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Nonlinear partial differential equations
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Secular terms for the kinetic McKean model

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Abstract. In this article, we investigate the kinetic McKean model. The perturbed solution of the Cauchy problem is sought in the form of Fourier series. The Fourier coefficients for the zero and nonzero modes are written out, respectively. The original system is reduced to an infinite system of differential equations. An approximation for the systems is constructed. Under certain assumptions, we find secular terms (non-integrable part). This, in turn, will allow us to prove for the first time the exponential stabilization of the solution in the future.

Keywords: kinetic model, Fourier series, Knudsen parameter, secular terms

1 Introduction

We consider the well-known kinetic McKean model [2, 5, 9]:

$$\partial_t u + \partial_x u = \frac{1}{\varepsilon}(w^2 - uw), \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$\partial_t w - \partial_x w = -\frac{1}{\varepsilon}(w^2 - uw), \tag{2}$$

$$u(x,0) = u^0, w(x,0) = w^0, \tag{3}$$

where $u^0(x) = u^0(x+2\pi), w^0(x) = w^0(x+2\pi)$ is periodic functions. This system describes a monatomic gas with two groups of particles with corresponding densities $u = u(x,t), w = w(x,t)$. The first group moves with the speed $c = 1$, the other $c = -1$, the parameter ε corresponds to the Knudsen number in the kinetic theory of gases. The McKean system is a non-integrable system, i.e. the Painlevé test is inapplicable.

The physical description of the Boltzmann equation is described in a fundamental article [1, 8]. The asymptotic stability of kinetic systems of Carleman, Godunov–Sultangazin and Broadwell for periodic initial data were studied in the works [7, 11, 12, 13, 14]. The proofs of the theoretical results were confirmed numerically in the works [17, 18]. The exact solutions of the systems are presented in [2, 3, 4, 5, 9, 10, 15, 16]. The secularity condition for the kinetic Carleman system was found in [6]. In this work, approaches and methods (see [6, 11, 13]) will be applied for our system as well as for the above systems. The McKean system has been largely unexplored. We will single out the non-dissipative part of the solution and reduce the problem of the existence of a global solution to a nonlinear equation in the Hilbert space. This will later allow us to prove for the first time the exponential stabilization of the solution.

2 Fourier solution for the McKean system

We study the Cauchy problem for small perturbations of the equilibrium state $w_e^2 = u_e w_e$, $u_e, w_e > 0$ of the system (1)-(3). Let be

$$u = u_e + w_e^{1/2}\varepsilon^2 \widehat{u},\ w = w_e + w_e^{1/2}\varepsilon^2 \widehat{w}. \tag{4}$$

Then

$$\partial_t \widehat{u} + \partial_x \widehat{u} - w_e \frac{1}{\varepsilon}(\widehat{w} - \widehat{u}) = \varepsilon w_e^{1/2}(\widehat{w}^2 - \widehat{u}\widehat{w}), \quad x \in \mathbb{R},\ t > 0, \tag{5}$$

$$\partial_t \widehat{w} - \partial_x \widehat{w} + w_e \frac{1}{\varepsilon}(\widehat{w} - \widehat{u}) = -\varepsilon w_e^{1/2}(\widehat{w}^2 - \widehat{u}\widehat{w}), \tag{6}$$

$$\widehat{u}\ |_{t=0} = \widehat{u}^0, \quad \widehat{w}\ |_{t=0} = \widehat{w}^0. \tag{7}$$

