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A new proof of the expansion of iterated Itô stochastic integrals with respect to the components of a multidimensional Wiener process based on generalized multiple Fourier series and Hermite polynomials

Dmitriy F. Kuznetsov

Peter the Great Saint-Petersburg Polytechnic University
e-mail: sde_kuznetsov@inbox.ru

Abstract. The article is devoted to a new proof of the expansion for iterated Itô stochastic integrals with respect to the components of a multidimensional Wiener process. The above expansion is based on Hermite polynomials and generalized multiple Fourier series in arbitrary complete orthonormal systems of functions in a Hilbert space. In 2006, the author obtained a similar expansion, but with a lesser degree of generality, namely, for the case of continuous or piecewise continuous complete orthonornal systems of functions in a Hilbert space. In this article, the author generalizes the expansion of iterated Itô stochastic integrals obtained by him in 2006 to the case of an arbitrary complete orthonormal systems of functions in a Hilbert space using a new approach based on the Itô formula. The obtained expansion of iterated Itô stochastic integrals is useful for constructing of high-order strong numerical methods for systems of Itô stochastic differential equations with multidimensional non-commutative noise.

Key words: iterated Itô stochastic integral, multiple Wiener stochastic integral, Itô stochastic differential equation, generalized multiple Fourier series, multidimensional Wiener process, Hermite polynomial, mean-square convergence, expansion.

Contents

1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_\tau, \tau \in [0,\bar{T}]\}$ be a non-decreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_\tau$ be a standard m-dimensional Wiener stochastic process, which is F_τ-measurable for any $\tau \in [0,\bar{T}]$. We assume that the components $\mathbf{w}_\tau^{(i)}$ $(i = 1,\ldots,m)$ of this process are independent. Consider an Itô stochastic differential equation (SDE) in the integral form

$$\mathbf{x}_s = \mathbf{x}_0 + \int\limits_0^s \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \sum_{j=1}^m \int\limits_0^s B_j(\mathbf{x}_\tau, \tau) d\mathbf{w}_\tau^{(j)}, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \quad (1)$$

Here $\mathbf{x}_s$ is n-dimensional stochastic process satisfying the equation (1). The nonrandom functions $\mathbf{a}(\mathbf{x}, \tau)$, $B_j(\mathbf{x}, \tau):\ \mathbf{R}^n \times [0, \bar{T}] \to \mathbf{R}^n$ $(j = 1, \ldots, m)$ guarantee the existence and uniqueness up to stochastic equivalence of the strong solution of equation (1) [1]. The second integral on the right-hand side of (1) is the Itô stochastic integral. Let $\mathbf{x}_0$ be an n-dimensional random variable, which is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$ (M denotes a mathematical expectation). We assume that $\mathbf{x}_0$ and $\mathbf{w}_\tau - \mathbf{w}_0$ are independent when $\tau > 0$. In addition to the above conditions, we will assume that the functions $\mathbf{a}(\mathbf{x}, \tau)$, $B_j(\mathbf{x}, \tau)$ $(j = 1, \ldots, m)$ are sufficiently smooth functions in both arguments.

It is well known [2]-[5] that Itô SDEs are adequate mathematical models of dynamic systems of various physical nature under the influence of random disturbances. One of the effective approaches to the numerical integration of Itô SDEs is an approach based on the Taylor–Itô and Taylor–Stratonovich expansions [2]-[10]. The most important feature of such expansions is a presence in them of the so-called iterated Itô and Stratonovich stochastic integrals, which play the key role for solving the problem of numerical integration of Itô SDEs and have the following form

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

$$J^*[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_t^{*T} \psi_k(t_k) \ldots \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{3}$$

where $\psi_1(\tau), \ldots, \psi_k(\tau)$ are nonrandom functions on $[t, T]$, $\mathbf{w}_\tau^{(i)}$ $(i = 1, \ldots, m)$ are independent standard Wiener processes and $\mathbf{w}_\tau^{(0)} = \tau$, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\int \ \text{and} \ \int\limits^{*}$$

denote Itô and Stratonovich stochastic integrals, respectively.

Generalization of the method of expansion of iterated Itô stochastic integrals (2) based on generalized multiple Fourier series (see Theorem 5.1 ([6], p. 236) and Sect. 5.1 ([6], pp. 235-245)) composes the subject of the article.

Note that another approaches to the expansion and mean-square approximation of iterated Itô and Stratonovich stochastic integrals (2) and (3) can be found in [2]-[5], [21]-[38].

Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Define the following function (the so-called factorized Volterra-type kernel) on the hypercube $[t,T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

where $t_1, \ldots, t_k \in [t,T]$ $(k \geq 2)$ and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t,T]$.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$.

