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Dynamical systems

Computer-oriented tests for hyperbolicity and structural stability of dynamical system

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Abstract. A diffeomorphism f is hyperbolic on a chain-recurrent set if the Morse spectrum does not contain zero. The symbolic image is a directed graph approximating a dynamical system. The chain-recurrent set is localized using this graph. The symbolic image of the differential allows us to estimate the Morse spectrum. A diffeomorphism f is structurally stable if the dual differential has only trivial bounded trajectories. The symbolic image of the dual differential makes it possible to check the absence of bounded trajectories of the dual differential.

Keywords: Lyapunov exponent, chain-recurrent set, projective bundle, pseudotrajectory, invariant decomposition, directed graph, adjacency matrix, strong component, extreme cycle.

1 Hyperbolicity

Consider a discrete dynamical system

$$x_{n+1} = f(x_n) \tag{1}$$

generated by a diffeomorphism $f : M \to M$ of a compact manifold M, $d = \dim M$. The differential Df is a linear map which in local coordinates is a partial derivative matrix

$$Df(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right).$$

The differential acts on tangent bundle $Df(x) : TM_x \to TM_{f(x)}$ and defines a linear expansion of the discrete system $x_{n+1} = f(x_n)$ to the tangent bundle according to the formula

$$v_{n+1} = Df(x_n)v_n. \tag{2}$$

Let E_1, E_2 be subspaces of linear space E. The vector direct sum $E = E_1 \oplus E_2$ is defined if $E = E_1 + E_2$ and $E_1 \cap E_2 = \{0\}$.

Definition 1 *Smooth dynamical system $x_{n+1} = f(x_n)$ is called hyperbolic on an invariant set $\Lambda \subset M$, if there exist invariant subbundles E^s and E^u of the tangent space $TM|_\Lambda$, the constants $K > 0$, and $\alpha > 0$ such that*

$$\begin{aligned} TM|_\Lambda &= E^s \oplus E^u, \\ |Df^n(x)v| &\le K\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^s(x),\ n > 0, \\ |Df^{-n}(x)v| &\le K\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^u(x),\ n > 0, \end{aligned}$$

where the invariance of the subbundle E^* means the equality $Df(x)E^*(x) = E^*(f(x))$. The number α is called the hyperbolicity exponent.

A doubly infinite sequence of points $T = \{x_k\}$ is a trajectory of the system (1) if $f(x_k) = x_{k+1}$. A doubly infinite sequence of points $\sigma = \{x_k,\ k \in \mathbb{Z}\}$ is called an ε-trajectory or a pseudotrajectory if

$$\rho(f(x_k), x_{k+1}) < \varepsilon$$

for any k. Let ε-trajectory $\{x_n\}$ be periodic, that is, there exists k¿ 0 such that $x_{n+k} = x_n$ for all n. Then the points x_n are called ε-periodic. The exact trajectory of the system is rarely known in practice, as a rule, we work with ε-trajectories for sufficiently small positive ε.

Definition 2 *A point x is called chain-recurrent if x is ε-periodic for any $\varepsilon > 0$, that is, there exists a periodic ε-trajectory passing through the point x.*

The chain-recurrent set consists of all chain-recurrent points and is denoted by CR. The chain-recurrent set CR is invariant, closed and contains all types of return trajectories: periodic, almost-periodic, nonwandering, homoclinic, etc. Note that if the chain-recurrent point is not periodic and $\dim M > 1$, then there is an arbitrarily small perturbation f in the C^0-topology for which this point is periodic [30]. We can say that chain-recurrent points generate periodic trajectories under C^0-perturbations. Therefore, in computer calculations, chain-recurrent points look like periodic points of sufficiently large period.

Definition 3 *Two chain-recurrent points are called equivalent if they can be connected by a periodic ε-trajectory for any $\varepsilon > 0$. The chain-recurrent set is divided into equivalence classes $\{\Omega_i\}$, which we will call the components of the chain-recurrent set.*

1.1 Lyapunov Exponent

Every nondegenerate linear mapping $A : \mathbb{R}^d \to \mathbb{R}^d$ can be represented in the form

$$Av = |Av| \cdot \frac{Av}{|Av|} = r|Ae| \cdot \frac{Ae}{|Ae|} = r|Ae| \cdot A_s(e),$$

where $r = |v|,\ |v| \neq 0,\ e = v/|v| \in S^{d-1}$, and S^{d-1} is the unit sphere. Thus, the linear mapping $A : v \to Av$ is the product of two mappings

$$e \to A_s(e) = \frac{Ae}{|Ae|} \ \ \textit{and} \ \ r \to r\ |Ae|,$$

where the first of them acts on the sphere S^{d-1}, and the other acts on the positive half-line $\mathbb{R}^+$. A projective space P^{d-1} is a set of the one-dimensional subspaces in $\mathbb{R}^d$. The projective space P^{d-1} can be obtained by identifying antipodal points of the sphere S^{d-1}. Symmetry of the mapping $A_s(e)$ with respect to the sign change $A_s(\pm e) = \pm A_s(e)$ enables us to define a mapping $PA(e) = Ae/|Ae|$ onto the projective space P^{d-1} by identifying the antipodal points on the sphere. We preserve the notation e for the points of the space P^{d-1}; that is, e stands for the one-dimensional subspace and (or) a unit vector on the subspace, which does not lead to confusion. Thus, for the diffeomorphism f we obtain the mapping

$$Pf(x, e) = (f(x), \frac{Df(x)e}{|Df(x)e|})$$

on the projective bundle $P = \{(x, e) : \ x \in M, \ e \in P^{d-1}(x)\}$, and this mapping generates discrete dynamic system of the form

$$x_{n+1} = f(x_n), \tag{3}$$

$$e_{n+1} = \frac{Df(x_n)e_n}{|Df(x_n)e_n|} \tag{4}$$

on the projective bundle P. A positive number $|Df(x)e|$ is the coefficient of length change on the subspace e under the action of the differential at the point x. The projective bundle P is a compact manifold with metric (see [7], pp. 531-539), which can be interpreted as sum of the distance on the manifold M and the angle between the subspaces.

Let $\xi = \{(x_k, e_k), \ k = 0, 1, 2, \cdots\}$ be a semitrajectory of the system (3, 4). The upper limit

$$\lambda(\xi) = \overline{\lim_{n\to\infty}} \frac{1}{n} \ln |Df^n(x_0)e_0| = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} \ln |Df(x_k)e_k| \tag{5}$$

is the Lyapunov exponent of the semitrajectory. If ξ is periodic of period p then $\lambda(\xi)$ is the average of the function $\ln |Df(x)e|$ over periodic ε-trajectory

$$\lambda(\xi) = \frac{1}{p} \sum_{k=0}^{p-1} \ln |Df(x_k)e_k|$$

The Morse spectrum is the limit set of the Lyapunov exponents of periodic ε-trajectories as $\varepsilon \to 0$

$$\Sigma = \{\lambda : \ \exists \ periodic \ \varepsilon_n - trajectories \ \{\xi_n\}, \ \varepsilon_n \to 0, \ \lambda = \lim_{n\to\infty} \lambda(\xi_n)\}.$$

Theorem 1 *[18] The following statements are equivalent:*

1. *the diffeomorphism f is hyperbolic on the chain-recurrent set CR;*
2. *the Morse spectrum does not contain zero.*

Theorem 2 *[4, 29] Let $x_{n+1} = f(x_n)$ be a smooth system on a compact manifold M, Ω is a component of the chain-recurrent set on M and $\pi : TM \to M$ is the projection from the tangent bundle to the manifold. Then*