For periodic solutions with zero means

$$\widehat{u}(t,x) = u_0(t) + \sum_{k \in \mathbb{Z}_0} u_k(t) e^{ikx}, \quad \widehat{w}(t,x) = w_0(t) + \sum_{k \in \mathbb{Z}_0} w_k(t) e^{ikx},$$

$$\mathbb{Z}_0 = \{k \in \mathbb{Z}, k \neq 0\},$$

we introduce weight spaces $L_{2,\gamma}(\mathbb{R}_+;\mathcal{H}_\sigma),\mathcal{H}_\sigma$ with norms:

$$\|\widehat{u}\|^2_{L_{2,\gamma}(\mathbb{R}_+;\mathcal{H}_\sigma)}=\int_0^\infty\ e^{2\gamma t}\mid u_0(t)\mid^2\ dt+$$

$$+\int_0^\infty\ e^{2\gamma t}\sum_{k\in\mathbb{Z}_0}\mid k\mid^{2\sigma}\mid u_k(t)\mid^2\ dt,\quad |||\ \widehat{u}\mid_{t=0}|||^2_{\mathcal{H}_\sigma}=\mid u_0^0\mid^2+\sum_{k\in\mathbb{Z}_0}\mid k\mid^{2\sigma}\mid u_k^0\mid^2.$$

Here $\gamma>0,\sigma=const.$

Theorem 1 *For any $\sigma>2$ and $w_e^2=u_ew_e>0$ there exist $\mu_0,q\in(0,1)$ such that for periodic initial data $(\widehat{u}^0,\widehat{w}^0)$ with zero averages satisfying the inequality*

$$|||\widehat{u}^0|||_{\mathcal{H}_\sigma}+|||\widehat{w}^0|||_{\mathcal{H}_\sigma}\le\varepsilon^2q,$$

there exists a global solution $(\widehat{u},\widehat{w})\in L_{2,\gamma}(R_+;\mathcal{H}_\sigma)$ to Cauchy problem (5)-(7), where $\gamma=\varepsilon\mu_0>0$.

Hence, the local equilibrium principle with an exponential stabilization to the equilibrium state holds.

Theorem 2 *Let $\sigma>2$ and let the condition of Theorem 1 be fulfilled. Then a positive equilibrium state ($u_e=const>0,w_e=const>0,w_e^2=u_ew_e$) is exponentially stable:*

$$|||u(x,t)-u_e|||_{\mathcal{H}_\sigma}\le c_1\Big(|||\widehat{u}^0|||_{\mathcal{H}_\sigma}+|||\widehat{w}^0|||_{\mathcal{H}_\sigma}\Big)e^{-2\gamma_1 t},$$

$$|||w(x,t)-w_e|||_{\mathcal{H}_\sigma}\le c_2\Big(|||\widehat{u}^0|||_{\mathcal{H}_\sigma}+|||\widehat{w}^0|||_{\mathcal{H}_\sigma}\Big)e^{-2\gamma_1 t},$$

where $c_1,c_2>0,\gamma_1>\gamma>0$.

These theorems will be proven in a future publication for the first time.

We assume that the average

$$u_0^0=\frac{1}{2\pi}\int_0^{2\pi}\widehat{u}^0(x)dx=w_0^0=\frac{1}{2\pi}\int_0^{2\pi}\widehat{w}^0(x)dx=0.$$

Let us rewrite the system (5)-(7) in terms of Fourier coefficients for $k\neq 0$

$$\frac{d}{dt}u_k+iku_k=-(\frac{d}{dt}w_k-ikw_k),\tag{8}$$

$$\frac{d}{dt}w_k-ikw_k+\frac{1}{\varepsilon}w_e(w_k-u_k)=-\varepsilon w_e^{1/2}\sum_{k_1+k_2=k,k,k_1,k_2\in\mathbb{Z}}(w_{k_1}w_{k_2}-u_{k_1}w_{k_1}),\tag{9}$$

$$u_k\mid_{t=0}=u_k^0,w_k\mid_{t=0}=w_k^0.\tag{10}$$

and the zero mode $k=0$

$$\frac{d}{dt}u_0 = -\frac{d}{dt}w_0, \tag{11}$$

$$\frac{d}{dt}w_0 + \frac{1}{\varepsilon}w_e(w_0-u_0) = -\varepsilon w_e^{1/2} \sum_{k_1+k_2=0, k_1\in\mathbb{Z}, k_2\in\mathbb{Z}} (w_{k_1}w_{k_2} - u_{k_1}w_{k_1}), \tag{12}$$

$$u_0 \mid_{t=0} = 0, w_0 \mid_{t=0} = 0. \tag{13}$$

Solving the equation (8), we find its solution

$$u_k = -w_k + (u_k^0 + w_k^0)e^{-ikt} + 2ik\int_0^t e^{ik(s-t)}w_k ds. \tag{14}$$