It is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t,T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t,T]^k$ in the mean-square sense, i.e.

$$\lim_{p_1, \ldots, p_k \to \infty} \left\| K - K_{p_1 \ldots p_k} \right\|_{L_2([t,T]^k)} = 0, \tag{5}$$

where

$$K_{p_1 \ldots p_k}(t_1, \ldots, t_k) = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l), \tag{6}$$

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k \tag{7}$$

is the Fourier coefficient, and

$$\|f\|_{L_2([t,T]^k)} = \left(\int\limits_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.$$

Consider the partition $\{\tau_j\}_{j=0}^{N}$ of $[t,T]$ such that

$$t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta\tau_j \to 0 \ \text{ if } \ N \to \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j. \tag{8}$$

Theorem 1 [6] (2006) (also see [7]-[20]). *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous nonrandom functions on* $[t,T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of continuous or piecewise continuous functions in the space* $L_2([t,T])$. *Then*

$$
J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-
$$

$$
-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{9}
$$

where

$$
\mathrm{G}_k=\mathrm{H}_k\backslash\mathrm{L}_k,\quad \mathrm{H}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\Bigr\},
$$

$$
\mathrm{L}_k=\Bigl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;\ l_g\ne l_r\ (g\ne r);\ g,r=1,\ldots,k\Bigr\},
$$

l.i.m. *is a limit in the mean-square sense*, $i_1,\ldots,i_k=0,1,\ldots,m$,

$$
\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)} \tag{10}
$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\ne 0$*),* $C_{j_k\ldots j_1}$ *is the Fourier coefficient* (7), $\Delta\mathbf{w}_{\tau_j}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)}$ $(i=0,1,\ldots,m)$, $\{\tau_j\}_{j=0}^{N}$ *is a partition of* $[t,T]$, *which satisfies the condition* (8).

A number of generalizations and modifications of Theorem 1 can be found in [10], Chapter 1 (see also bibliography therein).

Let us consider corollaries from Theorem 1 (see (9)) for $k=1,\ldots,5$ [6]

$$
J[\psi^{(1)}]_{T,t}^{(i_1)}=\underset{p_1\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}C_{j_1}\zeta_{j_1}^{(i_1)}, \tag{11}
$$

$$
J[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\Biggr), \tag{12}
$$

$$J[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \quad (13)$$

$$J[\psi^{(4)}]_{T,t}^{(i_1\ldots i_4)} = \underset{p_1,\ldots,p_4\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_4=0}^{p_4} C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \quad (14)$$

$$J[\psi^{(5)}]_{T,t}^{(i_1\ldots i_5)} = \underset{p_1,\ldots,p_5\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_5=0}^{p_5} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)} - \mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)} + \mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)} + \mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$\left.+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\right), \qquad (15)$$

where $\mathbf{1}_A$ is the indicator of the set A.

Consider a generalization of the formulas (11)–(15) for the case of arbitrary multiplicity k ($k \in \mathbf{N}$) of the iterated Itô stochastic integral (2).

In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \qquad (16)$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (16) is a partition and consider the sum with respect to all possible partitions

$$\sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}}, \qquad (17)$$

where $a_{g_1g_2,\ldots,g_{2r-1}g_{2r},q_1\ldots q_{k-2r}} \in \mathbf{R}$.

Below there are several examples of sums in the form (17)

$$\sum_{\substack{(\{g_1,g_2\})\\ \{g_1,g_2\}=\{1,2\}}} a_{g_1g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2,g_3g_4} = a_{12,34} + a_{13,24} + a_{23,14},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} =$$

$$= a_{12,345} + a_{13,245} + a_{14,235} + a_{15,234} + a_{23,145} + a_{24,135} +$$
$$+ a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} =$$

$$= a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{12,35,4} + a_{13,25,4} + a_{15,23,4} + a_{12,54,3} + a_{15,24,3} +$$
$$+ a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can formulate Theorem 1 (see (9)) in another form.

Theorem 2 [8] (2009) (also see [9]–[15]). *Suppose that* $\psi_1(\tau), \ldots, \psi_k(\tau)$ *are continuous nonrandom functions on* $[t, T]$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of continuous or piecewise continuous functions in the space* $L_2([t, T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \quad (18)$$

that converges in the mean-square sense is valid, where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ *is an integer part of a real number* x, $\prod\limits_{\emptyset} \stackrel{\text{def}}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\text{def}}{=} 0$; *another notations are the same as in Theorem* 1.

Further in this article, we will consider a generalization of the expansion (18) to the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Moreover, we will consider a modification of (18) based on the Hermite polynomials.