- *the chain-recurrent set of the restriction $Pf(x, e) = (f(x), \frac{Df(x)e}{|Df(x)e|})$ on $P|_\Omega$ has l components $\Omega_1, ..., \Omega_l, \ 1 \le l \le \dim M$,*

- *(!) every set Ω_k defines a (continuous, constant dimension) subbundle E_k over Ω of the form*

$$E_k = \{v \in \pi^{-1}(\Omega) : \ if \ v \neq 0 \ \Rightarrow \frac{v}{|v|} = e \in \Omega_k\},$$

- *decomposition of the tangent bundle takes the form*

$$TM|_{\Omega} = E_1 \oplus ... \oplus E_l, \tag{6}$$

- *each chain-recurrent component Ω^* in the bundle $P|_{\Omega}$ is projected onto the chain-recurrent component Ω and has the properties described in the item (!).*

According to the Theorem 2, the components $\Omega_1, ..., \Omega_l$ uniquely determine the decomposition (6). Since each chain-recurrent component is an invariant set, the decomposition is also invariant. Thus, in order to obtain the invariant decomposition (6) we must construct the components of the chain-recurrent set of the mapping $Pf(x,e) = (f(x), \frac{Df(x)e}{|Df(x)e|})$. The decomposition described in the Theorem 2 is called the finest Morse decomposition of the tangent bundle.

2 Symbolic image of dynamical system

Let $C = \{M(1), ..., M(n)\}$ be a finite covering of a manifold M by closed subsets. The set $M(i)$ will be referred to the cell of the index i.

Definition 4 *[17] Symbolic image of the dynamical system (1) for a covering C is a directed graph G with vertices $V = \{i\}$ corresponding to cells $\{M(i)\}$. Two vertices i and j are connected by a directed edge (an arc) $i \to j$ if and only if*

$$f(M(i)) \bigcap M(j) \neq \emptyset.$$

The symbolic image is a powerful tool for investigation of integral properties of dynamical system. The symbolic image generates the symbolic dynamics, which reflects the dynamics of system (1). We do not impose special restrictions on the coverage of C, but basing on the theorem about triangulation of compact manifold we may without loss of generality assume that cells $M(i)$ are polyhedrons that intersect in boundary disks. In numerical calculations (see [3]), M is a compact domain in R^d, and $M(i)$ are cubes or parallelepipeds.

Construction of a symbolic image based on numerical calculations is always only an estimation of the "real" symbolic image. The algorithms used for an efficient calculation of symbolic image are discussed in the papers [3, 9, 22]. There is a number of issues that need to be considered when building a cell image. Cells of covering, as a rule, are selected as rectangular parallelepipeds. To construct edge $i \to j$, we need to check that $f(M(i)) \cap M(j) \neq \emptyset$.

For our implementation we use the following simple way to construct an approximation of cell image $f(M(i))$. Each cell $M(i)$ is approximated by a finite set of so-called scan points. The number N of scan points for the cells as well as positions of these points are parameters of the described method which must be set by user. There is no general strategy how the scan points should be placed within the cell. Our practice shows that the good strategy is when the scan points uniformly distributed within the cell. Number N is determined by two factors: counting time and accuracy of calculating the cell image. Usually for smooth systems $10 \le N \le 100$ is quite sufficient. It is further assumed that arrow $i \to j$ exists if there is a scan point $x^* \in M(i)$ such that $f(x^*) \in M(j)$.

A doubly infinite sequence $\{z_k,\ k \in \mathbb{Z}\}$ of vertices of a graph G is called a path (or an admissible path) if, for each k, the graph G contains the directed edge $z_k \to z_{k+1}$. If the symbolic image is calculated as above, then a path $\omega = \{i_n,\ n \in \mathbb{Z}\}$ generates a point sequence $\sigma(\omega) = \{x_n,\ n \in \mathbb{Z}\}$, where $x_n \in M(i_n)$ is a scan point such that $f(x_n) \in M(i_{n+1})$. In this case, the distance

$$\rho(f(x_n), x_{n+1}) \le d,$$

where d is the maximal cell diameter. Thus, the sequence $\{x_n,\ n \in \mathbb{Z}\}$ is an ε-trajectory for any $\varepsilon > d$.

Five methods of constructing a cell image are considered in the dissertation [22]. There is a natural correspondence between admissible paths on the symbolic image G and ε-trajectories of the homeomorphism f.

Theorem 3 (Weak shadowing property) *[20]*

1. *If a sequence $\{z_k,\ k \in \mathbb{Z}\}$ is a path on the symbolic image G and a sequence $\{x_k,\ k \in \mathbb{Z}\}$ is such that $x_k \in M(z_k)$, then the sequence $\{x_k\}$ is an ε-trajectory of f for any $\varepsilon > q + d$.*

2. *If a sequence $\{z_k,\ k \in \mathbb{Z}\}$ is a path on the symbolic image G then there exists a sequence $\{x_k,\ k \in \mathbb{Z}\}$, $x_k \in M(z_k)$ that is an ε-trajectory of f for any $\varepsilon > d$.*

3. *There exists $r > 0$ such that if a sequence $\{x_k,\ k \in \mathbb{Z}\}$ is an ε-trajectory of f, $\varepsilon < r$, and $x_k \in M(z_k)$, then the sequence $\{z_k,\ k \in \mathbb{Z}\}$ is an admissible path on the symbolic image G.*

An admissible path in symbolic image can be thought of as a pseudotrajectory encoding. Each trajectory $Tr = \{x_n = f^n(x_0),\ n \in \mathbb{Z}\}$ generates the path $\omega(Tr) = \{i_n\}$ such that $x_n \in M(i_n)$, $\omega(Tr)$ is called the coding of the trajectory.

2.1 Transition Matrix

The directed graph G is uniquely determined by its (adjacency) matrix of transitions $\Pi = (\pi_{ij})$, where $\pi_{ij} = 1$ if and only if there is the directed edge $i \to j$, otherwise $\pi_{ij} = 0$. Let

$$\Pi^2 = \left(\pi^2_{ij}\right)$$

be the square of the transition matrix, where $\pi^2_{ij} = \sum_{k=1}^{n} \pi_{ik}\pi_{kj}$, and upper script 2 stands for an index (not for power). The sum $\sum_{k=1}^{n} \pi_{ik}\pi_{kj} = \pi^2_{ij}$ is the number of all admissible paths of length 2 from i to j. In the similar way one can verify that the entry π^p_{ij} is the number of all admissible paths of length p from i to j.

Definition 5 *A vertex of the symbolic image is called recurrent if there is a periodic path passing through it. The set of recurrent vertices is denoted by RV. Two recurrent vertices i and j are called equivalent if there is a periodic path containing i and j.*

The set of recurrent vertices RV is decomposed into classes $\{H_k\}$ of equivalent recurrent vertices. In the graph theory, the classes H_k are called strongly connected components. There are algorithms for constructing strongly connected components based on depth-first bypass of the graph [8, 31]. The most popular is algorithm of Robert Tarjan [32].

Let us introduce a quasi-order relation between the vertices of a symbolic image. We assume that $i \prec j$ if and only if there exists an admissible path of the form

$$i = i_0, i_1, i_2, ..., i_m = j.$$

Hence, a vertex i is recurrent if and only if $i \prec i$, and recurrent vertices $i,\ j$ are equivalent if and only if $i \prec j \prec i$.

