_\tau)^2\right\}d\tau<\infty,\quad \mathsf{M}\left\{(\xi_\tau)^2\right\}<\infty\ \text{ for all }\ \tau\in[t,T].$$

We introduce the class $\mathrm{Q}_4([t,T])$ of Itô processes $\eta_\tau^{(i)}$, $\tau\in[t,T]$, $i=1,\ldots,m$ of the form

$$\eta_\tau^{(i)}=\eta_t^{(i)}+\int\limits_t^\tau a_s ds+\int\limits_t^\tau b_s d\mathbf{w}_s^{(i)}\quad \text{w. p. 1},\tag{15}$$

where $(a_s)^4,(b_s)^4\in\mathrm{M}_2([t,T])$ and $\lim\limits_{s\to\tau}\mathsf{M}\bigl\{|b_s-b_\tau|^4\bigr\}=0$ for all $\tau\in[t,T]$. The second integral on the right-hand side of (15) is the Itô stochastic integral. Here and further, w. p. 1 means with probability 1.

Consider a function $F(x,\tau):\mathbb{R}\times[t,T]\to\mathbb{R}$ for fixed τ from the class $C_2(\mathbb{R})$ consisting of twice continuously differentiable in x functions on $\mathbb{R}$ such that the first two derivatives are bounded.

The mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}}\sum_{j=0}^{N-1}F\left(\frac{1}{2}\left(\eta_{\tau_j}^{(i)}+\eta_{\tau_{j+1}}^{(i)}\right),\tau_j\right)\left(\mathbf{w}_{\tau_{j+1}}^{(l)}-\mathbf{w}_{\tau_j}^{(l)}\right)\stackrel{\mathsf{def}}{=}\int\limits_t^{*T}F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}\tag{16}$$

is called [55] the Stratonovich stochastic integral with respect to the component $\mathbf{w}_\tau^{(l)}$ $(l=1,\ldots,m)$ of the multidimensional Wiener process $\mathbf{w}_\tau$, where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t,T]$, which satisfies the condition (6).

It is known [55] (also see [2]) that under proper conditions, the following relation between Stratonovich and Itô stochastic integrals holds

$$\int\limits_t^{*T}F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}=\int\limits_t^{T}F(\eta_\tau^{(i)},\tau)d\mathbf{w}_\tau^{(l)}+\frac{1}{2}\mathbf{1}_{\{i=l\}}\int\limits_t^T\frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau\tag{17}$$

w. p. 1, where $\mathbf{1}_A$ is the indicator of the set A and $i,l=1,\ldots,m$.

A possible variant of conditions under which the formula (17) is correct, for example, consists of the conditions $\eta_\tau^{(i)}\in\mathrm{Q}_4([t,T])$, $F(\eta_\tau^{(i)},\tau)\in\mathrm{M}_2([t,T])$, $F(x,\tau)\in C_2(\mathbb{R})$ (for fixed τ), where $i=1,\ldots,m$.

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k $(k\in\mathbb{N})$ Under the Condition of Convergence of Trace Series

In this section, we consider the expansion of iterated Stratonovich stochastic integrals (3) of arbitrary multiplicity k $(k\in\mathbb{N})$ under the condition of convergence of trace series [14] (we also note approaches to the expansion of multiple

Stratonovich stochastic integrals based on other definitions of these integrals [56], [57]).

Let us introduce some notations. Consider the Fourier coefficient

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{18}$$

corresponding to the kernel (4), where $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Biggr|_{(j_lj_l)\curvearrowright(\cdot)}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k=$$

$$=\sqrt{T-t}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\phi_0(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k, \tag{19}$$

i.e. (19) is again the Fourier coefficient of type (18) but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau)$, $\sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau),\ \psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ (see (14))).

Let

$$\bar C_{j_k\ldots j_q\ldots j_1}^{(p)}\Biggr|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggr|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \tag{20}$$

Introduce the following notation

$$S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}.$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}} \qquad (21)$$

as follows: S_l multiplies (21) by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation

$$\sum_{j_{g_{2l-1}}=p+1}^{\infty},$$

and replaces

$$C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}} \qquad (22)$$

with

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}. \qquad (23)$$

Note that we write

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}},$$

$$C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),(j_{g_3}j_{g_4})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}=$$

$$=C_{j_k\ldots j_1}\Biggl|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot)(j_{g_3}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}},\quad\ldots$$

Since (23) is again the Fourier coefficient, then the action of superposition S_lS_m on (22) is obvious. For example, for $r=3$

$$S_3S_2S_1\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$
= \frac{1}{2^3} \prod_{s=1}^{3} \mathbf{1}_{\{g_{2s}=g_{2s-1}+1\}} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},
$$

$$
S_3 S_1 \left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1} \Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6} \right\} =
$$

$$
= \frac{1}{2^2} \mathbf{1}_{\{g_6=g_5+1\}} \mathbf{1}_{\{g_2=g_1+1\}} \sum_{j_{g_3}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}},
$$

$$
S_2 \left\{ \bar{C}^{(p)}_{j_k\ldots j_q\ldots j_1} \Bigg|_{q\ne g_1,g_2,\ldots,g_5,g_6} \right\} =
$$

$$
= \frac{1}{2} \mathbf{1}_{\{g_4=g_3+1\}} \sum_{j_{g_1}=p+1}^{\infty} \sum_{j_{g_5}=p+1}^{\infty} C_{j_k\ldots j_1} \Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.
$$

Theorem 3 [14], [49], [50]. *Assume that the continuously differentiable functions $\psi_l(\tau) : [t,T] \to \mathbb{R}$ $(l = 1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x) = 1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied*:

1. *The equality*

$$
\frac{1}{2} \int_t^s \Phi_1(t_1)\Phi_2(t_1)dt_1 = \sum_{j=0}^{\infty} \int_t^s \Phi_2(t_2)\phi_j(t_2) \int_t^{t_2} \Phi_1(t_1)\phi_j(t_1)dt_1dt_2 \qquad (24)
$$

holds for all $s \in (t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (24) *converges absolutely.*

2. *The estimates*

$$
\left| \int_t^s \phi_j(\tau)\Phi_1(\tau)d\tau \right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}}, \qquad \left| \int_s^T \phi_j(\tau)\Phi_2(\tau)d\tau \right| \le \frac{\Psi_1(s)}{j^{1/2+\alpha}},
$$

$$\left|\sum_{j=p+1}^{\infty}\int\limits_t^s \Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right| \le \frac{\Psi_2(s)}{p^{\beta}}$$

hold for all $s \in (t,T)$ *and for some* $\alpha, \beta > 0$, *where* $\Phi_1(\tau)$, $\Phi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$, $j, p \in \mathbb{N}$, *and*

$$\int\limits_t^T \Psi_1^2(\tau)d\tau < \infty, \quad \int\limits_t^T |\Psi_2(\tau)|\, d\tau < \infty.$$

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2 = 0$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ *(see* (14)*) and* $l_1, l_2, \ldots, l_d$ *such that* $l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, r\}$, $l_1 > l_2 > \ldots > l_d$, $d = 0, 1, 2, \ldots, r-1$, *where* $r = 1, 2, \ldots, [k/2]$ *and*

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\} \stackrel{\mathsf{def}}{=} \bar{C}_{j_k\ldots j_q\ldots j_1}^{(p)}\Big|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$$

for $d = 0$.

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} \tag{25}$$

the following expansion

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} \tag{26}$$

that converges in the mean-square sense is valid, where

$$C_{j_k\ldots j_1} = \int\limits_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{27}$$

is the Fourier coefficient, l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k = 0,1,\ldots,m$,

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_\tau^{(0)} = \tau$.

The results presented below in this section show that Conditions 1 and 2 of Theorem 3 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$.

In [13] (Sect. 2.1.2), the following equality is proved

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j=0}^{\infty} C_{jj}, \tag{28}$$

where

$$C_{jj} = \int\limits_t^T \psi_2(t_2)\phi_j(t_2)\int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1)dt_1dt_2,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t,T]$.

Note that the following estimate

$$\left|\sum_{j=p+1}^{\infty} C_{jj}\right| \le \frac{C}{p} \tag{29}$$

holds under the above conditions, where constant C is independent of p [13] (Sect. 2.1.2).

The relations (28) and (29) have been modified as follows [13] (Sect. 2.7, 2.9)

$$\frac{1}{2}\int\limits_t^s \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j=0}^{\infty} C_{jj}(s), \tag{30}$$

$$\left|\sum_{j=p+1}^{\infty} C_{jj}(s)\right| \le \frac{C}{p\left(1-z^2(s)\right)^{1/4}}, \tag{31}$$

where (30) holds for the case of Legendre polynomials or trigonometric functions and (31) holds for the case of Legendre polynomials, $s \in (t, T)$ (s is fixed, the case $s = T$ corresponds to (28) and (29)), constant C does not depend p, the functions $\psi_1(\tau)$, $\psi_2(\tau)$ are continuously differentiable at the interval $[t, T]$,

$$C_{jj}(s) = \int\limits_t^s \psi_2(t_2)\phi_j(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_j(t_1)dt_1dt_2,$$

$$z(s) = \left(s - \frac{T+t}{2}\right)\frac{2}{T-t}. \tag{32}$$

For the trigonometric case, the estimate (31) is replaced by [13] (Sect. 2.9)

$$\left|\sum_{j=p+1}^{\infty} C_{jj}(s)\right| \le \frac{C}{p}, \tag{33}$$

where $s \in (t, T)$, constant C does not depend on p.

Note the well known estimate for the Legendre polynomials

$$|P_j(y)| < \frac{K}{\sqrt{j+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad j \in \mathbb{N}, \tag{34}$$

where $P_j(y)$ is the Legendre polynomial, constant K is independent of y and j.

