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Ordinary differential equations

On the stability and boundedness of solutions to certain second order differential equation

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Abstract. In this paper, we investigate by means of second method of Lyapunov, sufficient conditions that guarantee uniform-asymptotic stability of the trivial solution and ultimate boundedness of all solutions to a certain second order differential equation. We construct a complete Lyapunov function in order to discuss the qualitative properties mentioned earlier. The boundedness result in this paper is new and also complement some boundedness results in literature obtained by using an incomplete Lyapunov function together with a signum function. Finally, we demonstrate the correctness of our results with two numerical examples and graphical representation of the trajectories of solutions to the examples using Maple software.

Keywords: Second order, Stability, Boundedness, Lyapunov function.

1 Introduction

Our concern in this paper is to investigate sufficient conditions for the stability and boundedness of solutions to the second order differential equation of the form:

$$\dot{X} = H(Y),$$
$$\dot{Y} = -F(X,Y)Y - G(X) + P(t,X,Y), \tag{1}$$

where $X, Y : \mathbb{R}^+ \to \mathbb{R}^n$, $G, H : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, F is an $n \times n$ continuous symmetric positive definite matrix function depending on the arguments displayed explicitly, $\mathbb{R} = (-\infty, \infty)$ represents the real line, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ represent the real n-dimensional Euclidean space equipped with the usual norm $\|\cdot\|$ and the dots which appear in (1) indicate differentiation with respect to the independent variable t. The conditions for the existence and uniqueness of solutions of (1) with any predetermined initial conditions will be assumed (see, Rao [16]). We shall denote the scalar product $\langle X, Y \rangle$ of any vectors X, Y in $\mathbb{R}^n$ with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. Particularly, $\langle X, X \rangle = \|X\|^2$.

A. M. Lyapunov (1892) in his Ph.D. Thesis titled " The general problems of the stability of motion", studied sufficient conditions for the stability of solutions of non-linear system of differential equations by two distinct methods: the first(or indirect) and second(or direct) methods of Lyapunov (Lyapunov [12]). Since these two methods were introduced, the second method has gained wider acceptance and has been employed in all the listed papers in our reference to study qualitative behaviour of solutions of differential equations(see, [1]-[23]). This method (i.e. the second method) enables us to determine the qualitative properties of solutions of differential equations without necessarily finding the solutions themselves.

However, to use the second method of Lyapunov, one needs to construct a positive definite function, whose derivative with respect to variable t along the solution path of the equation being considered is negative semi-definite. This positive definite function is called Lyapunov function. Unfortunately, the method is not without any difficulty or challenge. One major challenge of the method has to do with, how to construct a suitable Lyapunov function especially for non-linear differential equations.

In [19], Tejumola gave necessary conditions for the boundedness of solutions to the following scalar differential equations:

$$\begin{aligned} \dot{x} &= h(y), \\ \dot{y} &= -f(x,y)y - g(x) + p(t,x,y). \end{aligned} \tag{2}$$

We note that system (2) is the scalar form of system (1). Later Omeike et. al. [14] with the aid of an incomplete Lyapunov function supplemented with signum function, extended the boundedness results obtained by Tejumola[19] to the corresponding n-dimensional system of equations given in (1).

Very recent, Adeyanju([2],[3]) examined some conditions for the stability of the zero solution, boundedness and uniform-ultimate boundedness of solutions of the second order non-linear differential equation:

$$\ddot{X} + F(X,\dot{X})\dot{X} + G(X) = P(t,X,\dot{X}), \tag{3}$$

where $X : \mathbb{R}^+ \to \mathbb{R}^n,\ G : \mathbb{R}^n \to \mathbb{R}^n,\ P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and F is an $n \times n$ continuous symmetric positive definite matrix function that depends on the arguments displayed explicitly. The boundedness property of solutions to (3) has also been studied by Omeike et. al.[15].