Proposition 1 **[1]** *The vertices of a symbolic image G can be renumbered so that*

- *the equivalent recurrent vertices are numbered with consecutive integers;*
- *the new numbers $i,\ j$ of other vertices are chosen so that $i < j$ if $i \prec j \not\prec i$.*

In other words, the transition matrix has the form

$$\Pi = \begin{array}{c} \left(\begin{array}{ccccc} (\Pi_1) & \cdots & \cdots & \cdots & \cdots \\ & \ddots & & & \\ 0 & & (\Pi_k) & \cdots & \cdots \\ & \ddots & & \ddots & \\ 0 & & 0 & & (\Pi_s) \end{array}\right) \end{array} \begin{array}{c} H_1 \\ \vdots \\ H_k \\ \vdots \\ H_s \end{array} \qquad (7)$$

where the elements under the diagonal blocks are zeros, each diagonal block Π_k corresponds to either a class of equivalent recurrent vertices H_k or a non-recurrent vertex. In the last case Π_k coincides with a single zero. A numbering $\{i\}$ of matrix rows of the Π_k diagonal block coincides with the vertices numbering of the H_k class. This kind of transition matrix can be called an upper triangular. In graph theory, the described renumbering of vertices is called *topological graph sorting.*

2.2 Localization of Chain-Recurrent Set

Let G be a symbolic image with respect to a covering C and d be diameter of the covering C. The multivalued mapping $h : M \to V$ associates with the point x all the vertices of i such that $x \in M(i)$.

Proposition 2 *[21] The map h has the following properties.*

1. *If x is a chain-recurrent point and $i \in h(x)$, then i is a recurrent vertex.*
2. *If x_1 and x_2 lie in the same component of the chain-recurrent set Ω, $i_1 \in h(x_1)$ and $i_2 \in h(x_2)$, then i_1 and i_2 are equivalent recurrent vertices.*
3. *If Ω is a component of a chain-recurrent set, then there exists a unique class $H(\Omega)$ of equivalent recurrent vertices such that $h(\Omega) \subset H(\Omega)$.*

Let Ω be a component of a chain-recurrent set. From the Proposition 2 it follows that the image $h(\Omega)$ lies in some class of equivalent recurrent vertices, which we denote $H(\Omega)$, the component Ω lies in $\{\bigcup M(i)|\ i \in H(\Omega)\}$. Note that the class $H(\Omega)$ is uniquely determined by Ω.

Denote by $P(d)$ the union of the cells $M(i)$ for which the vertex i is recurrent, i.e.,

$$P(d) = \{\bigcup M(i) : \ i \ is \ recurrent \ vertex\}. \tag{8}$$

Theorem 4 *[20]*

1. *The set $P(d)$ is a closed neighbourhood of the chain-recurrent set CR*

$$CR \subset P(d). \tag{9}$$

2. *For any neighbourhood V of the chain-recurrent set CR there exists $d > 0$ such that*

$$CR \subset P(d) \subset V.$$

3. *The chain-recurrent set CR coincides with the intersection of the sets $P(d)$ for all positive d:*

$$CR = \bigcap_{d>0} P(d). \tag{10}$$

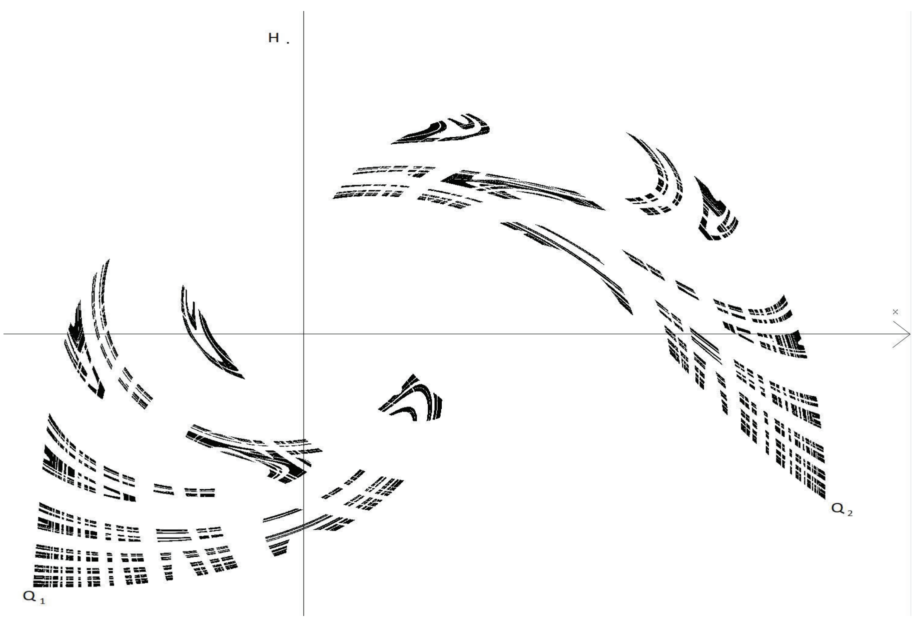

Figure 1: Localization of the chaotic chain-recurrent set.

Example 1 *Localization of a chaotic set.*

A component of the chain-recurrent set Ω is a chaotic invariant set if Ω contains a dense non-periodic trajectory. Japanese researchers led by Ikeda (see

[11]) described dynamics of a nonlinear optical system using a discrete dynamic system in the complex plane. The Ikeda mapping has the form

$$J:\ z \to R + C_2 z \exp(i(C_1 - \frac{C_3}{1+|z|^2})),$$

where z is a complex variable, $R,\ C_1,\ C_2,\ C_3$ are real constants. The Ikeda mapping models rather complex dynamics on the plane, see [12, 20]. Let us consider a modified Ikeda mapping of the form

$$J:(x,y)\mapsto (R + a(x\ \cos\tau - y\ \sin\tau), b(x\ \sin\tau + y\ \cos\tau)), \tag{11}$$

where $\tau = 0.4 - 6/(1+x^2+y^2)$, $R=1$, $a=-0.9$, $b=1.2$. The map J has a hyperbolic fixed point $H(-0.0950, 2.1937)$. There is a 2-periodic hyperbolic trajectory $Q(-1.5584,-1.9046)$, $(3.0088,-1.2438)$, which generates a homoclinic trajectory and a chaotic chain-recurrent set Ω. The Ω is a Cantor set, it is localized by the method above described. An obtained neighbourhood $P(\Omega)$ consists of 874618 cells, see Fig. 1. The set Ω consists of two parts; Ω_1 is the left part and Ω_2 is the right part. The Ikeda mapping is 2-periodic on Ω,

$$J:\ \Omega_1 \to \Omega_2;\ \Omega_2 \to \Omega_1.$$

2.3 Averaging on Graph

Let G be an arbitrary directed graph. Function on G is a distribution (framing) $\{a[ij]\}$ on the edges $\{i \to j\}$ of the graph. The averaging of the function over a periodic path $\omega = \{z_1, ..., z_p = z_0\}$ is the average of the function on the period.

Definition 6 *The averaging spectrum of the function $\{a[ij]\}$ over the graph G is the limit set of the averages over periodic paths, that is,*

$$\Sigma(G) = \{\lambda \in \mathbb{R}:\ \textit{there exists a sequence of periodic paths}$$

$$\omega_n\ \textit{such that}\ \ \lambda = \lim_{n\to\infty}\lambda(\omega_n)\}.$$

Thus, the spectrum of the graph is determined on the recurrent vertices RV.