It should be noted that there is a work [39] in which an expansion similar to (89) was obtained (see Sect. 4 for details). A comparison of our results with the results from [39] and with other publications will be given in Sect. 4.

2 Preliminary Results

2.1 Expansion of Iterated Itô Stochastic Integrals based on Generalized Multiple Fourier Series

Suppose that $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$, $i_1,\ldots,i_k=0,1,\ldots,m$, $d\mathbf{w}_\tau^{(0)}\stackrel{\sf def}{=}d\tau$.

Let us introduce the following notation for the sum of iterated Itô stochastic integrals

$$J''[\Phi]_{T,t}^{(i_1\ldots i_k)}\stackrel{\sf def}{=}\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{19}$$

where all permutations $(t_1,\ldots,t_k)$ when summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots\, d\mathbf{w}_{t_k}^{(i_k)}$. At the same time the indices near upper limits of integration in the iterated stochastic integrals are changed correspondently and if t_r swapped with t_q in the permutation $(t_1,\ldots,t_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$. In addition,

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$$

is the iterated Itô stochastic integral.

Let us give an exumple of the sum (19) for $k=3$

$$J''[\Phi]_{T,t}^{(i_1i_2i_3)}\stackrel{\sf def}{=}\sum_{(t_1,t_2,t_3)}\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_2}\int\limits_t^{t_3}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_1}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}+\int\limits_t^T\int\limits_t^{t_1}\int\limits_t^{t_3}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_1}^{(i_1)}+$$

$$+\int\limits_t^T\int\limits_t^{t_2}\int\limits_t^{t_1}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}\int\limits_t^{t_2}\Phi(t_1,t_2,t_3)d\mathbf{w}_{t_3}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}.$$

Theorem 3 [10], [15]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$$

converging in the mean-square sense is valid, where $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is the iterated Itô stochastic integral* (2), $J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (19) *or has the form*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. *At the same time if* j_r *swapped with* j_q *in the permutation* $(j_1,\ldots,j_k)$, *then* i_r *swapped with* i_q *in the permutation* $(i_1,\ldots,i_k)$. *Another notations are the same as in Theorems* 1 *and* 2.

Proof. Using (19), we have

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \int\limits_t^T \psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)} = J''[K]_{T,t}^{(i_1\ldots i_k)} \quad \text{w. p. 1}, \tag{20}$$

where $K = K(t_1,\ldots,t_k)$ is defined by (4).

Applying the linearity property of the Itô stochastic integral and (20), we obtain w. p. 1

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = J''[K]_{T,t}^{(i_1\ldots i_k)} = J''[K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} + J''[K-K_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} + J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{21}$$

where

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) = K(t_1,\ldots,t_k) - K_{p_1\ldots p_k}(t_1,\ldots,t_k),$$

$K(t_1,\ldots,t_k)$ and $K_{p_1\ldots p_k}(t_1,\ldots,t_k)$ are defined by (4) and (6), respectively; the Fourier coefficient $C_{j_k\ldots j_1}$ has the form (7).

Note that (see (19))

$$J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} =$$

$$= \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where notations are the same as in (19).

According to the standard moment properties of the Itô stochastic integral [1] and the properties of the Lebesgue integral, we get the following estimate

$$\mathsf{M}\left\{ \left(J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)} \right)^2 \right\} \le$$

$$\le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 \times$$

$$\times dt_1 \ldots dt_k = \tag{22}$$

$$= C_k \int\limits_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right)^2 dt_1 \ldots dt_k =$$

$$= C_k \left\| K - K_{p_1\ldots p_k} \right\|_{L_2([t,T]^k)}^2, \tag{23}$$

where constant C_k depends only on the multiplicity k of the iterated Itô stochastic integral $J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$, and permutations $(t_1,\ldots,t_k)$ when summing in (22) are performed in the expression $dt_1 \ldots dt_k$. At the same time the indices near upper limits of integration in the iterated integrals from (22) are changed correspondently.

Combining (5) and (23), we get

$$\lim_{p_1,\ldots,p_k\to\infty} \mathsf{M}\left\{\left(J''[R_{p_1\ldots p_k}]_{T,t}^{(i_1\ldots i_k)}\right)^2\right\} = 0. \tag{24}$$

From (21) and (24) we obtain the following expansion for the iterated Itô stochastic integral (2)

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}, \tag{25}$$

where $J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (19).

It is easy to see that $J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ can be written in the form

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{26}$$

where

$$\sum_{(j_1,\ldots,j_k)}$$

means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$.

The relations (25) and (26) complete the proof of Theorem 3. Theorem 3 is proved.