Motivated by the works of Tejumola [19] and Omeike et.al. [14], we will with the aid of a complete Lyapunov function, examine conditions that ensure the boundedness of solutions of (1) with less restricted assumptions. Furthermore, we will also prove a theorem on the stability of the trivial solution to the system (1) which was not discussed in [14] or elsewhere to the best of our knowledge in literature.

2 Preliminary Results

The following established algebraic results are required to prove our main results.

lemma 1 ([10],[20]). *Let A be a real $n \times n$ symmetric positive definite matrix. Then for $X \in \mathbb{R}^n$*

$$\delta_a \|X\|^2 \leq \langle AX, X \rangle \leq \Delta_a \|X\|^2,$$

where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.

lemma 2 ([9],[20]). *Let $G(0) = 0 = H(0)$ and assume that the matrices A, $J_g(X)$ and $J_h(Y)$ are symmetric and positive definite such that A commutes pairwise with each of $J_g(X)$ and $J_h(Y)$ for all $X, Y \in \mathbb{R}^n$. Then,*

$$\langle G(X), AX\rangle = \int_0^1 X^T A J_g(\sigma X) X d\sigma,$$
$$\langle H(Y), AY\rangle = \int_0^1 Y^T A J_h(\sigma Y) Y d\sigma,$$

where $J_g(X)$ and $J_h(Y)$ are respectively the Jacobian matrices $\frac{\partial g_i}{\partial x_j}$ and $\frac{\partial h_i}{\partial y_j}$ of $G(X)$ and $H(Y)$ $(j = 1, 2, ..., n)$.

lemma 3 [9]. *Let $G(0) = 0$ and assume that $J_g(X)$ is symmetric for all arbitrary $X \in \mathbb{R}^n$. Then*

$$\frac{d}{dt}\int_0^1 \langle G(\sigma X), X\rangle d\sigma = \langle G(X), \dot{X}\rangle.$$

Lemma 4 [11]. *Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$ and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\varphi \in C$, $u(|\phi(0)|) \le V(\varphi)$, $V(\varphi) \ge 0, \dot{V}(\varphi) \le 0$, then the zero solution of $\dot{x} = f(x_t)(t \ge 0)$, is stable.*

If we define $Z = \{\varphi \in C_H : \dot{V}(\varphi) = 0\}$, then the zero solution of $\dot{x} = f(x_t)(t \ge 0)$, is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Main Results

First, we introduce some basic assumptions to be used in proving our results.

Basic Assumptions:

Suppose that Δ_f, Δ_g, Δ_h, δ_f, δ_g, δ_h are some positive constants and let all the basic assumptions imposed on functions $F(X, Y)$, $G(X)$, and $H(Y)$ hold such that $G(0) = 0$ and $H(0) = 0$. In addition, let the following assumptions hold for all $X, Y \in \mathbb{R}^n$:

(i) the matrices $J_g(X)$, $J_h(Y)$ and $F(X,Y)$ are symmetric and positive definite such that their eigenvalues satisfy:

$$\delta_g \leq \lambda_i(J_g(X)) \leq \Delta_g,$$
$$\delta_h \leq \lambda_i(J_h(Y)) \leq \Delta_h,$$
$$\delta_f \leq \lambda_i(F(X,Y)) \leq \Delta_f,$$

$(i = 1, 2, ..., n)$,

(ii) there exists a function $a(t) \in C^1[0,\infty)$ such that $0 < a_0 < a(t) < a_1$, $-\beta \leq a'(t) \leq 0$, where

$$\beta > \max\left\{a_1\Delta_h, \frac{a_1\Delta_h}{\delta_h\delta_f}\right\}.$$

Now to the main results of the paper.

Theorem 1 *Suppose all the assumptions listed under the basic assumptions hold. Then the trivial solution of* (1) *is uniformly stable and uniformly-asymptotically stable when* $P(t,X,Y) \equiv 0$.

Proof

To prove this theorem, we depend on the continuously-differentiable scalar function $V = V(t,X,Y)$ define as

$$V = a(t)\langle X, X\rangle + \beta\int_0^1 \langle G(\sigma X), X\rangle d\sigma + \beta\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma, \tag{4}$$

where $a(t)$ and β are as defined under the basic assumptions.