Using (34), we obtain the following useful estimates for the case of Legendre polynomials [13] (Sect. 2.2.5)

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j(1-(z(x))^2)^{1/4}}, \tag{35}$$

$$\left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j(1-(z(x))^2)^{1/4}}, \tag{36}$$

where $j \in \mathbb{N}$, $z(x) \in (-1, 1)$ is defined by (32), $x \in (t, T)$, $\psi(\tau)$ is a continuously differentiable function at the interval $[t, T]$, constant C does not depend on j.

For the case of trigonometric functions, we note the following obvious estimates

$$\left|\int\limits_t^x \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \quad \left|\int\limits_x^T \psi(\tau)\phi_j(\tau)d\tau\right| < \frac{C}{j}, \tag{37}$$

where $j \in \mathbb{N}$, $x \in (t,T)$, the function $\psi(\tau)$ is continuously differentiable at the interval $[t,T]$, constant C does not depend on j.

Remark 1. *From* (28)–(31), (33), (35)–(37) *it follows that Conditions* 1 *and* 2 *of Theorem* 3 *are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space* $L_2([t,T])$ $(\alpha = 1/2,\ \beta = 1)$.

2.4 Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 5

The following three theorems were proved in [14] on the base of Theorem 3.

Theorem 4 [13], [14], [49], [50]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ $(i_1, i_2, i_3 = 0, 1, \ldots, m)$ *defined by* (3) *the following relations*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \tag{38}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}\right)^2\right\} \le \frac{C}{p} \tag{39}$$

are fulfilled, where $i_1, i_2, i_3 = 0, 1, \ldots, m$ *in* (38) *and* $i_1, i_2, i_3 = 1, \ldots, m$ *in* (39), *constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Theorem 5 [13], [14], [49], [50]. *Let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *be a complete orthonormal system of Legendre polynomials or trigonometric functions in the* *space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_4(\tau)$ *be continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)}$ $(i_1,\ldots,i_4=0,1,\ldots,m)$ *defined by* (3) *the following relations*

$$J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}, \tag{40}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} - \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_4}^{(i_4)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{41}$$

are fulfilled, where $i_1,\ldots,i_4=0,1,\ldots,m$ *in* (40) *and* $i_1,\ldots,i_4=1,\ldots,m$ *in* (41), *constant* C *does not depend on* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$; *another notations are the same as in Theorem* 1.

Theorem 6 [13], [14], [49], [50]. *Assume that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)}$ $(i_1,\ldots,i_5=0,1,\ldots,m)$ *defined by* (3) *the following relations*

$$J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}, \tag{42}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} - \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_5}^{(i_5)}\right)^2\right\} \le \frac{C}{p^{1-\varepsilon}} \tag{43}$$

are fulfilled, where $i_1,\ldots,i_5=0,1,\ldots,m$ *in* (42) *and* $i_1,\ldots,i_5=1,\ldots,m$ *in* (43), *constant* C *is independent of* p, ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of* *trigonometric functions in the space* $L_2([t,T])$; *another notations are the same as in Theorem* 1.

Also we note the following theorem for the case $k=2$.

Theorem 7 [13] (Sect. 2.8.1). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$ *and* $\psi_1(\tau),\psi_2(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$. *Then, for the iterated Stratonovich stochastic integral* $J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)}$ $(i_1,i_2=0,1,\ldots,m)$ *defined by* (3) *the following relations*

$$J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{44}$$

$$\mathsf{M}\left\{\left(J^*[\psi^{(2)}]_{T,t}^{(i_1 i_2)} - \sum_{j_1,j_2=0}^{p} C_{j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\right)^2\right\} \le \frac{C}{p} \tag{45}$$

are fulfilled, where $i_1,i_2=0,1,\ldots,m$ *in* (44) *and* $i_1,i_2=1,\ldots,m$ *in* (45), *constant* C *is independent of* p; *another notations are the same as in Theorem* 1.

Note that an analogue of Theorem 7 for the case $k=1$ follows from (8).

3 Main Result

3.1 Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6

In this section, we formulate the main result of the article.

Theorem 8 [13], [48]-[50]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$. *Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J_{T,t}^{*(i_1\ldots i_6)} = \int\limits_t^{*T} \ldots \int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)} \tag{46}$$

the following expansion

$$J_{T,t}^{*(i_1\ldots i_6)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p}C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m,$

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $\mathbf{w}_\tau^{(0)}=\tau.$

3.2 Proof of Theorem on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6

This section is devoted to the proof of Theorem 8.

As noted in Remark 1, Conditions 1 and 2 of Theorem 3 are satisfied for complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t,T])$. Let us verify Condition 3 of Theorem 3 for the iterated Stratonovich stochastic integral (46). Thus, we have to check the following conditions

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2},j_{q_3},j_{q_4}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2}}\right)^2=0, \tag{47}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4}}\right)^2=0, \tag{48}$$

$$\lim_{p\to\infty}\sum_{j_{q_1},j_{q_2}=0}^{p}\left(\sum_{j_{g_1}=p+1}^{\infty}C_{j_6\ldots j_1}\Bigg|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},g_4=g_3+1}\right)^2=0, \tag{49}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty}\sum_{j_{g_5}=p+1}^{\infty} C_{j_6\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}\right)^2=0, \tag{50}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty}\sum_{j_{g_3}=p+1}^{\infty} C_{j_6\ldots j_1}\Biggl|_{(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_6=g_5+1}\right)^2=0, \tag{51}$$

$$\lim_{p\to\infty}\left(\sum_{j_{g_1}=p+1}^{\infty} C_{j_6\ldots j_1}\Biggl|_{(j_{g_4}j_{g_3})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6},g_4=g_3+1,g_6=g_5+1}\right)^2=0, \tag{52}$$

where the expressions $(\{g_1,g_2\},\{g_3,g_4\},\{g_5,g_6\}\})$, $(\{g_1,g_2\},\{g_3,g_4\},\{q_1,q_2\}\})$, $(\{g_1,g_2\},\{q_1,q_2,q_3,q_4\})$ are partitions of the set $\{1,2,\ldots,6\}$ that is $\{g_1,g_2,g_3,g_4,g_5,g_6\}=\{g_1,g_2,g_3,g_4,q_1,q_2\}=\{g_1,g_2,q_1,q_2,q_3,q_4\}=\{1,2,\ \ldots,6\}$; braces mean an unordered set, and parentheses mean an ordered set.

The equalities (47), (49) were proved earlier (see the proof of equalities (2.786), (2.792) in [13], pp. 541, 543 or equalities (194), (200) in [14], pp. 164, 166). The relation (52) follows from the estimate (29) for the polynomial case and its analogue for the trigonometric case. It is easy to see that the equalities (48) and (51) are proved in complete analogy with the proof of equalities (166), (200) in [14], pp. 152, 166 (also see the proof of equalities (2.758), (2.792) in [13], p. 528, 543).

Thus, we have to prove the relation (50). The equality (50) is equivalent to the following equalities

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1}=0, \tag{53}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_1j_3j_2j_3j_2j_1}=0, \tag{54}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_2j_3j_1j_2j_1}=0, \tag{55}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=0, \tag{56}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_2j_3j_3j_1}=0, \tag{57}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=0, \tag{58}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=0, \tag{59}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=0, \tag{60}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_1j_2j_1}=0, \tag{61}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0, \tag{62}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_1j_3j_3j_2j_1}=0, \tag{63}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0, \tag{64}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0, \tag{65}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0, \tag{66}$$

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_2j_1}=0. \tag{67}$$

Consider in detail the case of Legendre polynomials (the case of trigonometric functions is considered in complete analogy).

First, we prove the following equality for the Fourier coefficients for the case $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

$$
C_{j_6j_5j_4j_3j_2j_1}+C_{j_1j_2j_3j_4j_5j_6}=C_{j_6}C_{j_5j_4j_3j_2j_1}-C_{j_5j_6}C_{j_4j_3j_2j_1}+
$$

$$
+C_{j_4j_5j_6}C_{j_3j_2j_1}-C_{j_3j_4j_5j_6}C_{j_2j_1}+C_{j_2j_3j_4j_5j_6}C_{j_1}. \tag{68}
$$

Using the integration order replacement, we have

$$
C_{j_6j_5j_4j_3j_2j_1}=
$$

$$
=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{t_6}\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5dt_6=
$$

$$
=\int\limits_t^T\phi_{j_6}(t_6)\int\limits_t^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6-
$$

$$
-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_4dt_5dt_6=
$$

$$
=C_{j_6}C_{j_5j_4j_3j_2j_1}-
$$

$$
-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_t^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6+
$$

$$
+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=
$$

$$
=C_{j_6}C_{j_5j_4j_3j_2j_1}-
$$

$$
-\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)dt_5dt_6\ C_{j_4j_3j_2j_1}+
$$

$$
+\int\limits_t^T\phi_{j_6}(t_6)\int\limits_{t_6}^{T}\phi_{j_5}(t_5)\int\limits_{t_5}^{T}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_3dt_4dt_5dt_6=
$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} +$$

$$+ \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^T \phi_{j_5}(t_5) \int\limits_{t_5}^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_3 dt_4 dt_5 dt_6 =$$

$$\ldots$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} - C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} -$$

$$- \int\limits_t^T \phi_{j_6}(t_6) \int\limits_{t_6}^T \phi_{j_5}(t_5) \ldots \int\limits_{t_2}^T \phi_{j_1}(t_1) dt_1 \ldots dt_5 dt_6 =$$

$$= C_{j_6}C_{j_5j_4j_3j_2j_1} - C_{j_5j_6}C_{j_4j_3j_2j_1} + C_{j_4j_5j_6}C_{j_3j_2j_1} -$$

$$- C_{j_3j_4j_5j_6}C_{j_2j_1} + C_{j_2j_3j_4j_5j_6}C_{j_1} - C_{j_1j_2j_3j_4j_5j_6}. \qquad (69)$$

The equality (69) completes the proof of the relation (68).