Consider the sequence of periodic paths $\{\omega_n\}$, $n \to \infty$, on which the limit $\lambda \in \Sigma(G)$ is reached. The set RV is divided into a finite number of $\{H_m\}$ equivalence classes of recurrent vertices. If the sequence $\{\omega_k\}$ lies in the class

H and $\lambda = \lim_{k\to\infty} \lambda(\omega_k)$, then we say that the value λ is realized in the class H. A spectrum of the class H is defined as

$$\Sigma(H) = \{\lambda \in \mathbb{R} : \textit{ there exists a sequence of periodic paths}$$

$$\omega_n \subset H \textit{ such that } \lambda = \lim_{n\to\infty} \lambda(\omega_n)\}.$$

A periodic path $\omega = \{z_1, ..., z_p = z_0\}$ is a simple path or a cycle if its vertices $z_1, ..., z_p$ are distinct. The number of simple paths is finite. Consider some class H of equivalent recurrent vertices. Let

$$\begin{aligned} \lambda_{\min}(H) &= \min\{\lambda(\phi_j),\ j = 1, ..., q\}, \\ \lambda_{\max}(H) &= \max\{\lambda(\phi_j),\ j = 1, ..., q\} \end{aligned}$$

be the minimum and maximum of the averages over simple periodic paths of the class H. Cycles on which $\lambda_{\min}(H)$ or $\lambda_{\max}(H)$ are reached are called extreme cycles. The algorithm for calculating extremal cycles is presented in [19].

Theorem 5 *[21] The spectrum $\Sigma(G)$ of the averaging over graph consists of the intervals $[\lambda_{\min}(H_k), \lambda_{\max}(H_k)]$, where $\{H_k\}$ is a complete family of classes of equivalent recurrent vertices of the graph G.*

2.4 Estimation of the Averaging Spectrum

To calculate the spectrum of the averaging of a function over pseudotrajectories of a dynamical system, we construct the function on the symbolic image by setting a value $a[ij]$ for each edge $i \to j$, as follows. Let $i \to j$ be an edge on G. Fixing a point x in each cell $M(i)$, we set $a[ij] = \varphi(x)$. Note that the value $a[ij] = b[i]$ is independent of j. If x^* is another point from $M(i)$ and $a^*[ji] = \varphi(x^*)$, then we have the estimate

$$|a^*[ij] - a[ij]| = |\varphi(x) - \varphi(x^*)| < \eta(d),$$

where $\eta(d)$ is the modulus of continuity of the function φ and d is the diameter of the covering.

Consider the constructed function on the symbolic image G. According to the previous theorem, we can find the averaging spectrum of the symbolic image in a finite number of steps. The task is to estimate the averaging spectrum Σ of the function φ through the spectrum of the symbolic image $\Sigma(G)$.

Theorem 6 *[21]*

1. *The spectrum Σ of averaging of the function φ lies in the extended spectrum Σ^* of the symbolic image, which consists of the intervals*

$$[\lambda_{\min}(H_k) - \eta(d), \lambda_{\max}(H_k) + \eta(d)],$$

where $\{H_k\}$ is the full family of equivalence classes of recurrent vertices, d is the diameter of the covering, and $\eta(\cdot)$ is the modulus of continuity of the function φ.

2. *If the diameter d of the covering converges to 0, then the extended spectrum Σ^* and the spectrum of the symbolic image $\Sigma(G)$ converge to Σ in the Hausdorff metric.*

The Hausdorff metric. Let X and Y be subsets of a compact set K and let $\rho(x, y)$ be a distance on K. The distance between X and Y is defined as

$$dist(X, Y) = \sup\{\rho(x, Y),\ \rho(y, X) :\ x \in X,\ y \in Y\},$$

where $\rho(x, Y) = \inf\{\rho(x, y) :\ y \in Y\}$, If $X \subset Y$, then $\rho(x, Y) = 0$ and

$$dist(X, Y) = \sup\{\rho(y, X) :\ y \in Y\}.$$

Thus, in order to estimate the averaging spectrum of the system, we need to find the spectrum of the equipped symbolic image. This task comes down to finding cycles with minimum and maximum averages. It's clear that the search for such cycles using continuous enumeration is impossible in our conditions, since during iterative constructions of a symbolic image, the number of cycles increases sharply. The Theorem 6 allows us to estimate the Morse spectrum using the technique of the symbolic image of the mapping Pf on the projective bundle.

Let G be a symbolic image of the mapping Pf with respect to a covering $C = \{M(i)\}$ of the projective space P, let $\{H_k\}$ be a complete family of classes of equivalent recurrent vertices of the graph G. To calculate the spectrum, we need to transform the function $\varphi(x, e) = \ln |Df(x)e|$ on P to a function $a[ij]$ on the symbolic image G. If $i \to j$ is an edge on G, fixing a point (x, e) in the cell $M(i)$, we set $a[ij] = \ln |Df(x)e|$. Recall that a periodic path $\omega = \{z_1, ..., z_p = z_0\}$ is a cycle (a simple path) if its vertices $z_1, ..., z_p$ are distinct. The number of simple paths is finite. Let H be a class of equivalent recurrent vertices and let

$$\begin{aligned} \lambda_{\min}(H) &= \min\{\lambda(\phi_j),\ j = 1, ..., q\}, \\ \lambda_{\max}(H) &= \max\{\lambda(\phi_j),\ j = 1, ..., q\} \end{aligned}$$

be the minimum and maximum values of averaging on cycles of the class H. Since the number of cycles is finite, the described values are calculated in a finite number of steps. To achieve the goal, we can use the linear programming method of the paper [19] or the least characteristic cycle calculation method, see [10].

According to the Theorem 5, spectrum of the symbolic image $\Sigma(G)$ consists of the intervals $[\lambda_{\min}(H_k), \lambda_{\max}(H_k)]$, where $\{H_k\}$ is the complete family of classes of equivalent recurrent vertices of the symbolic image of G. The following theorem is a corollary of Theorems 6.

Theorem 7 *The Morse spectrum lies in the extended spectrum of the symbolic image*

$$\Sigma^* = \bigcup_k [\lambda_{\min}(H_k) - \eta(d), \lambda_{\max}(H_k) + \eta(d)],$$

where $\{H_k\}$ is the complete family of classes of equivalent recurrent vertices of the symbolic image, d is the diameter of the covering, and $\eta(\cdot)$ is the modulus of continuity of the function $\varphi = \ln |Df(x)e|$.

From the theorems 1 and 7 it follows

Hyperbolicity Test

Theorem 8 *The following statements are equivalent:*

1. *the diffeomorphism f is hyperbolic on the chain-recurrent set CR;*
2. *there exists $d_0 > 0$ such that the extended Morse spectrum of the symbolic image for a covering (of projective bundle) with maximal diameter $d < d_0$ does not contain zero.*

Thus, the symbolic image of a dynamical system allows us to localize the chain-recurrent set and check its hyperbolicity. The article [19] contains an application of the hyperbolicity test.

3 Structural Stability

3.1 Definitions

A qualitative theory of the dynamical systems consists in the geometric description of a set of trajectories. This leads to the fact that it is necessary to determine when the spaces of trajectories of two systems has the same quality characteristics. It is natural to assume that the change of coordinates transforms the system into its equivalent. In other words, the dynamical system $x_{n+1} = f(x_n)$ is equivalent to the system $y_{n+1} = g(y_n)$, if there exists a change of coordinates $y = h(x)$, which maps the trajectories of the first system to the trajectories another system. The most natural change of coordinates is a homeomorphism. In this case, the equality $f(x) = h^{-1}(g(h(x)))$ holds. Thus, we come to the following definition.