Obviously, the Lyapunov function defined in (4) becomes zero when $X = 0$ and $Y = 0$. On applying Lemma 1, Lemma 2 and assumptions listed in (i) of basic assumptions to (4), we have for some positive constants D_0 and D_1

$$
\begin{aligned}
V =& a(t)\langle X, X\rangle + \beta\int_0^1 \langle G(\sigma X), X\rangle d\sigma + \beta\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma\\
=& a(t)\langle X, X\rangle + \beta\int_0^1\int_0^1 \langle J_G(\sigma_1\sigma_2 X)X, X\rangle \sigma_1 d\sigma_1 d\sigma_2\\
&+\beta\int_0^1\int_0^1 \langle J_H(\sigma_1\sigma_2 Y)Y, Y\rangle \sigma_1 d\sigma_1 d\sigma_2\\
\geq& a_0 \parallel X \parallel^2 + \frac{1}{2}\beta\delta_g \parallel X \parallel^2 + \frac{1}{2}\beta\delta_h \parallel Y \parallel^2\\
\geq& D_0(\|X\|^2 + \|Y\|^2),
\end{aligned}
$$

where, $D_0 = \frac{1}{2}\min\{2a_0 + \beta\delta_g, \beta\delta_h\}$.

Similarly, we have

$$
\begin{aligned}
V =& a(t)\langle X, X\rangle + \beta\int_0^1\int_0^1 \langle J_G(\sigma_1\sigma_2 X)X, X\rangle \sigma_1 d\sigma_1 d\sigma_2\\
&+\beta\int_0^1\int_0^1 \langle J_H(\sigma_1\sigma_2 Y)Y, Y\rangle \sigma_1 d\sigma_1 d\sigma_2\\
\leq& a_1 \parallel X \parallel^2 + \frac{1}{2}\beta\Delta_g \parallel X \parallel^2 + \frac{1}{2}\beta\Delta_h \parallel Y \parallel^2\\
\leq& D_1(\|X\|^2 + \|Y\|^2),
\end{aligned}
$$

where, $D_1 = \frac{1}{2}\min\{2a_1 + \beta\Delta_g, \beta\Delta_h\}$.

Hence,

$$D_0(\|X\|^2 + \|Y\|^2) \leq V \leq D_1(\|X\|^2 + \|Y\|^2). \tag{5}$$

Thus, the function V defined by (4) satisfies

$$V(X, Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{6}$$

Inequalities (5) and (6) show that the scalar function V is non-negative for all $X, Y \in \mathbb{R}^n$ and $V = 0$ if and only if $X = 0$ and $Y = 0$.

Next, we show that the derivative $\dot{V}$ of the function V in (4) exists, and that there are finite constants D_2, D_3 such that

$$\dot{V} \leq -D_2 < 0 \text{ if } \|X\|^2 + \|Y\|^2 \geq D_3^2. \tag{7}$$

Using Lemma 3, we obtain the time derivative of the scalar function defined in (4) along the system (1) as

$$\dot{V} = a'(t)\langle X, X\rangle + 2a(t)\langle X, H(Y)\rangle + \beta\langle H(Y), -F(X,Y)Y\rangle. \tag{8}$$

By Schwartz's inequality and (i) of Theorem 1, we have

$$2|\langle X, H(Y)\rangle| \leq \Delta_h\{\| X \|^2 + \| Y \|^2\}.$$

Again, we apply Lemma 1, Lemma 2 and the above inequality in (8) to obtain

$$\dot{V} \leq -(\beta - a_1\Delta_h) \| X \|^2 -(\beta\delta_h\delta_f - a_1\Delta_h) \| Y \|^2 . \tag{9}$$

From the definition of β, it is possible to get a positive constant D_4, such that

$$\dot{V} \leq -D_4\{\| X \|^2 + \| Y \|^2\} < 0, \text{whenever } \| X \|^2 + \| Y \|^2 > D_3^2, \tag{10}$$

where $D_4 = \min\{\beta - a_1\Delta_h, \beta\delta_h\delta_f - a_1\Delta_h\}$.
The two inequalities (5) and (10) guarantee uniform stability of the trivial solution of (1).