Under the Conditions 1 and 2 of Theorem 3 the following equalities are fulfilled [14] (pp. 108-112), [13] (pp. 485-489)

$$\sum_{j_l=0}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = 0 \quad (l-1 \geq s+1), \qquad (70)$$

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1} = \frac{1}{2} C_{j_k\ldots j_1}\Biggl|_{(j_lj_l)\curvearrowright(\cdot)} - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}, \qquad (71)$$

where $C_{j_k\ldots j_1}$ is defined by (4) and (5).

Obviously, the equality (70) is equivalent to the following relation

$$\sum_{j_l=0}^{p} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1} = - \sum_{j_l=p+1}^{\infty} C_{j_k\ldots j_{l+1}j_lj_{l-1}\ldots j_{s+1}j_lj_{s-1}\ldots j_1}. \qquad (72)$$

Let us consider (53). From (72) we obtain

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = - \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1}. \qquad (73)$$

Applying (68), we get

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} + \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_1j_2j_3} = 2\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =$$

$$= \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_3}C_{j_2j_1j_3j_2j_1} - C_{j_2j_3}C_{j_1j_3j_2j_1} + C_{j_1j_2j_3}C_{j_3j_2j_1} -$$

$$-C_{j_3j_1j_2j_3}C_{j_2j_1} + C_{j_2j_3j_1j_2j_3}C_{j_1} \Biggr). \qquad (74)$$

Note that

$$C_{j_2j_1} = \int\limits_t^T \phi_{j_2}(\tau) \int\limits_t^{\tau} \phi_{j_1}(\theta) d\theta d\tau =$$

$$= \frac{T-t}{2} \begin{cases} 1/\sqrt{(2j_1+1)(2j_1+3)} & \text{if } j_2 = j_1+1,\ j_1 = 0,1,2,\ldots \\ \\ -1/\sqrt{4j_1^2-1} & \text{if } j_2 = j_1-1,\ j_1 = 1,2,\ldots \\ \\ 1 & \text{if } j_1 = j_2 = 0 \\ \\ 0 & \text{otherwise} \end{cases}, \qquad (75)$$

$$C_{j_1} = \int\limits_t^T \phi_{j_1}(\tau) d\tau = \begin{cases} \sqrt{T-t} & \text{if } j_1 = 0 \\ \\ 0 & \text{if } j_1 \neq 0 \end{cases}. \qquad (76)$$

Moreover, the generalized Parseval equality gives

$$\lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3}C_{j_3j_2j_1} =$$

$$= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \int\limits_t^T \phi_{j_1}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_3}(t_1) dt_1 dt_2 dt_3 \times$$

$$\times \int_t^T \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 =$$

$$= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \int_t^T \phi_{j_3}(t_3) \int_{t_3}^T \phi_{j_2}(t_2) \int_{t_2}^T \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \times$$

$$\times \int_t^T \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 =$$

$$= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \int_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 \times$$

$$\times \int_{[t,T]^3} \mathbf{1}_{\{t_1<t_2<t_3\}} \prod_{l=1}^{3} \phi_{j_l}(t_l) dt_1 dt_2 dt_3 =$$

$$= \int_{[t,T]^3} \mathbf{1}_{\{t_3<t_2<t_1\}} \mathbf{1}_{\{t_1<t_2<t_3\}} dt_1 dt_2 dt_3 = 0. \qquad (77)$$

Using (72)–(74), (76) and (77), we get

$$-\lim_{p\to\infty} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3j_2j_1j_3j_2j_1} = \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1j_3j_2j_1} =$$

$$= \frac{1}{2} \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \Biggl(C_{j_3} C_{j_2j_1j_3j_2j_1} - C_{j_2j_3} C_{j_1j_3j_2j_1} -$$

$$- C_{j_3j_1j_2j_3} C_{j_2j_1} + C_{j_2j_3j_1j_2j_3} C_{j_1} \Biggr) =$$

$$= \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} \Bigl(C_{j_3} C_{j_2j_1j_3j_2j_1} - C_{j_3j_1j_2j_3} C_{j_2j_1} \Bigr) =$$

$$= \sqrt{T-t} \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2j_10j_2j_1} - \lim_{p\to\infty} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2j_3} C_{j_2j_1} =$$

$$= \sqrt{T-t} \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} C_{j_2j_10j_2j_1} + \lim_{p\to\infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3j_1j_2j_3} C_{j_2j_1}. \qquad (78)$$

By analogy with the proof of equality (130) in [14] (see the proof of Theorem 12 [14]) or equality (2.722) in [13] (see the proof of Theorem 2.33 [13]) we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_10j_2j_1} = \lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_2j_10j_2j_1} = 0, \tag{79}$$

where we used the following representation

$$C_{j_2j_10j_2j_1} =$$

$$= \frac{1}{\sqrt{T-t}}\int\limits_t^T \phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5 =$$

$$= \frac{1}{\sqrt{T-t}}\int\limits_t^T \phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}dt_3dt_2dt_4dt_5 =$$

$$= \frac{1}{\sqrt{T-t}}\int\limits_t^T \phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)(t_4-t)\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5 +$$

$$+ \frac{1}{\sqrt{T-t}}\int\limits_t^T \phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)(t-t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_4dt_5 \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \bar{C}_{j_2j_1j_2j_1} + \tilde{C}_{j_2j_1j_2j_1}.$$

Further, we have (see (75))

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_1j_2j_3}C_{j_2j_1} = \lim_{p\to\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{00}C_{j_300j_3} +$$

$$+\sum_{j_1=1}^{p} C_{j_1-1,j_1}C_{j_3j_1,j_1-1,j_3} + \sum_{j_1=1}^{p-1} C_{j_1+1,j_1}C_{j_3j_1,j_1+1,j_3} + C_{10}C_{j_301j_3}\Biggr). \tag{80}$$

Observe that

$$|C_{j_1-1,j_1}| + |C_{j_1+1,j_1}| \le \frac{K}{j_1} \quad (j_1 = 1,\ldots,p), \tag{81}$$

$$|C_{j_300j_3}| + |C_{j_3j_1,j_1-1,j_3}| + |C_{j_3j_1,j_1+1,j_3}| + |C_{j_301j_3}| \le$$

$$\le \frac{K_1}{j_3^2} \quad (j_3 \ge p+1), \tag{82}$$

where constants K, K_1 do not depend on j_1, j_3.

The estimate (81) follows from (75). At the same time, the estimate (82) can be obtained using the following reasoning. First note that the integration order replacement gives

$$C_{j_3j_1j_2j_3} = \int\limits_t^T \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_3}(t_1) dt_1 dt_2 dt_3 dt_4 =$$

$$= \int\limits_t^T \phi_{j_1}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \left(\int\limits_t^{t_2} \phi_{j_3}(t_1) dt_1 \right) dt_2 \left(\int\limits_{t_3}^{T} \phi_{j_3}(t_4) dt_4 \right) dt_3. \tag{83}$$

Further,

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2} \right) \frac{2}{T-t} \right), \quad j = 0, 1, 2, \ldots, \tag{84}$$

where $P_j(x)$ is the Legendre polynomial.

From (34) and (84) we get

$$|\phi_j(x)| \le \frac{C}{\left(1 - z^2(x)\right)^{1/4}}, \tag{85}$$

where $x \in (t, T)$, $z(x)$ is defined by (32), constant C does not depend on j, $j \in \mathbb{N}$. Applying the estimates (35), (36), and (85) to (83) gives the estimate (82).

Using (80), (81), and (82), we obtain

$$\left| \sum_{j_1, j_2 = 0}^{p} \sum_{j_3 = p+1}^{\infty} C_{j_3j_1j_2j_3} C_{j_2j_1} \right| \le K \sum_{j_3=p+1}^{\infty} \frac{1}{j_3^2} \left(1 + \sum_{j_1=1}^{p} \frac{1}{j_1} \right) \le$$

$$\le K \int\limits_p^{\infty} \frac{dx}{x^2} \left(2 + \int\limits_1^p \frac{dx}{x} \right) = \frac{K(2 + \ln p)}{p} \to 0 \tag{86}$$

if $p \to \infty$, where constant K is independent of p. Thus, the equality (53) is proved (see (78), (79), (86)).