Definition 7 *The diffeomorphisms $f : M \to M$ and $g : M \to M$ are called topologically equivalent if there exists a homeomorphism $h : M \to M$ such that $g \circ h = h \circ f$.*

This means that the mapping f is converted to g when the coordinates are changed. In this case, the dynamics of the system $x_{n+1} = f(x_n)$ is transformed into the dynamics of the system $y_{n+1} = g(y_n)$. If a dynamic system arises in an application, then its coefficients (parameters) are usually known only approximately. Therefore, the problem arises of studying the change in dynamics under small perturbations of the system. It is extremely important to know the conditions under which the dynamics of the system is conserved under small perturbations of the model system.

Definition 8 *A diffeomorphism f is called structurally stable if there exists a neighbourhood U of f in the space of diffeomorphisms such that any diffeomorphism g from U is topologically equivalent to the diffeomorphism f.*

Structural stability is a fundamental property of a dynamical system which means that the qualitative behaviour of the trajectories is unaffected by small perturbations.

Alexander Andronov, a specialist in the field of electrical engineering, radiophysics and applied mechanics, was the first to initiate the task of studying the conservation of the dynamics of the system under perturbations. In 1932, he turned to Lev Pontryagin with a proposal to begin joint scientific work in this direction. Their article "Rough Systems" was published in Reports of the USSR Academy of Sciences in 1937 [2]. An extensive theory of dynamical systems has grown out of this four-page article. They considered an autonomous system of differential equations on a two-dimensional disk. It was assumed that the vector field at the boundary is directed strictly inside the disk. It was necessary to find the conditions under which the trajectories of the perturbed system are translated by small homeomorphic transformation into the trajectories of the original system. The described property was called ¡¡system roughness¿¿. In order for the system to be rough, it is necessary and sufficient that the following Andronov-Pontryagin conditions are met:

- there are a finite number of singular points (equilibrium states),
- these singular points have non-zero Lyapunov exponents; that is, they can be stable and unstable equilibrium states or hyperbolic equilibrium states,
- there are finitely many periodic trajectories,
- these periodic trajectories have non-zero Lyapunov exponents,
- there are no trajectories (separatrices) connecting hyperbolic equilibrium states or trajectories beginning and ending in the same hyperbolic state of equilibrium.

The modern term ”structural stability” for the rough systems comes from S. Lefschetz, see [13]. Until the beginning of the sixties, there was a hope that the conditions of Andronov-Pontryagin are necessary and sufficient for the structural stability. However, in 1961 S. Smale constructed an example of a structurally stable diffeomorphism on the plane (Smale horseshoe) with an infinite set of periodic trajectories, see [114]

In the 80-s of the last century, the necessary and sufficient conditions for structural stability were discovered by the efforts of Robin, Robinson and Mane (see formulation below). This wonderful classic result is abstract in nature, but it is not very suitable for practical use. It should be noted that modern computer technology allows us to check the Andronov-Pontryagin conditions. Our goal is to transform Robin-Robinson-Mane conditions into equivalent conditions that can be verified with a computer.

3.2 Transversality Condition

Let f be a diffeomorphism of a compact manifold M. To describe the conditions of structural stability we define a stable $S(x)$ and an unstable $U(x)$ subspaces at each point x:

$$
\begin{aligned}
S(x) &= \{v \in TM(x): \ |Df^n(x)v| \to 0, \, n \to +\infty\}, \\
U(x) &= \{v \in TM(x): \ |Df^n(x)v| \to 0, \, n \to -\infty\},
\end{aligned}
$$

where $TM(x)$ is the tangent space to M at x.

Definition 9 *We say that the transversality condition is satisfied on the manifold M if*

$$TM(x) = S(x) + U(x)$$

at any point $x \in M$.

Summarizing the results of J. Robbin [23], C. Robinson [24] and R. Mane [15] we can state the following theorem.

Theorem 9 *The diffeomorphism f is structurally stable if and only if the transversality condition is satisfied on the manifold M.*

Theorem 9 describes the necessary and sufficient conditions. The practical application of this result is limited by the difficulties of verification of the transversality condition.

Relationship between the transversality condition and hyperbolicity. Suppose that the transversality condition is satisfied and the diffeomorphism f is hyperbolic on the invariant set Λ. Then the invariant subspaces $E^s(x)$ and $E^u(x)$ continuously depend on $x \in \Lambda$ and $\dim E^s \ + \ \dim E^u = dimM$. Therefore, the dimension of these subspaces is constant on each connected component of the set Λ. The differential contracts exponentially on E^s and extends exponentially on E^u. It is clear that $E^s(x) \subset S(x)$ and $E^u(x) \subset U(x)$.

Theorem 10 *[4, 14] If the transversality condition is satisfied, then*

- *the set $\Lambda = \{x \in M : \ TM(x) = S(x) \oplus U(x)\}$ is closed and invariant;*
- *the diffeomorphism f is hyperbolic on Λ and $S(x) = E^s(x)$, $U(x) = E^u(x)$, $x \in \Lambda$;*
- *the chain-recurrent set CR lies in Λ.*

3.3 Dual Differential

Let $\langle v, u\rangle$ be the scalar product (inner product) in the tangent space TM and A be a linear mapping, then the conjugate mapping A^* is defined by the equality

$$\langle Av, u\rangle = \langle v, A^*u\rangle \,.$$

If A is a matrix, then A^* is the transposed matrix. The matrix $(A^*)^{-1}$ is the inverse to the transposed matrix. The dual differential is determined according to the formula

$$\widehat{D}f(x) = ((Df(x))^*)^{-1}.$$

The dual differential acts from $TM(x)$ to $TM(f(x))$. In local coordinates, the elements of the Jacobi matrix $Df(x) = A(x)$ are partial derivatives $a_{ij} = \partial f_i/\partial x_j$. In this case, the dual differential has the form

$$\widehat{D}f(x) = ((Df(x))^*)^{-1} = \frac{1}{\det Df(x)}\left(A_{ij}(x)\right),$$

where A_{ij} is the algebraic complement to a_{ij}. The dual differential $\widehat{D}f$ generates a dynamical system

$$(x_{n+1}, v_{n+1}) = (f(x_n), \widehat{D}f(x_n)v_n) \equiv (\widehat{D}f)(x_n, v_n)$$

on the tangent bundle TM and a dynamical system of the form

$$(x_{n+1}, e_{n+1}) = (f(x_n), \frac{\widehat{D}f(x_n)e_n}{|\widehat{D}f(x_n)e_n|}) \equiv P(\widehat{D}f)(x_n, e_n) \tag{12}$$

on the projective bundle P.

Definition 10 *The dual differential $\widehat{D}f(x)v$ is called hyperbolic on an invariant set $\Lambda \subset M$, if there exist invariant subbundles E^s and E^u of the tangent space $TM|_\Lambda$, the constants $K > 0$ and $\alpha > 0$ such that*

$$\begin{aligned} TM|_\Lambda &= E^s \oplus E^u, \\ |\widehat{D}f^n(x)v| &\leq K\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^s(x),\ n > 0, \\ |\widehat{D}f^{-n}(x)v| &\leq K\,|v| \exp(-\alpha n),\ x \in \Lambda,\ v \in E^u(x),\ n > 0. \end{aligned}$$

The invariance of the subbundle E^* means the equality $\widehat{D}f(x)E^*(x) = E^*(f(x))$.

The next theorem asserts the equivalence of the hyperbolicity of the differential and of the dual differential.