2.2 Modification and Generalization of Itô's Theorem. Proof on the Base of the Itô Formula and Without Explicit Use of the Multiple Wiener Stochastic Integral

In this section, we generalize Theorem 3.1 from [40] (1951) which gives the relaionship between the multiple Wiener stochastic integral and the Hermite polynomials. Recall that in [40] the case $i_1 = \ldots = i_k \neq 0$ (the case of a scalar standard Wiener process) has been considered. In the main result of this section, we will consider the case $i_1,\ldots,i_k = 0,1,\ldots,m$ (the case of a multidimensional Wiener process). Moreover, our proof diffes from that given in [40] and is based on the Itô formula. Also, we do not explicitly use the multiple Wiener stochastic integral in the proof of Theorem 4. Although it should be noted that the sum (19), which plays a central role in the proof of Theorem 4, is equal w. p. 1 to the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process (see the proof in [10], Sect. 1.11 for details).

Let us introduce some notations.

We will say that the condition $(\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ $(i_1,\ldots,i_k = 0,1,\ldots,m)$ *if* $m_1,\ldots,m_k$ *are multiplicities of the elements* $i_1,\ldots,i_k$*, respectively, i.e.*

$$\{i_1,\ldots,i_k\} = \{\overbrace{i_1,\ldots,i_1}^{m_1},\overbrace{i_2,\ldots,i_2}^{m_2},\ldots,\overbrace{i_r,\ldots,i_r}^{m_r}\},$$

where $r = 1,\ldots,k$*, braces mean an unordered set, and parentheses mean an ordered set. At that,* $m_1+\ldots+m_k = k$*,* $m_1,\ldots,m_k = 0,1,\ldots,k$*, and all elements with nonzero multiplicities are pairwise different.*

It is not difficult to see that

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = J''\Biggl[\underbrace{\phi_{j_{g_1}}\ldots\phi_{j_{g_{m_1}}}}_{m_1}\underbrace{\phi_{j_{g_{m_1+1}}}\ldots\phi_{j_{g_{m_1+m_2}}}}_{m_2}\ldots$$

$$\ldots\underbrace{\phi_{j_{g_{m_1+m_2+\ldots+m_{k-1}+1}}}\ldots\phi_{j_{g_{m_1+m_2+\ldots+m_k}}}}_{m_k}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_1}^{m_1}\overbrace{i_2\ldots i_2}^{m_2}\ldots\overbrace{i_k\ldots i_k}^{m_k})}$$

w. p. 1, where we suppose that the condition $(\star)$ is fulfilled for the multi-index $(i_1\ldots i_k)$ and $\{j_{g_1},\ldots,j_{g_{m_1+m_2+\ldots+m_k}}\} = \{j_{g_1},\ldots,j_{g_k}\} = \{j_1,\ldots,j_k\}$.

Suppose that

$$\left\{j_{g_{m_1+m_2+\ldots+m_{l-1}+1}},\ldots,j_{g_{m_1+m_2+\ldots+m_l}}\right\} =$$

$$= \Biggl\{\underbrace{j_{h_{1,l}},\ldots,j_{h_{1,l}}}_{n_{1,l}},\underbrace{j_{h_{2,l}},\ldots,j_{h_{2,l}}}_{n_{2,l}},\ldots,\underbrace{j_{h_{d_l,l}},\ldots,j_{h_{d_l,l}}}_{n_{d_l,l}}\Biggr\}, \qquad (27)$$

where $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;\ n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;\ d_l=1,\ldots,m_l;$ $l=1,\ldots,k$. Note that the numbers $m_1,\ldots,m_k,\ g_1,\ldots,g_k$ depend on $(i_1,\ldots,i_k)$ and the numbers $n_{1,l},\ldots,n_{d_l,l},\ h_{1,l},\ldots,h_{d_l,l},\ d_l$ depend on $\{j_1,\ldots,j_k\}$. Moreover, $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$.

Let $H_n(x)$ be the Hermite polynomial of degree n

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

or

$$H_n(x)=n!\sum_{m=0}^{[n/2]}\frac{(-1)^m x^{n-2m}}{m!(n-2m)!2^m}\quad (n\in\mathbf{N}). \tag{28}$$

For example,

$$H_0(x)=1,$$
$$H_1(x)=x,$$
$$H_2(x)=x^2-1,$$
$$H_3(x)=x^3-3x,$$
$$H_4(x)=x^4-6x^2+3,$$
$$H_5(x)=x^5-10x^3+15x.$$

Let us formulate the following modification and generalization of Theorem 3.1 from [40] for the case $i_1,\ldots,i_k=0,1,\ldots,m$.