The relation (54) is proved in complete analogy with the proof of equality (53). For (54) we have (see (68))

$$\lim_{p\to\infty}\left(\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1}+\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_2j_3j_2j_3j_1}\right)=2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_1j_3j_2j_3j_2j_1}=$$

$$=\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_1}C_{j_3j_2j_3j_2j_1}-C_{j_3j_1}C_{j_2j_3j_2j_1}+C_{j_2j_3j_1}C_{j_3j_2j_1}-$$

$$-C_{j_3j_2j_3j_1}C_{j_2j_1}+C_{j_2j_3j_2j_3j_1}C_{j_1}\Biggr)=$$

$$=2\lim_{p\to\infty}\left(\sqrt{T-t}\sum_{j_2,j_3=0}^{p} C_{j_3j_2j_3j_20}-\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}\right)=$$

$$=-2\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_1}C_{j_3j_2j_3j_1}.$$

To estimate the Fourier coefficient $C_{j_3j_2j_3j_1}$, we use the following (see the proof of (53) for more details)

$$C_{j_3j_2j_3j_1}=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_3}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\int\limits_{t_1}^{t_3}\phi_{j_3}(t_2)dt_2dt_1dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_3}(t_2)dt_2\right)\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1dt_3dt_4-$$

$$-\int\limits_t^T\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)\left(\int\limits_t^{t_1}\phi_{j_3}(t_2)dt_2\right)dt_1dt_3dt_4=$$

$$
= \int\limits_t^T \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_3}(t_2) dt_2 \right) \int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \left(\int\limits_{t_3}^T \phi_{j_3}(t_4) dt_4 \right) dt_3 -
$$

$$
- \int\limits_t^T \phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) \left(\int\limits_t^{t_1} \phi_{j_3}(t_2) dt_2 \right) dt_1 \left(\int\limits_{t_3}^T \phi_{j_3}(t_4) dt_4 \right) dt_3.
$$

Let us prove (55). From (72) we obtain

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_2 j_3 j_1 j_2 j_1} = - \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{p} C_{j_3 j_2 j_3 j_1 j_2 j_1}. \tag{87}
$$

Applying (68) and (87), we get (we replaced j_3 by j_4)

$$
\sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4 j_1 j_2 j_1} + \sum_{j_1,j_2,j_4=0}^{p} C_{j_1 j_2 j_1 j_4 j_2 j_4} = 2 \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4 j_1 j_2 j_1} =
$$

$$
= \sum_{j_1,j_2,j_4=0}^{p} \Biggl(C_{j_4} C_{j_2 j_4 j_1 j_2 j_1} - C_{j_2 j_4} C_{j_4 j_1 j_2 j_1} + C_{j_4 j_2 j_4} C_{j_1 j_2 j_1} -
$$

$$
- C_{j_1 j_4 j_2 j_4} C_{j_2 j_1} + C_{j_2 j_1 j_4 j_2 j_4} C_{j_1} \Biggr) =
$$

$$
= 2 \sum_{j_1,j_2,j_4=0}^{p} \Bigl(C_{j_2 j_1 j_4 j_2 j_4} C_{j_1} - C_{j_1 j_4 j_2 j_4} C_{j_2 j_1} \Bigr) +
$$

$$
+ \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4} C_{j_1 j_2 j_1}. \tag{88}
$$

Further, we have (see (72))

$$
\lim_{p\to\infty} \sum_{j_1,j_2,j_4=0}^{p} C_{j_4 j_2 j_4} C_{j_1 j_2 j_1} = \lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_1 j_2 j_1} \right)^2 =
$$

$$
= \lim_{p\to\infty} \sum_{j_2=0}^{p} \left(\sum_{j_1=p+1}^{\infty} C_{j_1 j_2 j_1} \right)^2 = 0, \tag{89}
$$

where we applied the equality (103) from [14], p. 134 (also see [13], p. 510).

Furthermore, by analogy with the proof of (53), we have

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_4=0}^{p}\Biggl(C_{j_2j_1j_4j_2j_4}C_{j_1}-C_{j_1j_4j_2j_4}C_{j_2j_1}\Biggr)=0. \tag{90}$$

To estimate the Fourier coefficient $C_{j_1j_4j_2j_4}$ in (90), we use the following (see the proof of (53) for more details)

$$C_{j_1j_4j_2j_4}=\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_4}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_3dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)\int\limits_{t_2}^{t_4}\phi_{j_4}(t_3)dt_3dt_2dt_4=$$

$$=\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_t^{t_4}\phi_{j_4}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4-$$

$$-\int\limits_t^T\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_4}(t_3)dt_3\right)\left(\int\limits_t^{t_2}\phi_{j_4}(t_1)dt_1\right)dt_2dt_4.$$

The relations (87)–(90) complete the proof of equality (55).

Let us prove (56). Using (72), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}. \tag{91}$$

Applying (68) and (91), we obtain

$$2\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_1j_2j_3j_3j_2j_1}=$$

$$=\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_1}C_{j_2j_3j_3j_2j_1}-C_{j_2j_1}C_{j_3j_3j_2j_1}+\left(C_{j_3j_2j_1}\right)^2-$$

$$-C_{j_3j_3j_2j_1}C_{j_2j_1}+C_{j_2j_3j_3j_2j_1}C_{j_1}\Biggr)=$$

$$
= 2 \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Biggl(C_{j_1} C_{j_2 j_3 j_3 j_2 j_1} - C_{j_2 j_1} C_{j_3 j_3 j_2 j_1} \Biggr) +
$$

$$
+ \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \left(C_{j_3 j_2 j_1} \right)^2 . \tag{92}
$$

Using the estimate (1.217) from [13] (p. 158), we get

$$
\lim_{p \to \infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \left(C_{j_3 j_2 j_1} \right)^2 = 0. \tag{93}
$$

By analogy with the proof of (53), we have

$$
\lim_{p \to \infty} \sum_{j_1,j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Biggl(C_{j_1} C_{j_2 j_3 j_3 j_2 j_1} - C_{j_2 j_1} C_{j_3 j_3 j_2 j_1} \Biggr) = 0, \tag{94}
$$

where we applied the equality (131) in [14], p. 139 (also see the equality (2.723) in [13], p. 516). To estimate the Fourier coefficient $C_{j_3 j_3 j_2 j_1}$ in (94), we used the following (see the proof of (53) for more details)

$$
C_{j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= \int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^T \phi_{j_2}(t_2) \int\limits_{t_2}^T \phi_{j_3}(t_3) \int\limits_{t_3}^T \phi_{j_3}(t_4) dt_4 dt_3 dt_2 dt_1 =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_1}(t_1) \int\limits_{t_1}^T \phi_{j_2}(t_2) \left(\int\limits_{t_2}^T \phi_{j_3}(t_3) dt_3 \right)^2 dt_2 dt_1. \tag{95}
$$

Combining the equalities (91)–(94), we obtain (56).

Let us prove (57) (we replace j_2 by j_4 and j_3 by j_2 in (57)). As noted in [14] (p. 126), the sequential order of the series

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_4=p+1}^{\infty}
$$

is not important. This follows directly from the formulas (71) and (72).

Applying the mentioned property and (72), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = -\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{96}$$

Observe that (see the above reasoning)

$$\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} = \sum_{j_4=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1}. \tag{97}$$

Using (68) and (97), we obtain

$$\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1j_4j_4j_2j_2j_1}+C_{j_1j_2j_2j_4j_4j_1}\Biggr) = 2\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty} C_{j_1j_4j_4j_2j_2j_1} =$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}+C_{j_4j_4j_1}C_{j_2j_2j_1}-$$

$$-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Biggr)=$$

$$=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-C_{j_2j_4j_4j_1}C_{j_2j_1}+C_{j_2j_2j_4j_4j_1}C_{j_1}\Biggr)+$$

$$+\sum_{j_1=0}^{p}\Biggl(\sum_{j_2=p+1}^{\infty} C_{j_2j_2j_1}\Biggr)^2. \tag{98}$$

The equality

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\Biggl(\sum_{j_2=p+1}^{\infty} C_{j_2j_2j_1}\Biggr)^2 = 0 \tag{99}$$

follows from the relation (102) in [14], p. 134 (also see [13], p. 510).

By analogy with the proof of equality (53) we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_4=p+1}^{\infty}\Biggl(C_{j_1}C_{j_4j_4j_2j_2j_1}-C_{j_4j_1}C_{j_4j_2j_2j_1}-$$

$$
\left. -C_{j_2j_4j_4j_1}C_{j_2j_1} + C_{j_2j_2j_4j_4j_1}C_{j_1} \right) = 0, \tag{100}
$$

where we applied the equality (132) in [14], p. 140 (also see the equality (2.724) in [13], p. 516). To estimate the Fourier coefficient $C_{j_2j_4j_4j_1}$ in (100), we used the following (see the proof of (53) for more details)

$$
C_{j_2j_4j_4j_1} = \int\limits_t^T \phi_{j_2}(t_4) \int\limits_t^{t_4} \phi_{j_4}(t_3) \int\limits_t^{t_3} \phi_{j_4}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 =
$$

$$
= \int\limits_t^T \phi_{j_2}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) \int\limits_{t_1}^{t_4} \phi_{j_4}(t_2) \int\limits_{t_2}^{t_4} \phi_{j_4}(t_3) dt_3 dt_2 dt_1 dt_4 =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_2}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) \left(\int\limits_{t_1}^{t_4} \phi_{j_4}(t_2) dt_2 \right)^2 dt_1 dt_4 =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_2}(t_4) \left(\int\limits_t^{t_4} \phi_{j_4}(t_2) dt_2 \right)^2 \int\limits_t^{t_4} \phi_{j_1}(t_1) dt_1 dt_4 +
$$

$$
+ \frac{1}{2} \int\limits_t^T \phi_{j_2}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) \left(\int\limits_t^{t_1} \phi_{j_4}(t_2) dt_2 \right)^2 dt_1 dt_4 -
$$

$$
- \int\limits_t^T \phi_{j_2}(t_4) \left(\int\limits_t^{t_4} \phi_{j_4}(t_2) dt_2 \right) \int\limits_t^{t_4} \phi_{j_1}(t_1) \left(\int\limits_t^{t_1} \phi_{j_4}(t_2) dt_2 \right) dt_1 dt_4.
$$

The relation (57) follows from (96), (98)–(100).

Consider (58). Using the integration order replacement, we obtain

$$
C_{j_3j_3j_2j_2j_1j_1} =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 dt_3 dt_4 dt_5 dt_6 =
$$

$$
= \frac{1}{2} \int\limits_t^T \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 \int\limits_{t_3}^T \phi_{j_2}(t_4) \int\limits_{t_4}^T \phi_{j_3}(t_5) \int\limits_{t_5}^T \phi_{j_3}(t_6) dt_6 dt_5 dt_4 dt_3 =
$$

$$
= \frac{1}{4} \int\limits_t^T \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 \int\limits_{t_3}^T \phi_{j_2}(t_4) \left(\int\limits_{t_4}^T \phi_{j_3}(t_5) dt_5 \right)^2 dt_4 dt_3. \quad (101)
$$

Applying the estimates (35), (36), and (85) to (101) gives the following estimate

$$
|C_{j_3 j_3 j_2 j_2 j_1 j_1}| \le \frac{K}{j_1^2 j_3^2} \quad (j_1, j_3 > 0,\ j_2 \ge 0), \quad (102)
$$

where constant K does not depend on j_1, j_2, j_3.