Theorem 11 *[4, 26]. A diffeomorphism f is hyperbolic on a chain-recurrent set CR if and only if the dual differential $\widehat{D}f$ is hyperbolic on CR. Moreover, the stable subspace $\widehat{E}^s$ of $\widehat{D}f$ perpendicular to the stable subspace E^s of Df and the unstable subspace $\widehat{E}^u$ is perpendicular to the unstable subspace E^u.*

The spectrum of the dual differential $\Sigma(\widehat{D}f)$ is defined as the averaging spectrum of the function

$$\psi(x, e) = \ln|\widehat{D}f(x)e|$$

over the periodic pseudotrajectories of the system (12).

Consider the covering $C(P)$ of the projective bundle and construct a symbolic image G of the mapping $P(\widehat{D}f)$. Choosing a point (x_i, e_i) in each cell $M(i)$, we construct a function $a[ij] = \psi(x_i, e_i)$ on the edges of the symbolic image G. We can find the spectrum $\Sigma(G)$ of the symbolic image. By using the modulus of continuity of the function $\psi(x, e)$ we can find the the extension of the spectrum $\Sigma(G)$. From the theorems 1, 7, 8, and 11 it follows

Theorem 12 *The next statements are equivalent:*

- *the dual differential $\widehat{D}f$ is hyperbolic on the chain-recurrent set $CR(f)$,*
- *the spectrum of $\Sigma(\widehat{D}f)$ of the dual differential does not contain zero,*
- *there exists $d_0 > 0$ such that the extended spectrum of the symbolic image of the dual differential with respect to a covering $C(P)$ with a maximum diameter $d < d_0$ does not contain zero.*

Definition 11 *[4, 28] The dual differential is said to have only trivial bounded trajectories if any bounded trajectory*

$$\{(x_{n+1}, v_{n+1}) = (f(x_n), \widehat{D}f(x_n)v_n),\ n \in \mathbb{Z}\}$$

is a zero trajectory, i.e. $v_n = 0$.

The following theorem describes the transversality condition in terms of the dual differential, see Proposition 8.46 in [4].

Theorem 13 *[4, 15] The transversality condition holds if and only if the dual differential has only trivial bounded trajectories.*

At first glance, it seems that to verify the described property of the dual differential is just as difficult as to verify the transversality condition. However, we will show that the absence of nontrivial bounded trajectories can be constructively verified by using the symbolic image of the dual differential $P\widehat{D}f$.

3.4 Equivalent Conditions

Suppose that the spectrum of the dual differential does not contain zero, then $\Sigma(\widehat{D}f)$ consists of two parts: the positive part Σ^+ and the negative part Σ^-. The spectrum consists of the segments $[a, b]$, each of which is generated by a component of the chain-recurrent set Ω of the mapping $P\widehat{D}f$. Hence, each component Ω of the chain-recurrent set $CR(P\widehat{D}f)$ receives the sign ¡¡plus¿¿ or ¡¡minus¿¿, depending on which segment it generates — positive or negative. In this case, the chain-recurrent set is divided into two parts CR^+ and CR^- so that CR^+ includes all the positive components and CR^- includes all the negative components.

Let e be the point of the projective bundle, let $E(e)$ be the one-dimensional subspace of the tangent bundle spanned by e. If $v \in TM$ is the tangent vector, then the projection $Pv = e$ is the point of the projective bundle. By construction, the spectrum $\Sigma(\widehat{D}f|_{E(CR^+)})$ is positive, and the spectrum $\Sigma(\widehat{D}f|_{E(CR^-)})$ is negative. The sets CR^+ and CR^- are called positive and negative (respectively) chain-recurrent sets of the dual differential on the projective bundle. We denote

$$\widehat{S} = \{v \in TM_x : \ lim_{n\to+\infty}\widehat{D}f^n(x)v = 0\},$$
$$\widehat{U} = \{v \in TM_x : \ lim_{n\to-\infty}\widehat{D}f^n(x)v = 0\},$$

and $P\widehat{S}$, $P\widehat{U}$ are the projections of these subspaces into the projective bundle.

Theorem 14 *[4, 7, 27] The dual differential has only trivial bounded trajectories if and only if the projections $P\widehat{U}$ is an attractor, and $P\widehat{S}$ is its dual repeller on the projective bundle.*

The theorem was proved by I.U. Bronstein (see [4, p. 132]) and, independently, D. Salamon, E. Zehnder (see [27, Theorem 2.7]). If a diffeomorphism is structurally stable, then from the Theorems 9, 10, 11, 12 it follows that the spectrum of the dual differential does not contain zero.

Consider a diffeomorphism f whose dual differential spectrum does not contain zero. Suppose a trajectory $\xi = \{(x_n, e_n) = P\widehat{D}f^n(x, e), \ n \in \mathbb{Z}\}$ is not chain-recurrent. Recall that the α-limit set of a trajectory is a limit set of the trajectory points for $n \to -\infty$, The ω-limit set of a trajectory is a limit set of the trajectory points for $n \to +\infty$. Since the trajectory ξ is not chain-recurrent, then its α-limit set lies in some component Ω_1 of the chain recurrence set and its ω-limit set lies in another component of Ω_2. We can say that the trajectory ξ starts at Ω_1 and ends at Ω_2. Let $C = \{M(i)\}$ be a closed covering of the projective bundle P and G is the symbolic image of the mapping $P\widehat{D}f$. We construct a function $a[ij]$ on the edges of the graph G by setting $\{a[ij] = \ln|\widehat{D}f(x_i)e_i|\}$, where $(x_i, e_i) \in M(i)$. The extended spectrum of the symbolic image of G contains the spectrum of the dual differential. According to Theorem 6, if the diameter of the covering converges to zero, then the extended spectrum converges to the spectrum of the dual differential in the Hausdorff metric. Let the diameter of the covering be so small such that the extended spectrum does not contain zero. The next proposition follows from the t heorem 142 in [20].

Proposition 3 *Let H be a class of equivalent recurrent vertices on G and $[a, b]$ is an extended spectrum of the class H. Then there are positive constants K_* and K^* such that:*

- *for any positive segment of trajectory of the form $\xi = \{(x_k, e_k) = P\widehat{D}f^k(x, e),\ k = 0, 1, ..., p\}$, whose admissible path $\omega = \{z_k :\ (x_k, e_k) \in M(z_k),\ k = 0, 1, ..., p\}$ lies in the class H, the inequalities*

$$K_* \exp(pa)|v| \leq |\widehat{D}f^p(x)v| \leq K^* \exp(pb)|v|$$

  hold, where vector v lies in the subspace e_0,

- *for any negative segment of trajectory of the form $\xi = \{(x_k, e_k) = P\widehat{D}f^k(x, e),\ k = 0, -1, ..., -p\}$, whose admissible path $\omega = \{z_k :\ (x_k, e_k) \in M(z_k),\ k = 0, -1, ..., -p\}$ lies in the class H, the inequalities*

$$K_* \exp(-pb)|v| \leq |\widehat{D}f^{-p}(x)v| \leq K^* \exp(-pa)|v|$$

  hold, where vector v lies in the subspace e_0.

Definition 12 *Suppose that the spectrum of the dual differential does not contain zero. We say that there exists a connection $CR^+ \to CR^-$, if there is a trajectory ξ of the map $P\widehat{D}f$ such that its α-limit set lies in CR^+, and its ω-limit set lies in CR^-.*

Theorem 15 *A diffeomorphism f is structurally stable if and only if the spectrum of the dual differential does not contain zero and there is no connection $CR^+ \to CR^-$.*

Proof. *Sufficiency.* Suppose that the spectrum of the dual differential does not contain zero and there is no link $CR^+ \to CR^-$. According to Theorems 9 and 13, we have to show that the dual differential has only trivial bounded trajectories. If the spectrum of the dual differential does not contain zero, then the chain-recurrent set $CR(P\widehat{D}f)$ decomposes into the sum of the positive and negative parts,

$$CR(P\widehat{D}f) = CR^+ + CR^-.$$

Since the spectrum of $\Sigma(\widehat{D}f)$ is closed, there exists a segment $[-\beta,\beta],\ \beta>0$, which does not contain points of the spectrum.