To prove the asymptotic stability of the trivial solution, we consider a set defines by

$$S \equiv \{(X,Y) : \dot{V}(X,Y) = 0\}.$$

Applying the famous LaSalle's invariance principle to this set, we note that $(X,Y) \in S$ implies that $X = Y = 0$. This shows that the largest invariant set contained in S is $(0,0)$. Hence, we conclude by Lemma 4 that the trivial solution of (1) is uniformly-asymptotically stable when $P(t,X,Y) \equiv 0$. This completes the proof of the theorem.

Theorem 2 *Further to the assumptions of Theorem 1, let $P(t,X,Y)$ in (1) satisfies*

(iii)

$$\| P(t,X,Y) \| \leq \gamma \| Y \|,$$

uniformly in $t \geq 0$ and $0 < \gamma < \frac{\delta_h\delta_f}{\Delta_h}$. Then there exists a finite constant K whose magnitude depends only on the constants Δ_f, Δ_g, Δ_h, δ_f, δ_g, δ_h such that every solution $(X(t), Y(t))$ of (1) ultimately satisfies

$$\|X(t)\| \leq K, \quad \|Y(t)\| \leq K. \tag{11}$$

proof
The prove of this theorem still rests on the Lyapunov function defined in (4). In the proof of Theorem 1, we have demonstrated that the Lyaounov function satisfies (5) which is also correct when $P(t,X,Y) \neq 0$. However, the derivative of V when $P(t,X,Y) \neq 0$ is

$$\dot{V} = a'(t)\langle X, X\rangle + a(t)\langle X, H(Y)\rangle + \beta\langle H(Y), -F(X,Y)Y\rangle + \beta\langle H(Y), P(t,X,Y)\rangle.$$

From the proof of Theorem 1 and (iii) of Theorem 2, we have

$$\begin{aligned}\dot{V} &\leq -(\beta - a_1\Delta_h) \parallel X \parallel^2 - (\beta\delta_h\delta_f - a_1\Delta_h) \parallel Y \parallel^2 + \beta\Delta_h\gamma \parallel Y \parallel^2 \\ &\leq -(\beta - a_1\Delta_h) \parallel X \parallel^2 - \big(\beta(\delta_h\delta_f - \Delta_h\gamma) - a_1\Delta_h\big) \parallel Y \parallel^2 \\ &\leq -D_5\{\parallel X \parallel^2 + \parallel Y \parallel^2\}, \end{aligned} \tag{12}$$

where $D_5 = \min\{\beta - a_1\Delta_h, \beta(\delta_h\delta_f - \Delta_h\gamma) - a_1\Delta_h\}$. Then, there exists a positive constant D_6 such that

$$\dot{V} \leq -1 \text{ provided } \parallel X \parallel^2 + \parallel Y \parallel^2 \geq D_6. \tag{13}$$

By following the Yoshizawa-type technique used in [[13], [23]], it can be shown from inequalities (5) and (12) that we can always find a positive constant D_7 such that every solution $(X(t), Y(t))$ of (1) satisfies

$$\parallel X \parallel^2 + \parallel Y \parallel^2 \leq D_7. \tag{14}$$

On setting $K = \sqrt{D_7}$ in (14), we obtain (11) with little simplification. This completes the proof of the theorem.

4 Examples

The following examples are given as special cases of (1) when $n = 2$.