Theorem 4 [10], [15]. *Suppose that the condition* $(\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *and the condition* (27) *is also fulfilled. Furthermore, let* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases}H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right)$$

w. p. 1, where $H_n(x)$ *is the Hermite polynomial* (28), $\mathbf{1}_A$ *is the indicator of the set* A, $i_1,\ldots,i_k=0,1,\ldots,m;$ $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;$ $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;$ $d_l=1,\ldots,m_l;$ $l=1,\ldots,k;$ $m_1+\ldots+m_k=k;$ *the numbers* $m_1,\ldots,m_k$, $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l}$, $h_{1,l},\ldots,h_{d_l,l}$, d_l *depend on* $\{j_1,\ldots,j_k\}$; *moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\}$;

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}\quad (i=0,1,\ldots,m;\quad j=0,1,2,\ldots)$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$) *and* $d\mathbf{w}_\tau^{(0)}=d\tau$.

Proof. First, consider the case $i_1=\ldots=i_k=1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup\mathbf{N}$. This case has been considered in [40], but we give a different proof here. By induction, we prove the following equality

$$p!\int\limits_t^T\phi_l(t_p)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_p}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$=\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{p})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_p')\ldots\int\limits_t^{t_2'}\phi_l(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_p'}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\qquad(29)$$

w. p. 1, where $p\in\mathbf{N}$, $l\neq j_1,\ldots,j_q$, and

$$\sum_{(q_1,\ldots,q_n)}$$

means the sum with respect to all possible permutations $(q_1,\ldots,q_n)$.

Consider the case $p=1$. Using the Itô formula, we get w. p. 1 for $s\in[t,T]$

$$\int\limits_t^s\phi_l(\tau)d\mathbf{w}_\tau^{(1)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +$$

$$+ \int\limits_t^s \phi_l(\tau) \int\limits_t^{\tau} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} d\mathbf{w}_{\tau}^{(1)} +$$

$$+ \int\limits_t^s \phi_{j_q}(\tau) \left(\int\limits_t^{\tau} \phi_l(\theta) d\mathbf{w}_{\theta}^{(1)} \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} \right) d\mathbf{w}_{\tau}^{(1)}. \tag{30}$$

Hereinafter in this section always $s \in [t, T]$. Differentiating by the Itô formula the expression in parentheses on the right-hand side of equality (30) and combining the result of differentiation with (30), we obtain w. p. 1

$$J_{(l)s,t} J_{(j_q \ldots j_1)s,t} =$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +$$

$$+ J_{(l j_q \ldots j_1)s,t} +$$

$$+ \int\limits_t^s \phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_l(\theta)\phi_{j_{q-1}}(\theta) \int\limits_t^{\theta} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} d\theta d\mathbf{w}_{\tau}^{(1)} +$$

$$+ J_{(j_q l j_{q-1} \ldots j_1)s,t} +$$

$$+ \int\limits_t^s \phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(\theta) \times$$

$$\times \left(\int\limits_t^{\theta} \phi_l(u)\, d\mathbf{w}_u^{(1)} \int\limits_t^{\theta} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} \right) d\mathbf{w}_{\theta}^{(1)} d\mathbf{w}_{\tau}^{(1)},$$

where

$$\int\limits_t^s \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} \stackrel{\mathsf{def}}{=} J_{(j_q \ldots j_1)s,t}.$$

Continuing the process of iterative application of the Itô formula, we have w. p. 1

$$J_{(l)s,t}J_{(j_q\ldots j_1)s,t}=$$

$$=J_{(lj_q\ldots j_1)s,t}+J_{(j_qlj_{q-1}\ldots j_1)s,t}+\ldots+J_{(j_q\ldots j_1l)s,t}+$$

$$+\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots$$

$$\ldots+\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_l(\tau)\phi_{j_1}(\tau)d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \qquad (31)$$

Summing the equality (31) over permutations $(j_1,\ldots,j_q)$, we get

$$\sum_{(j_1,\ldots,j_q)}J_{(l)s,t}J_{(j_q\ldots j_1)s,t}=\sum_{(j_1,\ldots,j_q,l)}J_{(lj_q\ldots j_1)s,t}+S(s) \qquad (32)$$

w. p. 1, where

$$S(s)=$$

$$=\sum_{(j_1,\ldots,j_q)}\left(\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)\int\limits_t^{\tau}\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}d\tau+\ldots\right.$$

$$\left.\ldots+\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_l(\tau)\phi_{j_1}(\tau)d\tau d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\right). \qquad (33)$$

Consider

$$\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^s\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}.$$

Applying the Itô formula, we get w. p. 1

$$\int\limits_t^s\phi_l(\tau)\phi_{j_q}(\tau)d\tau\int\limits_t^s\phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}=$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^\tau \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau +$$

$$+ \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \times$$

$$\times \left(\int\limits_t^{t_{q-1}} \phi_l(\tau)\phi_{j_q}(\tau) d\tau \int\limits_t^{t_{q-1}} \phi_{j_{q-2}}(t_{q-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-2}}^{(1)} \right) d\mathbf{w}_{t_{q-1}}^{(1)}.$$