Further, we get (see (71))

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1} = \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1} =
$$

$$
= \frac{1}{2} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1} \Bigg|_{(j_2 j_2) \curvearrowright (\cdot)} - \sum_{j_2=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} C_{j_3 j_3 j_2 j_2 j_1 j_1}, \quad (103)
$$

where

$$
C_{j_3 j_3 j_2 j_2 j_1 j_1} \Bigg|_{(j_2 j_2) \curvearrowright (\cdot)} =
$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \int\limits_t^{t_4} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_4 dt_5 dt_6 =
$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 \int\limits_{t_2}^{t_5} dt_4 dt_2 dt_5 dt_6 =
$$

$$
= \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5)(t_5 - t) \int\limits_t^{t_5} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_5 dt_6 +
$$

$$
+ \int\limits_t^T \phi_{j_3}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_2)(t - t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_5 dt_6 \stackrel{\text{def}}{=}
$$

$$
\stackrel{\text{def}}{=} C'_{j_3 j_3 j_1 j_1} + C''_{j_3 j_3 j_1 j_1}. \quad (104)
$$

Let us substitute (104) into (103)

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}=\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}+$$

$$+\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}. \qquad (105)$$

The relation (132) from [14], p. 140 (also see the equality (2.724) from [13], p. 516) implies that

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C'_{j_3j_3j_1j_1}=0,\quad \lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C''_{j_3j_3j_1j_1}=0. \qquad (106)$$

From the estimate (102) we get

$$\left|\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_2j_2j_1j_1}\right|\le K(p+1)\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_3=p+1}^{\infty}\frac{1}{j_3^2}\le$$

$$\le K(p+1)\left(\int\limits_{p}^{\infty}\frac{dx}{x^2}\right)^2\le\frac{K(p+1)}{p^2}\ \to\ 0 \qquad (107)$$

if $p\to\infty$, where constant K is independent of p.

The relations (105)–(107) complete the proof of (58).

Let us prove (59). Using the integration order replacement, we get

$$C_{j_2j_3j_3j_2j_1j_1}=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2dt_3dt_4dt_5dt_6=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6dt_5dt_4dt_3=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_2}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3=$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^T \phi_{j_3}(t_5)\left(\int\limits_{t_5}^T \phi_{j_2}(t_6)dt_6\right)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\times
$$

$$
\times dt_5dt_3-
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T \phi_{j_3}(t_5)\left(\int\limits_{t_5}^T \phi_{j_2}(t_6)dt_6\right)\times
$$

$$
\times\, dt_5dt_3. \tag{108}
$$

Applying (72) and (71), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=-\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_2=0}^{p}\sum_{j_3=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}=
$$

$$
=\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_1=p+1}^{\infty}C_{0000j_1j_1}-
$$

$$
-\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{0j_3j_30j_1j_1}-\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_200j_2j_1j_1}-
$$

$$
-\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}. \tag{109}
$$

The equality

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=0 \tag{110}
$$

follows from the inequality similar to (158) in [14], p. 149 (also see the relation (2.750) in [13], p. 526), where we used the following representation

$$
C_{j_2j_3j_3j_2j_1j_1}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=
$$

$$= \int\limits_t^T \phi_{j_2}(t_6) \int\limits_t^{t_6} \int\limits_t^{t_4} \phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4 dt_6 =$$

$$= \int\limits_t^T \phi_{j_2}(t_6) \int\limits_t^{t_6} \phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 \int\limits_{t_3}^{t_6} dt_4 dt_3 dt_6 =$$

$$+ \int\limits_t^T \phi_{j_2}(t_6)(t_6 - t) \int\limits_t^{t_6} \phi_{j_2}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_6 +$$

$$+ \int\limits_t^T \phi_{j_2}(t_6) \int\limits_t^{t_6} \phi_{j_2}(t_3)(t - t_3) \int\limits_t^{t_3} \phi_{j_1}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_6 \stackrel{\sf def}{=}$$

$$\stackrel{\sf def}{=} C^{*}_{j_2 j_2 j_1 j_1} + C^{**}_{j_2 j_2 j_1 j_1}. \tag{111}$$

The following estimate

$$\left| \int\limits_t^s \psi(\tau)\phi_j(\tau) d\tau \right| < \frac{C}{j^{1-\varepsilon/2}(1 - z^2(s))^{1/4-\varepsilon/4}} \tag{112}$$

is proved in [14], p. 137 (also see [13], p. 513), where ε is an arbitrary small positive real number, $\psi(\tau)$ is a continuously differentiable function at the interval $[t, T]$, $j \in \mathbb{N}$, $s \in (t, T)$, $z(s)$ is defined by (32), constant C does not depend on j.

Applying the estimates (35), (36), (85), and (112) ($\varepsilon = 1/2$) to (108) gives the following estimates

$$|C_{j_2 j_3 j_3 j_2 j_1 j_1}| \le \frac{K}{j_1^2 j_2 j_3^{3/4}} \quad (j_1, j_2, j_3 > 0), \tag{113}$$

$$|C_{j_2 0 0 j_2 j_1 j_1}| \le \frac{K}{j_1^2 j_2} \quad (j_1, j_2 > 0), \tag{114}$$

$$|C_{0 j_3 j_3 0 j_1 j_1}| \le \frac{K}{j_1^2 j_3} \quad (j_1, j_3 > 0), \tag{115}$$

$$|C_{0000 j_1 j_1}| \le \frac{K}{j_1^2} \quad (j_1 > 0). \tag{116}$$

Using the estimate (113), we have

$$\left|\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_2j_3j_3j_2j_1j_1}\right| \le K\sum_{j_1=p+1}^{\infty}\frac{1}{j_1^2}\sum_{j_2=1}^{p}\frac{1}{j_2}\sum_{j_3=1}^{p}\frac{1}{j_3^{3/4}} \le$$

$$\le K\int\limits_p^{\infty}\frac{dx}{x^2}\left(1+\int\limits_1^p\frac{dx}{x}\right)\left(1+\int\limits_1^p\frac{dx}{x^{3/4}}\right) \le K_1\frac{1+\ln p}{p^{3/4}} \to 0 \qquad (117)$$

if $p\to\infty$, where constants K, K_1 do not depend on p.

Similarly we get (see (114)–(116))

$$\left|\sum_{j_1=p+1}^{\infty} C_{0000j_1j_1}\right| + \left|\sum_{j_3=1}^{p}\sum_{j_1=p+1}^{\infty} C_{0j_3j_30j_1j_1}\right| + \left|\sum_{j_2=1}^{p}\sum_{j_1=p+1}^{\infty} C_{j_200j_2j_1j_1}\right| \to 0 \quad (118)$$

if $p\to\infty$.

The relations (109), (110), (117), (118) prove (59).

Consider (60). Using the integration order replacement, we get

$$C_{j_3j_2j_3j_2j_1j_1} =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_2}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2 dt_3dt_4dt_5dt_6 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_3}(t_4)\int\limits_{t_4}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4dt_5dt_3 =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\int\limits_{t_3}^{T}\phi_{j_2}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\left(\int\limits_{t_5}^{T}\phi_{j_3}(t_6)dt_6\right)\times$$

$$\times dt_5dt_3-$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)^2\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)\int\limits_{t_3}^T\phi_{j_2}(t_5)\left(\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6\right)\times
$$
$$
\times\, dt_5dt_3. \tag{119}
$$

Applying (72), we obtain

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}=
$$
$$
=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_2j_3j_2j_1j_1}. \tag{120}
$$

Further proof of the equality (60) is based on the relations (119), (120) and is similar to the proof of the formula (59).

Let us prove (61). Applying the integration order replacement, we obtain

$$
C_{j_3j_3j_2j_1j_2j_1}=
$$
$$
=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=
$$
$$
=\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\int\limits_{t_4}^T\phi_{j_3}(t_5)\int\limits_{t_5}^T\phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=
$$
$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_1)\int\limits_{t_1}^T\phi_{j_2}(t_2)\int\limits_{t_2}^T\phi_{j_1}(t_3)\int\limits_{t_3}^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2dt_4dt_3dt_2dt_1=
$$
$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_1}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=
$$
$$
=\frac{1}{2}\int\limits_t^T\phi_{j_2}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_1}(t_3)dt_3dt_2dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_1}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\times
$$

$$
\times dt_2dt_4-
$$

$$
-\frac{1}{2}\int\limits_t^T \phi_{j_2}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)^2\times
$$

$$
\times\, dt_2dt_4. \qquad (121)
$$

Using (72), we get

$$
\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1}=\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1}=
$$

$$
=-\sum_{j_2=0}^{p}\sum_{j_1=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_2j_1j_2j_1}. \qquad (122)
$$

Further proof of the equality (61) is based on the relations (121), (122) and is similar to the proof of the relations (59), (60).