Example 1
Our first example is when $P(t,X,Y) \equiv 0$. Suppose in (1) we have

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = H(Y) = \begin{bmatrix} 3y_1 + \cos y_1 - 1 \\ 3y_2 + \cos y_2 - 1 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; G(X) = \begin{bmatrix} 5x_1 + \sin x_1 \\ 5x_2 + \sin x_2 \end{bmatrix},$$

$$F(X,Y) = \begin{bmatrix} 2 + \frac{1}{x_1^2\sin^2 y_1 + 1} & 1 \\ 1 & 2 + \frac{1}{x_2^2 \sin^2 y_2 + 1} \end{bmatrix} \text{ and } a(t) = 1 + \exp^{-8.5t}.$$

We obtain the following Jacobian matrices $J_h(Y)$ and $J_g(X)$ of $H(Y)$ and $G(X)$ respectively as,

$$J_h(Y) = \begin{bmatrix} 3-\sin y_1 & 0 \\ 0 & 3-\sin y_2 \end{bmatrix}, \ J_g(X) = \begin{bmatrix} 5+\cos x_1 & 0 \\ 0 & 5+\cos x_2 \end{bmatrix}.$$

It is evident from the above that matrices $F(X,Y)$, $J_h(Y)$ and $J_g(X)$ are symmetric and positive definite with the following as eigenvalues.

$$\lambda_1(F(X,Y)) = 3 + \frac{1}{x_1^2\sin^2 y_1 + 1}, \quad \lambda_2(F(X,Y)) = 1 + \frac{1}{x_2^2\sin^2 y_2 + 1},$$
$$\lambda_1(J_h(Y)) = 3-\sin y_1, \quad \lambda_2(J_h(Y)) = 3-\sin y_2,$$
$$\lambda_1(J_g(X)) = 5+\cos x_1, \quad \lambda_2(J_g(X)) = 5+\cos x_2.$$

Thus, with some elementary calculations, we obtain for $i = 1, 2$,

$$\delta_f = 1 \le \lambda_i(F(X,Y)) \le 4 = \Delta_f$$
$$\delta_h = 2 \le \lambda_i(J_h(Y)) \le 4 = \Delta_h,$$
$$\delta_g = 4 \le \lambda_i(J_g(X)) \le 6 = \Delta_g.$$

Also, $0 < a_0 = 1 < a(t) = 1 + \exp^{-8.5t} < a_1 = 2$ and $-8.5 \le a'(t) = -8.5\exp^{-8.5t} \le 0$. Thus,

$$\beta = 8.5 > \max\left\{a_1\Delta_h, \frac{a_1\Delta_h}{\delta_h\delta_f}\right\} = \max\{8,4\} = 8.$$

Hence, all the assumptions of Theorem 1 hold and, the trivial solution of this example is uniform-asymptotically stable.

The next example is when $P(t,X,Y) \neq 0$.

Example 2

Suppose in addition to Example 1, we have

$$P(t,X,Y) = \frac{1}{4+\sin t}\begin{bmatrix} \frac{|y_1|-1}{1+x_1^2} \\ \frac{|y_2|-1}{1+x_2^2} \end{bmatrix}.$$

Hence,

$$\| P(t,X,Y) \| = |\frac{1}{4+\sin t}| \ \| \begin{bmatrix} \frac{|y_1|-1}{1+x_1^2} \\ \frac{|y_2|-1}{1+x_2^2} \end{bmatrix} \|$$
$$\le \frac{1}{\sqrt{5}} \| Y \| .$$

Therefore,

$$0 < \gamma = \frac{1}{\sqrt{5}} < \frac{\delta_h \delta_f}{\Delta_h} = \frac{1}{2}.$$

Again, all the assumptions of Theorem 2 hold. Hence, solutions of equation in Example 2 are uniform-ultimately bounded.

Graphically, with the aid of Maple software, we illustrate the stability and boundedness of solutions to the Example 2 as shown below.

5 Conclusion

By constructing a new and complete Lyapunov function, we have been able to prove some results on the uniform-asymptotic stability of the trivial solution and uniform-ultimate boundedness of all solutions to certain systems of second order differential equations studied in this paper. With these new results, we have improved on some of the existing results in the literature.

Statements and Declarations
There is no competing interests as regard this paper.

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