Let $\xi=\{(x_k,v_k)\}$ be a nonzero trajectory of the dual differential $\widehat{D}f$ and $P\xi=\{(x_k,e_k),\ e_k=Pv_k\}$ is the projection of the trajectory ξ into the projective bundle P.

If $P\xi$ is a chain-recurrent trajectory, then $P\xi$ lies in $CR^+ + CR^-$ and, hence, ξ is hyperbolic and unbounded. Suppose that $P\xi$ is not a chain-recurrent trajectory. Then its α- and ω-limit sets lie in $CR^+ + CR^-$. We say that $P\xi$ starts at CR^+, if its α-limit set lies in CR^+, and we say that $P\xi$ starts at CR^-, if its α-limit set lies in CR^-. Similarly, $P\xi$ ends at CR^+, if the ω-limit set lies in CR^+, and $P\xi$ ends at CR^-, if the ω-limit set lies in CR^-.

Let $P\xi$ end in CR^+. Then according to the Proposition 3, we have the inequalities

$$K_*\exp(pa)|v_0|\le|\widehat{D}f^p(x_0)v_0|,$$

for positive semitrajectori, where $a\ge\beta>0$ and $p>0$. Consequently, the positive semitrajectory ξ^+ is unbounded. Similarly, if $P\xi$ ends in CR^-, then

$$|\widehat{D}f^p(x_0)v_0|\le K^*\exp(pb)|v_0|,$$

for the positive semitrajectory, where $b\le-\beta<0$ and $p>0$. In this case, the positive semitrajectory ξ^+ is bounded. In the same way, we can show that if $P\xi$ starts at CR^+, then the negative semitrajectory ξ^- is bounded, and if $P\xi$ starts at CR^-, then the negative semitrajectory ξ^- is unbounded. Thus, a nonzero trajectory ξ is bounded if and only if $P\xi$ starts at CR^+ and ends at CR^-. In this case, we have the connection $CR^+\to CR^-$. This contradicts the assumptions of the theorem. Hence, the trajectory ξ is unbounded. Thus, any nonzero trajectory of the dual differential is unbounded. Then it follows from the Theorems 9 and 13 that the diffeomorphism f is structurally stable.

Necessity. Assume that f is structurally stable. According to the Theorems 9 and 10, f is hyperbolic on the chain-recurrent set $CR(f)$. By the Theorem 11, the spectrum of the dual differential does not contain zero and, due to its closure, the spectrum does not contain some segment $[-\beta,\beta],\ \beta>0$. The spectrum is divided into two parts Σ^+ and Σ^-. Similarly, the chain-recurrent set $CR(P\widehat{D}f)$ of the dual differential also splits into two parts CR^+ and CR^-, as described above. According to Theorem 13, a dual differential should have only trivial bounded trajectories.

Let us show by contradiction that the link $CR^+\to CR^-$ is absent. Suppose that the link $CR^+\to CR^-$ takes place. The connection $CR^+\to CR^-$ generates the trajectory $\omega = \{(x_k, e_k)\}$ of the map $P\widehat{D}f$, which starts at CR^+ and ends at CR^-. The trajectory ω generates a trajectory of the dual differential $\widehat{D}f$ of the form

$$\xi = \{(x_k, v_k),\ x_k = f^k(x_0),\ v_k = \widehat{D}f^k(x_0)v_0,\ v_0 \in e_0,\ k \in \mathbb{Z}\}$$

such that $P\xi = \omega$. The trajectory ω starts at CR^+ and, hence, the negative semitrajectory ξ^- is bounded. The trajectory ω ends at CR^- and, hence, the positive semitrajectoryis ξ^+ is bounded. In this case, ξ is a bounded trajectory, that leads to a contradiction. Hence, there is no connection $CR^+ \to CR^-$. The theorem is proved.

3.5 Verification of Structural Stability

We will show that the conditions of Theorems 15 can be constructively verified by means of the symbolic image of the dual differential. Consider the symbolic image G of the dual differential $P\widehat{D}f : P \to P$ on the projective bundle. Suppose that the spectrum of the dual differential does not contain zero. Then there exists $d_0 > 0$ such that for any covering of the projective bundle with a diameter $d < d_0$ the extended spectrum of the symbolic image does not contain zero. In this case, each class of equivalent recurrent vertices H_k receives an extended spectral interval

$$[a_k, b_k] = [\lambda_{\min}(H_k) - \eta(d), \lambda_{\max}(H_k) + \eta(d)],$$

which is positive for $a_k > 0$ or negative for $b_k < 0$. Denote by H^+ the union of the classes with positive spectral intervals $H^+ = \{\bigcup H_m,\ a_m > 0\}$ and $H^- = \{\bigcup H_m,\ b_m < 0\}$ with negative spectral intervals. The sets H^+ and H^- are naturally called positive and negative classes of recurrent vertices. It is known that the set

$$P = \{\bigcup M(i) :\ i \text{ are recurrent}\} = \{\bigcup M(i) :\ i \in H^+ \cup H^-\}$$

is a closed neighbourhood of the chain-recurrent set. In particular, the set

$$P^+ = \{\bigcup M(i) :\ i \in H^+\}$$

is a neighbourhood of CR^+ and the set

$$P^- = \{\bigcup M(j) :\ j \in H^-\}$$

is a neighbourhood of CR^-.

Definition 13 *We say that on the symbolic image of G there exists a connection $H^+ \to H^-$ if there exists an admissible path from H^+ to H^-.*

Theorem 16 *A diffeomorphism f is structurally stable if and only if there exists $d_0 > 0$ such that for any covering C of the projective bundle P with diameter $d < d_0$ the extended spectrum of symbolic image of the dual differential does not contain zero and there is no link $H^+ \to H^-$.*

Proof. *Sufficiency.* Suppose that there exists $d_0 > 0$ such that for the covering $C(P)$ of the projective bundle of diameter $d < d_0$, the extended spectrum of the symbolic image $G(P\widehat{D}f)$ of the dual differential does not contain zero and there is no link $H^+ \to H^-$. Let us fix the described covering and construct a symbolic image. Then according to Theorems 12, the spectrum of the dual differential does not contain zero and the dual differential is hyperbolic on the chain-recurrent set $CR(f)$. The chain-recurrent set $CR(P\widehat{D}f)$ is divided into two parts $CR^+ + CR^-$, the set

$$P^+ = \{\bigcup M(i) : \ i \in H^+\}$$

is a neighbourhood of CR^+ and the set

$$P^- = \{\bigcup M(j) : \ j \in H^-\}$$

is a neighbourhood of CR^-.

Let us show by contradiction that there is no link $CR^+ \to CR^-$. In fact, if there exists a trajectory $\xi = \{(x_k, e_k)\}$ of the mappings $P\widehat{D}f$, which starts at CR^+ and ends at CR^-, then ξ generates an admissible path $\omega = \{z_k, \ (x_k, e_k) \in M(z_k)\}$ from H^+ to H^- on G. This gives the link $H^+ \to H^-$, which contradicts the assumption of the theorem. So there is no link $CR^+ \to CR^-$. From the Theorems 15 it follows that the diffeomorphism f is structurally stable.