By iterative application of the Itô formula (as above), we obtain w. p. 1

$$\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} =$$

$$= \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^\tau \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots$$

$$\ldots + \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(\tau)\phi_{j_q}(\tau) d\tau d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)}. \tag{34}$$

Summing the equality (34) over permutations $(j_1, \ldots, j_q)$, we get

$$\sum_{(j_1,\ldots,j_q)} \int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) d\tau \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} = S_1(s), \tag{35}$$

w. p. 1, where

$$S_1(s) =$$

$$= \sum_{(j_1,\ldots,j_q)} \left(\int\limits_t^s \phi_l(\tau)\phi_{j_q}(\tau) \int\limits_t^\tau \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} d\tau + \ldots \right.$$

$$
\ldots + \int\limits_t^s \phi_{j_{q-1}}(t_{q-1}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(\tau)\phi_{j_q}(\tau) d\tau d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{q-1}}^{(1)} \Biggr) . \qquad (36)
$$

It is not difficult to see that

$$
S(s) = S_1(s) \quad \text{w. p. 1.} \qquad (37)
$$

Moreover, due to the orthogonality of $\{\phi_j(x)\}_{j=0}^{\infty}$ and (35), (37), we have

$$
S(T) = S_1(T) = 0 \quad \text{w. p. 1.} \qquad (38)
$$

Thus (see (32), (38)), the equality (29) is proved for the case $p = 1$. Let us assume that the equality (29) is true for $p = 2, 3, \ldots, k-1$, and prove its validity for $p = k$.

From (32) for the case $q = k-1$, $j_1 = \ldots = j_{k-1} = l$ we obtain

$$
(J_1)_{s,t} \, (k-1)! \, (J_{k-1})_{s,t} = k! \, (J_k)_{s,t} + S_2(s) \qquad (39)
$$

w. p. 1, where

$$
S_2(s) = S(s)\Biggr|_{j_1=\ldots=j_q=l, \ q=k-1} \quad (k \ge 2) \quad \text{and} \quad S_2(s) \stackrel{\mathsf{def}}{=} 0 \ \ (q = k-1, \ k = 1),
$$

$$
\int\limits_t^s \phi_l(t_r) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_r}^{(1)} \stackrel{\mathsf{def}}{=} (J_r)_{s,t} \quad (r \in \mathbf{N}) \quad \text{and} \quad (J_0)_{s,t} \stackrel{\mathsf{def}}{=} 1.
$$

Taking into account (33), (35)–(37) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we have

$$
S_2(T) = (k-1)! \, (J_{k-2})_{T,t} \, . \qquad (40)
$$

Combining (39) and (40), we obtain the following recurrence relation

$$
k! \, (J_k)_{T,t} = (J_1)_{T,t} \, (k-1)! \, (J_{k-1})_{T,t} - (k-1)! \, (J_{k-2})_{T,t} \qquad (41)
$$

w. p. 1.

Using (41) and the induction hypothesis, we get w. p. 1

$$k!\int\limits_t^T\phi_l(t_k)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$=\int\limits_t^T\phi_l(\tau)\ d\mathbf{w}_\tau^{(1)}\Biggl((k-1)!\int\limits_t^T\phi_l(t_{k-1})\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}\Biggr)-$$

$$-(k-1)!\int\limits_t^T\phi_l(t_{k-2})\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}=$$

$$=\int\limits_t^T\phi_l(\tau)\ d\mathbf{w}_\tau^{(1)}\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t'_{k-1})\ldots\int\limits_t^{t'_2}\phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}-$$

$$-(k-1)\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t'_{k-2})\ldots\int\limits_t^{t'_2}\phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}.\qquad(42)$$

Let $\boxed{l}$ be the symbol l which does not participate in the following sum with respect to permutations