For $k=0$ we have

$$u_0 = -w_0.$$

We rewrite out the sum for $k \neq 0$

$$\sum_{k_1+k_2=k, k_1\in\mathbb{Z}, k_2\in\mathbb{Z}} (w_{k_1}w_{k_2} - u_{k_1}w_{k_1}) =$$

$$= w_0w_k - u_0w_k + w_kw_0 - u_kw_0 + \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} (w_{k_1}w_{k_2} - u_{k_1}w_{k_1}) =$$

$$= 4w_0w_k - w_0\Big((u_k^0 + w_k^0)e^{-ikt} + 2ik\int_0^t e^{ik(s-t)}w_k ds\Big) +$$

$$+ \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} \Big(2w_{k_1}w_{k_2} - w_{k_2}\Big((u_{k_1}^0 + w_{k_1}^0)e^{-ik_1t} +$$

$$+2ik_1\int_0^t e^{ik_1(s-t)}w_{k_1}ds\Big)\Big). \tag{15}$$

Substituting (14), (15) to (9), we have an infinite system of ordinary differential equations (ODEs)

$$\frac{d}{dt}w_k - ikw_k + 2w_e\frac{1}{\varepsilon}w_k - 2ikw_e\frac{1}{\varepsilon}\int_0^t e^{ik(s-t)}w_k ds = \tag{16}$$

$$= w_e^{1/2}\frac{1}{\varepsilon}d_k e^{-ikt} +$$

$$+\varepsilon w_e^{1/2}\Big(l_k(w) - 2B_k(w,w)\Big) - \varepsilon w_e^{1/2}T_k^{add}(w),$$

$$w_k \mid_{t=0} = w_k^0, \quad k \in \mathbb{Z}_0 = \{k\in\mathbb{Z}, k\neq 0\}.$$

Here $T_k^{add}(w)$ is the perturbation operator of the base system

$$\frac{d}{dt}w_k - ikw_k + 2w_e\frac{1}{\varepsilon}\ w_k - 2ikw_e\frac{1}{\varepsilon}\int_0^t\ e^{ik(s-t)}w_k ds = \tag{17}$$
$$= w_e^{1/2}\frac{1}{\varepsilon}d_k e^{-ikt} + \varepsilon w_e^{1/2}\Big(l_k(w) - 2B_k(w,w)\Big),$$
$$w_k\mid_{t=0} = w_k^0, \quad k \in \mathbb{Z}_0 = \{k \in \mathbb{Z}, k \neq 0\},$$

where

$$d_k = w_e^{1/2}(u_k^0 + w_k^0), T_k^{add}(w) = w_0\Big(4w_k - (u_k^0 + w_k^0)e^{-ikt} - 2ik\int_0^t e^{ik(s-t)}w_k ds\Big),$$

$$l_k(w) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}(u_{k_1}^0 + w_{k_1}^0)e^{-ik_1 t}w_{k_2},$$

$$B_k(w,w) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} w_{k_2}\Big(w_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}w_{k_1}ds\Big).$$

Let's make a replacement for the transition to zero initial data

$$w_k = w_k^0 e^{(ik-2w_e\frac{1}{\varepsilon})t} + y_k, y_k \in L_{2,\gamma}(\mathbb{R}; \mathcal{H}_\sigma). \tag{18}$$