Necessity. Let the diffeomorphism f be structurally stable. Then from the Theorems 15 it follows that the spectrum of the dual differential does not contain zero and there is no connection $CR^+ \to CR^-$. From the Theorem 12 it follows that there exists $d_1 > 0$ such that the extended spectrum of the symbolic image of the dual differential for a covering with a diameter $d < d_1$ does not contain zero. The classes of equivalent recurrent vertices are divided into positive and negative parts H^+ and H^-, so that the set

$$P^+ = \{\bigcup M(i) : \ i \in H^+\}$$

is a neighbourhood of CR^+ and the set

$$P^- = \{\bigcup M(j) : \ j \in H^-\}$$

is a neighbourhood of CR^-. Let us denote

$$\widehat{S} = \{v \in TM_x : \ lim_{n\to+\infty}\widehat{D}f^n(x)v = 0\},$$

$$\widehat{U} = \{v \in TM_x : \ lim_{n\to-\infty}\widehat{D}f^n(x)v = 0\}$$

and let $P\widehat{S}$, $P\widehat{U}$ be the projections of the subspaces $\widehat{S}$, $\widehat{U}$ into the projective bundle. Suppose that a point (x, v) is such that $lim_{n\to+\infty}\widehat{D}f^n(x)v = 0$. The projection of the trajectory $(x_n, v_n) = (f^n(x), \widehat{D}f^n(x)v)$ onto the projective bundle has the form

$$(x_n, e_n) = (f^n(x), \widehat{D}f^n(x)e/|\widehat{D}f^n(x)e|),$$

where $e_0 = v/|v|$. The ω-limit set of this trajectory lies in some component of the chain-recurrent set $CR(P\widehat{D}f) = CR^+ + CR^-$. The equality

$$\lim_{n\to+\infty} \widehat{D}f^n(x)v = 0$$

holds if only the ω-limit set of the trajectory (x_n, e_n) lies in CR^-. The stable manifold of the set CR^- is of the form

$$W^s(CR^-) = \{(x, e) : \ P\widehat{D}f^n(x)e \to CR^-, \ n \to \infty\}.$$

Hence, the stable manifold $W^s(CR^-)$ coincides with the projection $P\widehat{S}$. Similarly $P\widehat{U} = W^u(CR^+)$. From the Theorems 9, 13, 14 it follows that $P\widehat{E}^u$ is an attractor, and $P\widehat{E}^s$ is its dual repellor for the dual differential. Thus, the unstable manifold $W^u(CR^+)$ is an attractor, and the stable manifold $W^s(CR^-)$ is its dual repellor. It follows that there exists $d_2 > 0$ such that every symbolic image G, for a covering with a diameter $d < d_2$, has an attractor L^+ and a dual repellor L^- such that

$$U^+ = \{\bigcup M(i) : i \in L^+\}$$

is a closed neighbourhood of $W^u(CR^+)$ and

$$U^- = \{\bigcup M(i) : i \in L^-\}$$

is a closed neighbourhood of $W^s(CR^-)$. The set U^+ is an absorbing neighbourhood for the attractor $W^u(CR^+)$ and U^- is a repelling neighbourhood for the dual repellor $W^s(CR^-)$, moreover, the trajectories outside $U^- \cup U^+$ go from U^-

to U^+. If we choose $d_0 = \min(d_1,\ d_2)$, $d < d_0$ then there are L^+ is the attractor, and L^- is the dual repellor on the symbolic image described above. In this case (see [20] for details), the transition matrix is reduced by renumbering of the vertices to the form

$$\begin{pmatrix} (L^-) & * & * & * & * \\ . & 0 & * & * & * \\ 0 & . & \ddots & * & * \\ . & 0 & . & 0 & * \\ 0 & . & 0 & . & (L^+) \end{pmatrix},$$

where there are zeros under the diagonal, the diagonal zeros correspond to nonrecurrent (transient) vertices, the diagonal blocks (L^-) and (L^+) correspond to the vertices L^- and L^+, ones may be above the diagonal or in (L^-) and (L^+).

Such transition matrix only admits paths from L^- to L^+. Since $H^+ \subset L^+$ and $H^- \subset L^-$, there is no path from H^+ to H^-. The theorem is proved.

Structural Stability Test.

Structural stability can be verified by the following method:

1) the dual differential $\widehat{D}f : TM \to TM$ is constructed;

2) a covering C of the projective bundle P is chosen and the symbolic image G is constructed for the dual differential;

3) the strong components $\{H_k\}$ of the symbolic image G are determined;

4) a function $a[ij]$ on the edges of the graph G is constructed for the function $\varphi = \ln|\widehat{D}f(x)e|$ on the projective bundle;

5) the extended spectrum of the symbolic image of G is found;

6) if the extended spectrum does not contain zero, then the positive and negative classes of recurrent vertices H^+ and H^- is determined;

7) the absence of a link $H^+ \to H^-$ is checked. If such a link is absent, then the diffeomorphism f structurally stable.

Theorem 16 guarantees that if the diffeomorphism f is structurally stable, then there exists a sufficiently small diameter of the covering for which the described method checks the structural stability.

Computer implementation.

At present, all algorithms for computer implementation of the presented method are known. Let us consider these graph theory algorithms and their applications to dynamical systems. The book [20] and the article [3] describe the computer technique for constructing symbolic images. Please note that these calculations were performed on personal computers and do not require the use of supercomputers. The main technical difficulty is the huge number of coverage cells.

We start from the localization of chain-recurrent set. This process is described in the book [20]. Localization is reduced to calculating classes of equivalent recurrent vertices H_k (strong connected components) of symbolic image. There are many algorithms (see the book [31]) for solving this problem and the Tarjan algorithm (see the paper [32]) is the most popular of them. The article [3] presents the results of localization of chain-recurrent sets for 2- and 3-dimensional dynamical systems with chaotic dynamics.

The next task is to determine the extended spectrum of the symbolic image. The solution is reduced to calculating the largest (smallest) average over periodic simple paths of the classes H_k. These paths are named the extreme cycles. There are many algorithms for finding extreme cycles on a graph (see [5, 10, 6]), and Romanovsky (see [25]) were the first to solve this problem. The article [19] describes the results of a computer test for the hyperbolicity of a nontrivial chain-recurrent set which is homeomorphic to a Cantor set. The article calculates the extended Morse spectrum that does not contain zero.

The last task is to check the absence of the path $H^+ \to H^-$. The solution is based on topological sorting of the symbolic image (see the book [31]). The topological sorting determines the renumbering of vertices so that the transition matrix of the symbolic image G takes the form

$$\Pi = \begin{pmatrix} (\Pi_1) & \cdots & \cdots & \cdots & \cdots \\ & \ddots & & & \\ 0 & & (\Pi_k) & \cdots & \cdots \\ & \ddots & & \ddots & \\ 0 & & 0 & & (\Pi_s) \end{pmatrix},$$

where each diagonal block Π_k either corresponds to the equivalence class of recurrent vertices H_k, or corresponds to some nonrecurrent vertex (block consists of one zero), under the diagonal blocks are only zeros. With this numbering of vertices, only paths from the upper blocks (classes) to the lower blocks (classes) are possible, which allows us to check the absence of path $H^+ \to H^-$.

According to the Theorem 16, for a structurally stable diffeomorphism f there exists $d_0 > 0$ such that the described method checks the structural stability for any covering with cell diameter $d < d_0$. If C is a covering with arbitrary diameter, then a finite number of subdivisions gives the covering with cell diameter $d < d_0$. Thus, the described method implements a constructive verification of the structural stability in a finite number of steps.

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