Consider (62). Using the integration order replacement, we have

$$
C_{j_3j_3j_1j_2j_2j_1}=
$$

$$
=\int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4dt_5dt_6=
$$

$$
=\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\int\limits_{t_4}^T \phi_{j_3}(t_5)\int\limits_{t_5}^T \phi_{j_3}(t_6)dt_6dt_5dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_1)\int\limits_{t_1}^T \phi_{j_2}(t_2)\int\limits_{t_2}^T \phi_{j_2}(t_3)\int\limits_{t_3}^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2 dt_4dt_3dt_2dt_1=
$$

$$
=\frac{1}{2}\int\limits_t^T \phi_{j_1}(t_4)\left(\int\limits_{t_4}^T \phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1dt_2dt_3dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\int\limits_{t_2}^{t_4}\phi_{j_2}(t_3)dt_3dt_2dt_4=
$$

$$
=\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\left(\int\limits_t^{t_4}\phi_{j_2}(t_3)dt_3\right)\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\times
$$

$$
\times dt_2dt_4-
$$

$$
-\frac{1}{2}\int\limits_t^T\phi_{j_1}(t_4)\left(\int\limits_{t_4}^T\phi_{j_3}(t_5)dt_5\right)^2\int\limits_t^{t_4}\phi_{j_2}(t_2)\left(\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\right)\left(\int\limits_t^{t_2}\phi_{j_2}(t_3)dt_3\right)\times
$$

$$
\times\ dt_2dt_4. \tag{123}
$$

Applying (72) and (71), we obtain

$$
-\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=-\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\sum_{j_1=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=
$$

$$
=\sum_{j_1=0}^{p}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_1j_2j_2j_1}=
$$

$$
=\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}-\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}. \tag{124}
$$

The equality

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}\Biggr|_{(j_2j_2)\curvearrowright(\cdot)}=0 \tag{125}
$$

follows from the the inequality (158) in [14], p. 149 (also see the inequality (2.750) in [13], p. 526), where we proceed similarly to the proof of equality (110) (see (111)).

The relation

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}C_{j_3j_3j_1j_2j_2j_1}=0 \tag{126}
$$

is proved on the basis of (123) and similarly with the proof of (59). The equalities (124)–(126) prove (62).

Let us prove (63). Using (72) and (71), we get

$$\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2 j_1 j_3 j_3 j_2 j_1} = \sum_{j_3=p+1}^{\infty}\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1} =$$

$$= \frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} - \sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1}. \qquad (127)$$

Using the equality (130) in [14], p. 139 (also see the equality (2.722) in [13], p. 516) we have

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1}\Bigg|_{(j_3 j_3)\curvearrowright(\cdot)} = 0, \qquad (128)$$

where we proceed similarly to the proof of equality (110) (see (111)).

Further, we will prove the following relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1} = 0 \qquad (129)$$

using the equality (68). From (68) we have

$$\sum_{j_1,j_2,j_3=0}^{p} C_{j_2 j_1 j_3 j_3 j_2 j_1} = \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2 j_1 j_3 j_3 j_2 j_1} + C_{j_1 j_2 j_3 j_3 j_1 j_2}\Bigr) =$$

$$= \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2} C_{j_1 j_3 j_3 j_2 j_1} - C_{j_1 j_2} C_{j_3 j_3 j_2 j_1} + C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} -$$

$$- C_{j_3 j_3 j_1 j_2} C_{j_2 j_1} + C_{j_2 j_3 j_3 j_1 j_2} C_{j_1}\Bigr) =$$

$$= \sum_{j_1,j_2,j_3=0}^{p}\Bigl(C_{j_2 j_3 j_3 j_1 j_2} C_{j_1} - C_{j_3 j_3 j_1 j_2} C_{j_2 j_1}\Bigr) +$$

$$+ \frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1}. \qquad (130)$$

The generalized Parseval equality gives (by analogy with (77))

$$\lim_{p\to\infty}\frac{1}{2}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_1j_2}C_{j_3j_2j_1}=0. \tag{131}$$

Let us prove the following equality

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p}\Biggl(C_{j_2j_3j_3j_1j_2}C_{j_1}-C_{j_3j_3j_1j_2}C_{j_2j_1}\Biggr)=0. \tag{132}$$

The relation

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_2j_3j_3j_1j_2}C_{j_1}=0 \tag{133}$$

is proved by the same methods as in the proof of equality (53) and also using Theorem 12 from [14] (also see Theorem 2.33 from [13]) and (71).

Further, we have (see (71))

$$\sum_{j_3=0}^{p} C_{j_3j_3j_1j_2}=\frac{1}{2}C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}-\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}. \tag{134}$$

Moreover,

$$C_{j_3j_3j_1j_2}\Biggl|_{(j_3j_3)\curvearrowright(\cdot)}=\int\limits_t^T\int\limits_t^{t_3}\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2dt_3=$$

$$=\int\limits_t^T\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1\int\limits_{t_2}^{T}dt_3dt_2=\int\limits_t^T(T-t_2)\phi_{j_1}(t_2)\int\limits_t^{t_2}\phi_{j_2}(t_1)dt_1dt_2=$$

$$=\int\limits_t^T\phi_{j_2}(t_1)\int\limits_{t_1}^{T}(T-t_2)\phi_{j_1}(t_2)dt_2dt_1=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_{t_2}^{T}(T-t_1)\phi_{j_1}(t_1)dt_1dt_2=$$

$$=\int\limits_{[t,T]^2}(T-t_1)\mathbf{1}_{\{t_2<t_1\}}\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\stackrel{\mathsf{def}}{=}\tilde C_{j_2j_1}. \tag{135}$$

Using (134), (135), and the generalized Parseval equality, we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1} = \frac{1}{2}\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p} C_{j_2j_1}\tilde{C}_{j_2j_1}-$$

$$-\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1} = -\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1}. \quad (136)$$

We have (see (95))

$$C_{j_3j_3j_1j_2} = \frac{1}{2}\int\limits_t^T \phi_{j_2}(t_1)\int\limits_{t_1}^T \phi_{j_1}(t_2)\left(\int\limits_{t_2}^T \phi_{j_3}(t_3)dt_3\right)^2 dt_2dt_1. \quad (137)$$

By analogy with (86) and also using (137), we get

$$\lim_{p\to\infty}\sum_{j_1,j_2=0}^{p}\sum_{j_3=p+1}^{\infty} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \quad (138)$$

Combining (136) and (138), we obtain

$$\lim_{p\to\infty}\sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_3j_1j_2}C_{j_2j_1} = 0. \quad (139)$$

The relation (132) follows from (133) and (139). From (130)–(132) we get (129). The equalities (127)–(129) complete the proof of (63).

For the proof of (64)–(67) we will use a new idea. More precisely, we will consider the sums of expressions (64)–(67) with the expressions already studied throughout this proof.

Let us begin from (64). Applying the integration order replacement, we obtain

$$C_{j_3j_1j_2j_3j_2j_1} + C_{j_3j_1j_2j_3j_1j_2} =$$

$$= \int\limits_t^T \phi_{j_3}(t_6)\int\limits_t^{t_6} \phi_{j_1}(t_5)\int\limits_t^{t_5} \phi_{j_2}(t_4)\int\limits_t^{t_4} \phi_{j_3}(t_3)\left(\int\limits_t^{t_3} \phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3} \phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3 dt_4 dt_5 dt_6 =$$

$$= \int_t^T \phi_{j_3}(t_6) \int_t^{t_6} \phi_{j_1}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2\right) \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1\right) \int_{t_3}^{t_5} \phi_{j_2}(t_4) dt_4 \times$$

$$\times dt_3 dt_5 dt_6 =$$

$$= \int_t^T \phi_{j_3}(t_6) \int_t^{t_6} \phi_{j_1}(t_5) \left(\int_t^{t_5} \phi_{j_2}(t_4) dt_4\right) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2\right) \times$$

$$\times \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1\right) dt_3 dt_5 dt_6 -$$

$$- \int_t^T \phi_{j_3}(t_6) \int_t^{t_6} \phi_{j_1}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2\right)^2 \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1\right) \times$$

$$\times dt_3 dt_5 dt_6 =$$

$$= \int_t^T \phi_{j_1}(t_5) \left(\int_t^{t_5} \phi_{j_2}(t_4) dt_4\right) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2\right) \times$$

$$\times \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1\right) dt_3 \left(\int_{t_5}^{T} \phi_{j_3}(t_6) dt_6\right) dt_5 -$$

$$- \int_t^T \phi_{j_1}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2\right)^2 \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1\right) dt_3 \times$$

$$\times \left(\int_{t_5}^{T} \phi_{j_3}(t_6) dt_6\right) dt_5. \tag{140}$$

Using (72), we get

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_2 j_3 j_2 j_1} + C_{j_3 j_1 j_2 j_3 j_1 j_2}\Bigr) =$$

$$= \sum_{j_1=0}^{p} \sum_{j_3=0}^{p} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_2 j_3 j_2 j_1} + C_{j_3 j_1 j_2 j_3 j_1 j_2}\Bigr). \tag{141}$$

Further, by analogy with the proof of equality (59) and using (140), we obtain

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0. \qquad (142)$$

From (141) and (142) we get

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_2j_3j_2j_1}+C_{j_3j_1j_2j_3j_1j_2}\right)=0. \qquad (143)$$

Moreover (see (53)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_1j_2}=0. \qquad (144)$$

Combining (143) and (144), we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_2j_3j_2j_1}=0.$$

The equality (64) is proved.