In what follows, we will consider the system (16) without the perturbation operator. Substituting (18) to (17) and taking into account that

$$ik\int_0^t e^{ik(s-t)}\Big(w_k^0 e^{(ik-2w_e\frac{1}{\varepsilon})t} + y_k\Big)ds = \frac{w_k^0 ik}{2(ik - w_e\frac{1}{\varepsilon})}\Big(e^{(ik-2w_e\frac{1}{\varepsilon})t} - e^{-ikt}\Big),$$

then we have for y_k an infinite system of ordinary differential equations (ODEs)

$$T_k(y_k) \equiv \frac{d}{dt}y_k - iky_k + 2w_e\frac{1}{\varepsilon}\,y_k - 2ikw_e\frac{1}{\varepsilon}\int_0^t\ e^{ik(s-t)}y_k ds = \tag{19}$$
$$= w_e^{1/2}\frac{1}{\varepsilon}D_k e^{-ikt} + f_k(t)e^{-2w_e\frac{1}{\varepsilon}t} + \varepsilon w_e^{1/2}\Big(L_k(y) - 2B_k(y,y)\Big),$$
$$y_k\mid_{t=0} = 0,\ k \in \mathbb{Z}_0 = \{k \in \mathbb{Z}, k \neq 0\},$$

where

$$D_k = w_e^{1/2}(u_k^0 + w_k^0) - \frac{ikw_e^{1/2}}{ik - w_e\frac{1}{\varepsilon}}w_k^0,$$

$$B_k(y,y) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} y_{k_2}\Big(y_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}y_{k_1}ds\Big),$$

$$f_k(t) = \frac{ikw_e}{ik - w_e\frac{1}{\varepsilon}}\frac{1}{\varepsilon}w_k^0 e^{ikt} + \varepsilon w_e^{1/2}\Big(f_k^L(t) - 2f_k^B(t)\Big),$$

$$f_k^L(t) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}(u_{k_1}^0 + w_{k_1}^0)e^{-ik_1t}w_{k_2}^0 e^{ik_2t},$$

$$f_k^B(t) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} w_{k_2}^0 e^{ik_2t}\Big(w_{k_1}^0 e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - \frac{ik_1w_{k_1}^0}{2(ik_1 - w_e\frac{1}{\varepsilon})}\times$$

$$\times\Big(e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - e^{-ik_1t}\Big)\Big),$$

$$L_k(y) = \sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}(u_{k_1}^0 + w_{k_1}^0)e^{-ik_1t}y_{k_2} +$$

$$+\sum_{k_1+k_2=k, k\in\mathbb{Z}_0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}\Big(w_{k_2}^0 e^{(ik_2-2w_e\frac{1}{\varepsilon})t}\Big(y_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}y_{k_1}ds\Big)+$$

$$+y_{k_2}\Big(w_{k_1}^0 e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - \frac{ik_1w_{k_1}^0}{2(ik_1 - w_e\frac{1}{\varepsilon})}\Big(e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - e^{-ik_1t}\Big)\Big)\Big).$$

3 Equation for zero mode

From the system (12) for the zero mode, we obtain

$$\frac{d}{dt}w_0 + 2\frac{1}{\varepsilon}w_e w_0 = -\varepsilon w_e^{1/2}\Big(2w_0w_0 + \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}(w_{k_1}w_{k_2} - u_{k_1}w_{k_2})\Big), \quad (20)$$

$$w_0 \,|_{t=0} = 0. \quad (21)$$

Rewrite (20) as the Riccati equation

$$\frac{d}{dt}w_0 + 2\frac{1}{\varepsilon}w_e w_0 = -\varepsilon w_e^{1/2}\Big(2w_0w_0 - l_0(w) + 2B_0(w,w)\Big), \quad (22)$$

$$w_0 \,|_{t=0} = 0, \quad (23)$$

where

$$l_0(w) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0}(u_{k_1}^0 + w_{k_1}^0)e^{-ik_1t}w_{k_2},$$

$$B_0(w,w) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} w_{k_2}\Big(w_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}w_{k_1}ds\Big).$$

Let be

$$w_k = w_k^0 e^{(ik-2w_e\frac{1}{\varepsilon})t} + y_k, y_k \in L_{2,\gamma}(\mathbb{R}).$$

Here

$$\|y\|^2_{L_{2,\gamma}(\mathbb{R}_+)} = \int_0^\infty \; e^{2\gamma t} \mid y \mid^2 \; dt.$$