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.$$

Using (32), we have w. p. 1

$$\int\limits_t^s \phi_l(\tau)\ d\mathbf{w}_\tau^{(1)} \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^s \phi_{\boxed{l}}(\tau)\ d\mathbf{w}_\tau^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_l(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(J_{(\boxed{l} j_q\ldots j_1 \underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q \boxed{l} j_{q-1}\ldots j_1 \underbrace{l\ldots l}_{k-1})s,t} + \ldots$$

$$\ldots + J_{(j_q\ldots j_1 \boxed{l} \underbrace{l\ldots l}_{k-1})s,t} + J_{(j_q\ldots j_1 l \boxed{l} \underbrace{l\ldots l}_{k-2})s,t} + \ldots + J_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k-1} \boxed{l})s,t} \Biggr) + S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} J_{(j_q\ldots j_1 \underbrace{l\ldots l}_{k})s,t} + S_3(s), \tag{43}$$

where

$$S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(\int\limits_t^s \phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau) \int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}^{(1)}_{t'_1} \ldots d\mathbf{w}^{(1)}_{t'_{k-1}} d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_{q-1}} d\tau + \ldots$$

$$+ \ldots \int\limits_t^{s} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{\boxed{l}}(\tau) \phi_{j_1}(\tau) \times$$

$$\times \int\limits_t^{\tau} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}^{(1)}_{t'_1} \ldots d\mathbf{w}^{(1)}_{t'_{k-1}} d\tau d\mathbf{w}^{(1)}_{t_2} \ldots d\mathbf{w}^{(1)}_{t_q} +$$

$$+ \int\limits_t^{s} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{\boxed{l}}(\tau) \phi_l(\tau) \times$$

$$\times \int\limits_t^{\tau} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}^{(1)}_{t'_1} \ldots d\mathbf{w}^{(1)}_{t'_{k-2}} d\tau d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_q} + \ldots$$

$$\ldots + \int\limits_t^{s} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-1}) \ldots \int\limits_t^{t'_3} \phi_l(t'_2) \int\limits_t^{t'_2} \phi_{\boxed{l}}(\tau) \phi_l(\tau) d\tau d\mathbf{w}^{(1)}_{t'_2} \ldots d\mathbf{w}^{(1)}_{t'_{k-1}} d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_q} \Biggr).$$

Using (33), (35)–(37), we get w. p. 1

$$S_3(s) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \int\limits_t^{s} \phi_{\boxed{l}}(\tau) \phi_l(\tau) d\tau \int\limits_t^{s} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}^{(1)}_{t'_1} \ldots d\mathbf{w}^{(1)}_{t'_{k-2}} d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_q} =$$

$$= (k-1) \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})} \int_t^s \phi_{\boxed{l}}(\tau)\phi_l(\tau)d\tau \int_t^s \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int_t^{t_1} \phi_l(t'_{k-2})\ldots \int_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}+$$

$$+ \sum_{(j_1,\ldots,j_{q-1},\underbrace{l,\ldots,l}_{k-1})} \int_t^s \phi_{\boxed{l}}(\tau)\phi_{j_q}(\tau)d\tau \int_t^s \phi_{j_{q-1}}(t_{q-1})\ldots \int_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int_t^{t_1} \phi_l(t'_{k-1})\ldots \int_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-1}}^{(1)}+$$

$$+ \sum_{(j_1,\ldots,j_{q-2},j_q\underbrace{l,\ldots,l}_{k-1})} \int_t^s \phi_{\boxed{l}}(\tau)\phi_{j_{q-1}}(\tau)d\tau \int_t^s \phi_{j_q}(t_q) \int_t^{t_q} \phi_{j_{q-2}}(t_{q-2})\ldots \int_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int_t^{t_1} \phi_l(t'_{k-1})\ldots \int_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{q-2}}^{(1)} d\mathbf{w}_{t_q}^{(1)}+$$

$$\ldots$$

$$+ \sum_{(j_2,\ldots,j_q\underbrace{l,\ldots,l}_{k-1})} \int_t^s \phi_{\boxed{l}}(\tau)\phi_{j_1}(\tau)d\tau \int_t^s \phi_{j_q}(t_q)\ldots \int_t^{t_3} \phi_{j_2}(t_2)\times$$

$$\times \int_t^{t_2} \phi_l(t'_{k-1})\ldots \int_t^{t'_2} \phi_l(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_q}^{(1)}. \qquad (44)$$

Applying (44) and the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we finally have

$$S_3(T) = (k-1) \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-2})} \int_t^T \phi_{j_q}(t_q)\ldots \int_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_l(t'_{k-2}) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_{k-2}}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}. \tag{45}$$

Combining (42), (43), (45), we obtain w. p. 1

$$k! \int\limits_t^{T} \phi_l(t_k) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_k}^{(1)} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^{T} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(\underbrace{l,\ldots,l}_{k})} \int\limits_t^{T} \phi_l(t_k) \ldots \int\limits_t^{t_2} \phi_l(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_k}^{(1)} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^{T} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} \int\limits_t^{T} \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(t'_k) \ldots \int\limits_t^{t'_2} \phi_l(t'_1) \times$$

$$\times d\mathbf{w}_{t'_1}^{(1)} \ldots d\mathbf{w}_{t'_k}^{(1)} d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_q}^{(1)}, \tag{46}$$

where $l \neq j_1, \ldots, j_q$.