Consider (65). Using the integration order replacement, we have

$$C_{j_2j_3j_1j_3j_2j_1}+C_{j_2j_3j_1j_3j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_1}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_1}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_3}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$
\times \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 dt_5 dt_6 -
$$

$$
- \int_t^T \phi_{j_2}(t_6) \int_t^{t_6} \phi_{j_3}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 \times
$$

$$
\times dt_3 dt_5 dt_6 =
$$

$$
= \int_t^T \phi_{j_3}(t_5) \left(\int_t^{t_5} \phi_{j_1}(t_4) dt_4 \right) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \times
$$

$$
\times \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 \left(\int_{t_5}^T \phi_{j_2}(t_6) dt_6 \right) dt_5 -
$$

$$
- \int_t^T \phi_{j_3}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \left(\int_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int_t^{t_3} \phi_{j_1}(t_1) dt_1 \right)^2 dt_3 \times
$$

$$
\times \left(\int_{t_5}^T \phi_{j_2}(t_6) dt_6 \right) dt_5. \tag{145}
$$

Using (72), we obtain

$$
- \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_2 j_3 j_1 j_3 j_2 j_1} + C_{j_2 j_3 j_1 j_3 j_1 j_2} \Bigr) =
$$

$$
= \sum_{j_3=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_2 j_3 j_1 j_3 j_2 j_1} + C_{j_2 j_3 j_1 j_3 j_1 j_2} \Bigr). \tag{146}
$$

By analogy with the proof of (59) and applying (145), we get

$$
\lim_{p \to \infty} \sum_{j_3=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_2 j_3 j_1 j_3 j_2 j_1} + C_{j_2 j_3 j_1 j_3 j_1 j_2} \Bigr) = 0. \tag{147}
$$

From (146) and (147) we have

$$
\lim_{p \to \infty} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_2 j_3 j_1 j_3 j_2 j_1} + C_{j_2 j_3 j_1 j_3 j_1 j_2} \Bigr) = 0. \tag{148}
$$

Moreover (see (54)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_1j_2}=0. \tag{149}$$

Combining (148) and (149), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_1j_3j_2j_1}=0.$$

The equality (65) is proved.

Now consider (66). Using the integration order replacement, we obtain

$$C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$\times\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)dt_3dt_5dt_6-$$

$$-\int\limits_t^T\phi_{j_3}(t_6)\int\limits_t^{t_6}\phi_{j_1}(t_5)\int\limits_t^{t_5}\phi_{j_2}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times\left(\int\limits_t^{t_3}\phi_{j_3}(t_4)dt_4\right)dt_3dt_5dt_6=$$

$$
= \int\limits_t^T \phi_{j_1}(t_5) \left(\int\limits_t^{t_5} \phi_{j_3}(t_4) dt_4 \right) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \times
$$

$$
\times \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6) dt_6 \right) dt_5 -
$$

$$
- \int\limits_t^T \phi_{j_1}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) \times
$$

$$
\times \left(\int\limits_t^{t_3} \phi_{j_3}(t_4) dt_4 \right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_3}(t_6) dt_6 \right) dt_5. \tag{150}
$$

Applying (72) and (71), we obtain

$$
\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_3 j_2 j_2 j_1} + C_{j_3 j_1 j_3 j_2 j_1 j_2} \Bigr) =
$$

$$
= - \sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_3 j_2 j_2 j_1} + C_{j_3 j_1 j_3 j_2 j_1 j_2} \Bigr) =
$$

$$
= \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=p+1}^{\infty} \Bigl(C_{j_3 j_1 j_3 j_2 j_2 j_1} + C_{j_3 j_1 j_3 j_2 j_1 j_2} \Bigr) -
$$

$$
- \frac{1}{2} \sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_3 j_2 j_2 j_1} \Bigg|_{(j_2 j_2) \curvearrowright (\cdot)}. \tag{151}
$$

The equality

$$
\lim_{p\to\infty} \frac{1}{2} \sum_{j_1=0}^{p} \sum_{j_3=p+1}^{\infty} C_{j_3 j_1 j_3 j_2 j_2 j_1} \Bigg|_{(j_2 j_2) \curvearrowright (\cdot)} = 0 \tag{152}
$$

follows from the equality (130) in [14], p. 139 (also see the equality (2.722) in [13], p. 516), where we proceed similarly to the proof of equality (110) (see (111)).

By analogy with the proof of (59) and applying (150), we get

$$\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_2=0}^{p}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{153}$$

From (151)–(153) we have

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\left(C_{j_3j_1j_3j_2j_2j_1}+C_{j_3j_1j_3j_2j_1j_2}\right)=0. \tag{154}$$

Moreover (see (55)),

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_1j_2}=0. \tag{155}$$

Combining (154) and (155), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_3j_1j_3j_2j_2j_1}=0.$$

The equality (66) is proved.

Finally consider (67). Using the integration order replacement, we have

$$C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\times$$

$$\times dt_3dt_4dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\left(\int\limits_t^{t_3}\phi_{j_1}(t_1)dt_1\right)\int\limits_{t_3}^{t_5}\phi_{j_3}(t_4)dt_4\times$$

$$\times dt_3dt_5dt_6=$$

$$=\int\limits_t^T\phi_{j_2}(t_6)\int\limits_t^{t_6}\phi_{j_3}(t_5)\left(\int\limits_t^{t_5}\phi_{j_3}(t_4)dt_4\right)\int\limits_t^{t_5}\phi_{j_1}(t_3)\left(\int\limits_t^{t_3}\phi_{j_2}(t_2)dt_2\right)\times$$

$$\times \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 dt_5 dt_6 -$$

$$- \int\limits_t^T \phi_{j_2}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) \times$$

$$\times \left(\int\limits_t^{t_3} \phi_{j_3}(t_4) dt_4 \right) dt_3 dt_5 dt_6 =$$

$$= \int\limits_t^T \phi_{j_3}(t_5) \left(\int\limits_t^{t_5} \phi_{j_3}(t_4) dt_4 \right) \int\limits_t^{t_5} \phi_{j_1}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \times$$

$$\times \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_2}(t_6) dt_6 \right) dt_5 -$$

$$- \int\limits_t^T \phi_{j_3}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_3) \left(\int\limits_t^{t_3} \phi_{j_2}(t_2) dt_2 \right) \left(\int\limits_t^{t_3} \phi_{j_1}(t_1) dt_1 \right) \times$$

$$\times \left(\int\limits_t^{t_3} \phi_{j_3}(t_4) dt_4 \right) dt_3 \left(\int\limits_{t_5}^{T} \phi_{j_2}(t_6) dt_6 \right) dt_5. \qquad (156)$$

Using (72) and (71), we get

$$\sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \sum_{j_3=p+1}^{\infty} \left(C_{j_2 j_3 j_3 j_1 j_2 j_1} + C_{j_2 j_3 j_3 j_1 j_1 j_2} \right) =$$

$$= \frac{1}{2} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \left(C_{j_2 j_3 j_3 j_1 j_2 j_1} \Bigr|_{(j_3 j_3) \curvearrowright (\cdot)} + C_{j_2 j_3 j_3 j_1 j_1 j_2} \Bigr|_{(j_3 j_3) \curvearrowright (\cdot)} \right) -$$

$$- \sum_{j_3=0}^{p} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \left(C_{j_2 j_3 j_3 j_1 j_2 j_1} + C_{j_2 j_3 j_3 j_1 j_1 j_2} \right) =$$

$$= \frac{1}{2} \sum_{j_1=p+1}^{\infty} \sum_{j_2=p+1}^{\infty} \left(C_{j_2 j_3 j_3 j_1 j_2 j_1} \Bigr|_{(j_3 j_3) \curvearrowright (\cdot)} + C_{j_2 j_3 j_3 j_1 j_1 j_2} \Bigr|_{(j_3 j_3) \curvearrowright (\cdot)} \right) +$$

$$
+\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Biggr)-
$$

$$
-\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}. \qquad (157)
$$

The equalities

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\left(C_{j_2j_3j_3j_1j_2j_1}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}+C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_3j_3)\curvearrowright(\cdot)}\right)=0, \qquad (158)
$$

$$
\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=
$$

$$
=\lim_{p\to\infty}\frac{1}{4}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)(j_3j_3)\curvearrowright(\cdot)}-
$$

$$
-\lim_{p\to\infty}\frac{1}{2}\sum_{j_3=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}\Biggr|_{(j_1j_1)\curvearrowright(\cdot)}=0 \qquad (159)
$$

follows from the equalities (130), (131) in [14], p. 139 (also see the equalities (2.722), (2.723) in [13], p. 516), where we used the same technique as in (111). When proving (159), we also applied (71) and (29).

By analogy with the proof of (59) and applying (156), we obtain

$$
\lim_{p\to\infty}\sum_{j_1=0}^{p}\sum_{j_3=0}^{p}\sum_{j_2=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Biggr)=0. \qquad (160)
$$

From (157)–(160) we have

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}\Biggl(C_{j_2j_3j_3j_1j_2j_1}+C_{j_2j_3j_3j_1j_1j_2}\Biggr)=0. \qquad (161)
$$

Furthermore (see (57)),

$$
\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty}C_{j_2j_3j_3j_1j_1j_2}=0. \qquad (162)
$$

Combining (161) and (162), we finally obtain

$$\lim_{p\to\infty}\sum_{j_1=p+1}^{\infty}\sum_{j_2=p+1}^{\infty}\sum_{j_3=p+1}^{\infty} C_{j_2j_3j_3j_1j_2j_1}=0.$$

The equality (67) is proved. Theorem 8 is proved.

4 Conclusion

In the first part [14] of this work, expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 5 with respect to components of the multidimensional Wiener process were obtained. The second part of the work (this article) is devoted to the development of the approach from [14]. More precisely, in this article we have obtained the expansion of iterated Stratonovich stochastic integrals of multiplicity 6 with respect to components of the multidimensional Wiener process. The noted results make it possible to construct efficient procedures for the mean-square approximation of iterated Stratonovich stochastic integrals that appear in strong methods with orders 1.0, 1.5, 2.0, 2.5, and 3.0 of convergence for Itô SDEs with multidimensional non-commutative noise (in the framework of the approach based on the Taylor–Stratonovich expansion). The above procedures based on multiple Fourier–Legendre series have been successfully implemented as part of the software package in the Python programming language in [52].

References

[1] Gihman, I.I., Skorohod, A.V. Stochastic Differential Equations and its Applications. Kiev, Naukova Dumka, 1982, 612 pp.