Then we have

$$\frac{d}{dt}y_0 + 2\frac{1}{\varepsilon}w_e y_0 = -\varepsilon w_e^{1/2}\Big(2y_0y_0 - f_0(t)e^{-2w_e\frac{1}{\varepsilon}t} - l_0(y) + 2B_0(y,y)\Big),$$

$$y_0 \mid_{t=0} = 0,$$

where

$$B_0(y,y) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} y_{k_2}\Big(y_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}y_{k_1}ds\Big),$$

$$f_0(t) = f_0^L(t) - 2f_0^B(t),$$

$$f_0^L(t) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} (u^0_{k_1} + w^0_{k_1})e^{-ik_1t}w^0_{k_2}e^{ik_2t},$$

$$f_0^B(t) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} w^0_{k_2}e^{ik_2t}\Big(w^0_{k_1}e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - \frac{ik_1w^0_{k_1}}{2(ik_1 - w_e\frac{1}{\varepsilon})}\times$$

$$\times\Big(e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - e^{-ik_1t}\Big)\Big),$$

$$L_0(y) = \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} (u^0_{k_1} + w^0_{k_1})e^{-ik_1t}y_{k_2} +$$

$$+ \sum_{k_1+k_2=0, k_1\in\mathbb{Z}_0, k_2\in\mathbb{Z}_0} \Big(w^0_{k_2}e^{(ik_2-2w_e\frac{1}{\varepsilon})t}\Big(y_{k_1} - ik_1\int_0^t e^{ik_1(s-t)}y_{k_1}ds\Big) +$$

$$+ y_{k_2}\Big(w^0_{k_1}e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - \frac{ik_1w^0_{k_1}}{2(ik_1 - w_e\frac{1}{\varepsilon})}\Big(e^{(ik_1-2w_e\frac{1}{\varepsilon})t} - e^{-ik_1t}\Big)\Big)\Big).$$

4 Finite approximation

To construct an approximation solution of the Cauchy problem (5)-(7), a finite approximation of the infinite system (19) is introduced for $m \in \mathbb{N}$:

$$T_k(y_k^{(m)}) \equiv \frac{d}{dt}y_k^{(m)} - iky_k^{(m)} + 2w_e\frac{1}{\varepsilon}\,y_k^{(m)} - 2ikw_e\frac{1}{\varepsilon}\int_0^t \; e^{ik(s-t)}y_k^{(m)}ds = \quad (24)$$

$$= w_e^{1/2}\frac{1}{\varepsilon}D_k^{(m)}e^{-ikt} + f_k^{(m)}(t)e^{-2w_e\frac{1}{\varepsilon}t} + \varepsilon w_e^{1/2}\Big(L_k^{(m)}(y^{(m)}) - 2B_k^{(m)}(y^{(m)}, y^{(m)})\Big),$$

$$y_k^{(m)} \mid_{t=0} = 0, \;\; k \in \mathbb{Z}_0, \mid k \mid \le m,$$

Here

$$D_k^{(m)} = w_e^{1/2}(u_k^0 + w_k^0) - \frac{ikw_e^{1/2}}{ik - w_e\frac{1}{\varepsilon}} w_k^0,$$

$$B_k^{(m)}(y^{(m)}, y^{(m)}) = \sum_{k_1+k_2=k, |k_1|\le m, |k_2|\le m} y_{k_2}\Big(y_{k_1} - ik_1 \int_0^t e^{ik_1(s-t)} y_{k_1} ds\Big),$$

$$f_k^{(m)}(t) = \frac{ikw_e}{ik - w_e\frac{1}{\varepsilon}} \frac{1}{\varepsilon} w_k^0 e^{ikt} + \varepsilon w_e^{1/2}\Big(f_{k,m}^L(t) - 2f_{k,m}^B(t)\Big),$$

$$f_{k,m}^L(t) = \sum_{k_1+k_2=k, |k_1|\le m, |k_2|\le m} (u_{k_1}^0 + w_{k_1}^0) e^{-ik_1 t} w_{k_2}^0 e^{ik_2 t},$$