[2] Kloeden, P.E., Platen, E. Numerical Solution of Stochastic Differential Equations. Berlin, Springer, 1992, 632 pp.

[3] Milstein, G.N. Numerical Integration of Stochastic Differential Equations. Sverdlovsk, Ural University Press, 1988, 225 pp.

[4] Milstein, G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Berlin, Springer, 2004, 616 pp.

[5] Kloeden, P.E., Platen, E., Schurz, H. Numerical Solution of SDE Through Computer Experiments. Berlin, Springer, 1994, 292 pp.

[6] Platen, E., Wagner, W. On a Taylor formula for a class of Ito processes. Prob. Math. Stat. 3 (1982), 37-51.

[7] Kloeden, P.E., Platen, E. The Stratonovich and Ito-Taylor expansions. Math. Nachr. 151 (1991), 33-50.

[8] Averina, T.A. Statistical Modeling of Solutions of Stochastic Differential Equations and Systems with Random Structure. Siberian Branch of the Russian Academy of Sciences, Novosibirsk, 2019, 350 pp.

[9] Kulchitskiy, O.Yu., Kuznetsov, D.F. The unified Taylor–Ito expansion. J. Math. Sci. (N. Y.), 99, 2 (2000), 1130-1140.
DOI: http://doi.org/10.1007/BF02673635

[10] Kuznetsov, D.F. New representations of the Taylor–Stratonovich expansion. J. Math. Sci. (N. Y.), 118, 6 (2003), 5586-5596.
DOI: http://doi.org/10.1023/A:1026138522239

[11] Kuznetsov, D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg. 2006, 764 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-227

[12] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian].

Polytechnical University Publishing House, Saint-Petersburg. 2009, 768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230

[13] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs. arXiv:2003.14184v34 [math.PR], 2022, 928 pp.
DOI: http://doi.org/10.48550/arXiv.2003.14184

[14] Kuznetsov, D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Differencialnie Uravnenia i Protsesy Upravlenia, 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[15] Allen, E. Approximation of triple stochastic integrals through region subdivision. Commun. Appl. Anal. (Special Tribute Issue to Professor V. Lakshmikantham), 17 (2013), 355-366.

[16] Li, C.W., Liu, X.Q. Approximation of multiple stochastic integrals and its application to stochastic differential equations. Nonlinear Anal. Theor. Meth. Appl., 30, 2 (1997), 697-708.

[17] Tang, X., Xiao, A. Asymptotically optimal approximation of some stochastic integrals and its applications to the strong second-order methods. Adv. Comp. Math., 45 (2019), 813-846.

[18] Gaines, J.G., Lyons, T.J. Random generation of stochastic area integrals. SIAM J. Appl. Math., 54 (1994), 1132-1146.

[19] Wiktorsson, M. Joint characteristic function and simultaneous simulation of iterated Ito integrals for multiple independent Brownian motions. Ann. Appl. Prob., 11, 2 (2001), 470-487.

[20] Ryden, T., Wiktorsson, M. On the simulation of iterated Ito integrals. Stoch. Proc. and their Appl., 91, 1 (2001), 151-168.

[21] Averina, T.A., Prigarin, S.M. Calculation of stochastic integrals of Wiener processes. [In Russian]. Preprint 1048. Novosibirsk, Inst. Comp. Math. Math. Geophys. Siberian Branch Russ. Acad. Sci., 1995, 15 pp.

[22] Prigarin, S.M., Belov, S.M. One application of series expansions of Wiener process. [In Russian]. Preprint 1107. Novosibirsk, Inst. Comp. Math. Math. Geophys. Siberian Branch Russ. Acad. Sci., 1998, 16 pp.

[23] Kloeden, P.E., Platen, E., Wright, I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl., 10, 4 (1992), 431-441.

[24] Rybakov, K.A. Applying spectral form of mathematical description for representation of iterated stochastic integrals. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2019), 1-31. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.1.html

[25] Rybakov, K.A. Using spectral form of mathematical description to represent Stratonovich iterated stochastic integrals. Smart Innovation, Systems and Technologies, vol. 217. Springer, 2021, pp. 287-304. DOI: http://doi.org/10.1007/978-981-33-4826-4_20

[26] Rybakov, K.A. Using spectral form of mathematical description to represent Ito iterated stochastic integrals. Smart Innovation, Systems and Technologies, vol. 274. Springer, 2022, pp. 331-344. DOI: http://doi.org/10.1007/978-981-16-8926-0_22

[27] Kuznetsov, D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2019), 32-52. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html

[28] Foster, J., Habermann, K. Brownian bridge expansions for Lévy area approximations and particular values of the Riemann zeta function. Combin. Prob. Comp. (2022), 1-28.
DOI: http://doi.org/10.1017/S096354832200030X

[29] Kastner, F., Rößler, A. An analysis of approximation algorithms for iterated stochastic integrals and a Julia and MATLAB simulation toolbox. arXiv:2201.08424v1 [math.NA], 2022, 43 pp.
DOI: http://doi.org/10.48550/arXiv.2201.08424

[30] Malham, S.J.A., Wiese A. Efficient almost-exact Lévy area sampling. Stat. Prob. Letters, 88 (2014), 50-55.

[31] Stump, D.M., Hill J.M. On an infinite integral arising in the numerical integration of stochastic differential equations. Proc. Royal Soc. Series A. Math. Phys. Eng. Sci. 461, 2054 (2005), 397-413.

[32] Platen, E., Bruti-Liberati, N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin, Heidelberg. 2010, 868 pp.

[33] Kuznetsov, D.F. Multiple Ito and Stratonovich Stochastic Integrals: Approximations, Properties, Formulas. Polytechnical University Publishing House, Saint-Petersburg. 2013, 382 pp.
DOI: http://doi.org/10.18720/SPBPU/2/s17-234

[34] Kuznetsov, D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2018), A.1-A.1073. Available at:
http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[35] Kuznetsov, D.F. Development and application of the Fourier method for the numerical solution of Ito stochastic differential equations. Comp.

Math. Math. Phys., 58, 7 (2018), 1058-1070.
DOI: http://doi.org/10.1134/S0965542518070096

[36] Kuznetsov, D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence. Autom. Remote Control, 79, 7 (2018), 1240-1254.
DOI: http://doi.org/10.1134/S0005117918070056

[37] Kuznetsov, D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 2.5 order of strong convergence. Autom. Remote Control, 80, 5 (2019), 867-881.
DOI: http://doi.org/10.1134/S0005117919050060

[38] Kuznetsov, D.F. Explicit one-step mumerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multidimensional nonadditive noise based on the Taylor–Stratonovich expansion. Comp. Math. Math. Phys., 60, 3 (2020), 379-389.
DOI: http://doi.org/10.1134/S0965542520030100

[39] Kuznetsov, D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Comp. Math. Math. Phys., 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[40] Kuznetsov, D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. arXiv:1905.03724v15 [math.GM], 2022, 41 pp. DOI: http://doi.org/10.48550/arXiv.1905.03724

[41] Kuznetsov, D.F. Application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations.

arXiv:1912.02612v6 [math.PR], 2022, 32 pp.
DOI: http://doi.org/10.48550/arXiv.1912.02612

[42] Kuznetsov, D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (1997), 18-77. Available at: http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html

[43] Kuznetsov, D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. J. Autom. Inform. Sci., 32, 12 (2000), 69-86.
DOI: http://doi.org/10.1615/JAutomatInfScien.v32.i12.80

[44] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Math. J., 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49

[45] Kuznetsov, D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. arXiv:1712.09746v24 [math.PR]. 2022, 111 pp.
DOI: http://doi.org/10.48550/arXiv.1712.09746

[46] Kuznetsov, D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. arXiv:1801.01079v14 [math.PR]. 2022, 68 pp.
DOI: http://doi.org/10.48550/arXiv.1801.01079

[47] Kuznetsov, D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor–Ito and Taylor–Stratonovich expansions using Legendre polynomials. arXiv:1801.00231v19 [math.PR]. 2022, 106 pp.
DOI: http://doi.org/10.48550/arXiv.1801.00231

[48] Kuznetsov, D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. arXiv:1801.03195v18 [math.PR]. 2022, 138 pp.
DOI: http://doi.org/10.48550/arXiv.1801.03195

[49] Kuznetsov, D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. arXiv:1712.09516v21 [math.PR]. 2022, 204 pp.
DOI: http://doi.org/10.48550/arXiv.1712.09516

[50] Kuznetsov, D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. arXiv:1802.00643v14 [math.PR]. 2022, 129 pp.
DOI: http://doi.org/10.48550/arXiv.1802.00643

[51] Kuznetsov, D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Differencialnie Uravnenia i Protsesy Upravlenia, 2 (2020), 89-117. Available at:
http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

[52] Kuznetsov, M.D., Kuznetsov, D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. Differencialnie Uravnenia i Protsesy Upravlenia, 1 (2021), 93-422. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[53] Kuznetsov, D.F., Kuznetsov, M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods

and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol. 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32.
DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[54] Rybakov, K.A. Orthogonal expansion of multiple Itô stochastic integrals. Differencialnie Uravnenia i Protsesy Upravlenia, 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[55] Stratonovich, R.L. Conditional Markov Processes and Their Application to the Theory of Optimal Control. Elsevier, N. Y., 1968, 350 pp.

[56] Johnson, G.W., Kallianpur, G. Homogeneous chaos, p-forms, scaling and the Feynman integral. Trans. Amer. Math. Soc., 340 (1993), 503-548.

[57] Rybakov, K.A. Orthogonal expansion of multiple Stratonovich stochastic integrals. Differencialnie Uravnenia i Protsesy Upravlenia, 4 (2021), 81–115. Available at:
http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.5.html