$$f_{k,m}^B(t) = \sum_{k_1+k_2=k, |k_1|\le m, |k_2|\le m} w_{k_2}^0 e^{ik_2 t}\Big(w_{k_1}^0 e^{(ik_1 - 2w_e\frac{1}{\varepsilon})t} - \frac{ik_1 w_{k_1}^0}{2(ik_1 - w_e\frac{1}{\varepsilon})}\times$$

$$\times\Big(e^{(ik_1 - 2w_e\frac{1}{\varepsilon})t} - e^{-ik_1 t}\Big)\Big),$$

$$L_k^{(m)}(y^{(m)}) = \sum_{k_1+k_2=k, |k_1|\le m, |k_2|\le m} (u_{k_1}^0 + w_{k_1}^0) e^{-ik_1 t} y_{k_2} +$$

$$+ \sum_{k_1+k_2=k, |k_1|\le m, |k_2|\le m} \Big(w_{k_2}^0 e^{(ik_2 - 2w_e\frac{1}{\varepsilon})t}\Big(y_{k_1} - ik_1 \int_0^t e^{ik_1(s-t)} y_{k_1} ds\Big) +$$

$$+ y_{k_2}\Big(w_{k_1}^0 e^{(ik_1 - 2w_e\frac{1}{\varepsilon})t} - \frac{ik_1 w_{k_1}^0}{2(ik_1 - w_e\frac{1}{\varepsilon})}\Big(e^{(ik_1 - 2w_e\frac{1}{\varepsilon})t} - e^{-ik_1 t}\Big)\Big)\Big).$$

The solution of the system (24) will be sought in the form

$$y_k^{(m)} = Q_k^{(m)} T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)}), \quad z_k^{(m)} |_{t=0} = 0,$$

$$Q^{(m)} \in \mathcal{H}_\sigma^{(m)}, \quad z^{(m)} \in L_{2,\gamma}(R_+; \mathcal{H}_\sigma^{(m)}),$$

where $z^{(m)} = (z_k^{(m)},\ |\ k\ | \le m, k \ne 0)$. Then

$$z_k^{(m)} = \Big(w_e^{1/2} \frac{1}{\varepsilon} D_k^{(m)} - Q_k^{(m)}\Big) e^{-ikt} + f_k^{(m)}(t) e^{-2w_e\frac{1}{\varepsilon}t} + \tag{25}$$

$$+ \varepsilon w_e^{1/2}\Big(L_k^{(m)}\Big(Q_k^{(m)} T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)})\Big) -$$

$$-2B_k^{(m)}\Big(Q_k^{(m)} T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)}), Q_k^{(m)} T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)})\Big)\Big).$$

In the variables $(z_k^{(m)}, Q_k^{(m)})$, the system (25) under the secularity condition

$$w_e^{1/2} \frac{1}{\varepsilon} D_k^{(m)} - Q_k^{(m)} = 0,\ |\ k\ | = 1, \ldots, m, \tag{26}$$

will be written as:

$$z_k^{(m)} = f_k^{(m)}(t)e^{-2w_e\frac{1}{\varepsilon}t} + \varepsilon w_e^{1/2}\Big(L_k^{(m)}\Big(Q_k^{(m)}T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)})\Big) -$$
$$-2B_k^{(m)}\Big(Q_k^{(m)}T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)}), Q_k^{(m)}T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)})\Big)\Big).$$

We get the system in the Hilbert space $L_{2,\gamma}(\mathbb{R}_+;\mathcal{H}_\sigma^{(m)})$. For zero mode we set $y_0 = z_0$. In this case

$$z_0 = -\varepsilon w_e^{1/2}\int_0^t e^{2w_e\frac{1}{\varepsilon}(s-t)}\Big(2z_0z_0 - f_0(t)e^{-2w_e\frac{1}{\varepsilon}t} - l_0(y) + 2B_0(y,y)\Big)ds. \quad (27)$$

Here is no secularity condition for the zero mode.

5 Local equilibrium

We will find a solution to the secularity condition from the principle of local equilibrium. Taking into account (14) and (18), we have

$$u_k^{(m)} = -y_k^{(m)} - w_k^0 e^{(ik-2w_e\frac{1}{\varepsilon})t} + (u_k^0 + w_k^0)e^{-ikt} + 2ik\int_0^t e^{ik(s-t)}y_k ds +$$
$$+\frac{ik}{ik - w_e\frac{1}{\varepsilon}}\Big(w_k^0 e^{(ik-2w_e\frac{1}{\varepsilon})t} - e^{-ikt}\Big) =$$
$$= -y_k^{(m)} + e^{(ik-2w_e\frac{1}{\varepsilon})t}\frac{w_e\frac{1}{\varepsilon}w_k^0}{ik - w_e\frac{1}{\varepsilon}} +$$
$$+2ik\int_0^t e^{ik(s-t)}y_k ds + (u_k^0 - \frac{w_e\frac{1}{\varepsilon}}{ik - w_e\frac{1}{\varepsilon}}w_k^0)e^{-ikt}, \quad (28)$$

We separate the non-integrable part using (24)

$$ik\int_0^t e^{ik(s-t)}Q_k^{(m)}T_k^{-1}(e^{-iks})ds = -\frac{\varepsilon}{2w_e}Q_k^{(m)}e^{-ikt} + R_k, \quad (29)$$

$$R_k = \frac{\varepsilon}{2w_e}\Big(\frac{d}{dt}T_k^{-1}(e^{-ikt}) - ikT_k^{-1}(e^{-ikt}) + 2w_e\frac{1}{\varepsilon}T_k^{-1}(e^{-ikt})\Big) \in L_{2,\gamma}(\mathbb{R};\mathcal{H}_\sigma^{(m)}).$$

Applying the formula (29), we get

$$u_k^{(m)} = -Q_k^{(m)}T_k^{-1}(e^{-ikt}) - T_k^{-1}(z_k^{(m)}) + 2ik\int_0^t e^{ik(s-t)}T_k^{-1}(z_k^{(m)})ds + \quad (30)$$
$$+(u_k^0 - \frac{w_e\frac{1}{\varepsilon}}{ik - w_e\frac{1}{\varepsilon}}w_k^0 - \frac{\varepsilon}{w_e}Q_k^{(m)})e^{-ikt} + e^{(ik-2w_e\frac{1}{\varepsilon})t}\frac{w_e\frac{1}{\varepsilon}w_k^0}{ik - w_e\frac{1}{\varepsilon}} +$$
$$+\frac{\varepsilon}{w_e}\Big(\frac{d}{dt}T_k^{-1}(e^{-ikt}) - ikT_k^{-1}(e^{-ikt}) + 2w_e\frac{1}{\varepsilon}T_k^{-1}(e^{-ikt})\Big).$$

If

$$Q_k^{(m)} = \frac{w_e}{\varepsilon}\Big(u_k^0 - \frac{w_e\frac{1}{\varepsilon}w_k^0}{ik - w_e\frac{1}{\varepsilon}}\Big), \mid k \mid \leq m, \tag{31}$$

then we have $u_k^{(m)} \to 0$, when $t \to \infty$. For the second component in $\mathcal{H}_\sigma^{(m)}$, we have

$$w_k^{(m)} = Q_k^{(m)} T_k^{-1}(e^{-ikt}) + T_k^{-1}(z_k^{(m)}) + w_k^0 e^{(ik - 2w_e\frac{1}{\varepsilon})t} \to 0, t \to \infty.$$

Thus, under the condition (31), we have the local equilibrium.

6 Conclusion

The one-dimensional McKean system was investigated. Secular terms were found that do not belong to our space $L_{2,\gamma}$. As a result, we obtain a nonlinear equation in the Hilbert space. In what follows, we obtain priori estimates for one, an existence theorem for a solution using the fixed point theorem. We will also prove the weak convergence of the approximative solution to the weak solution and just the classical solution. From here, the exponential stabilization of the solution to a positive equilibrium state will follow (see theorems 1, 2).

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