The equality (29) is proved. From the other hand, (46) means that

$$J''[\phi_{j_1} \ldots \phi_{j_q} \underbrace{\phi_l \ldots \phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{q+n})} = J''[\underbrace{\phi_l \ldots \phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{n})} \cdot J''[\phi_{j_1} \ldots \phi_{j_q}]_{T,t}^{(\overbrace{1\ldots1}^{q})} \tag{47}$$

w. p. 1, where $n, q = 0, 1, 2 \ldots$; $l \neq j_1, \ldots, j_q$ and

$$J''[\phi_{j_1} \ldots \phi_{j_q}]_{T,t}^{(\overbrace{1\ldots1}^{q})} \stackrel{\text{def}}{=} 1$$

for $q = 0$.

Consider polynomials $H_n(x,y)$, $n=0,1,\ldots$ defined by [41]

$$H_n(x,y)=\left(\frac{d^n}{d\alpha^n}e^{\alpha x-\alpha^2 y/2}\right)\Bigg|_{\alpha=0}\qquad (H_0(x,y)\stackrel{\rm def}{=}1). \tag{48}$$

It is well known that polynomials $H_n(x,y)$ are connected with the Hermite polynomials (28) by the formula [41]

$$H_n(x,y)=y^{n/2}H_n\left(\frac{x}{\sqrt{y}}\right)=n!\sum_{i=0}^{[n/2]}\frac{(-1)^i x^{n-2i}y^i}{i!(n-2i)!2^i}. \tag{49}$$

For example,

$$H_1(x,y)=x,\quad H_2(x,y)=x^2-y,\quad H_3(x,y)=x^3-3xy,$$
$$H_4(x,y)=x^4-6x^2y+3y^2,\quad H_5(x,y)=x^5-10x^3y+15xy^2.$$

From (28) and (49) we get

$$H_n(x,1)=H_n(x). \tag{50}$$

Note that [41] (also see [10] (Chapter 6, Sect. 6.6) for details)

$$\int\limits_t^T\phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)}=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)},\int\limits_t^T\phi_l^2(\tau)d\tau\right)=$$
$$=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)},1\right)=\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right) \tag{51}$$

w. p. 1, where $n\in\mathbf{N}$, $H_n(x,y)$ is defined by (48) (also see (49)), and $H_n(x)$ is the Hermite polynomial (28).

From (51) we have w. p. 1

$$J''[\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots 1}^{n})}=n!\int\limits_t^T\phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_n}^{(1)}=$$
$$=n!\frac{1}{n!}H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right)=H_n\left(\int\limits_t^T\phi_l(\tau)d\mathbf{w}_{\tau}^{(1)}\right), \tag{52}$$

where $n \in \mathbf{N}$.

Combining (47) and (52), we obtain

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{1\ldots1}^{q+n})} = H_n\left(\int\limits_t^T \phi_l(\tau)d\mathbf{w}_\tau^{(1)}\right)\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{1\ldots1}^{q})} \quad (53)$$

w. p. 1, where $n, q = 0, 1, 2\ldots$; $l \ne j_1, \ldots, j_q$.

The iterated application of the formula (53) completes the proof of Theorem 4 for the case $i_1 = \ldots = i_k = 1, \ldots, m$ and $j_1, \ldots, j_k \in \{0\} \cup \mathbf{N}$.

To prove Theorem 4 for the case $i_1 = \ldots = i_k = 0, 1, \ldots, m$ and $j_1, \ldots, j_k \in \{0\} \cup \mathbf{N}$, we need to prove the following formula in addition to the previous proof

$$p!\int\limits_t^T \phi_l(t_p)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_p \sum_{(j_1,\ldots,j_q)}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{p})}\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_l(t_p')\ldots\int\limits_t^{t_2'}\phi_l(t_1')dt_1'\ldots dt_p'dt_1\ldots dt_q, \quad (54)$$

where $p \in \mathbf{N}$,

$$\sum_{(j_1,\ldots,j_d)}$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_d)$.

First, consider the case $p = 1$. We have

$$d\left(\int\limits_t^s \phi_l(\theta)d\theta\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q\right) =$$

$$= \phi_l(s)\int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_q ds +$$

$$+\phi_{j_q}(s)\left(\int\limits_t^s \phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{q-1}\cdot\int\limits_t^s \phi_l(\theta)d\theta\right)ds.$$

Then

$$\int\limits_t^s \phi_l(\theta)d\theta \int\limits_t^s \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_q =$$

$$= I_{(lj_q\ldots j_1)s,t} +$$

$$+ \int\limits_t^s \phi_{j_q}(\tau)\left(\int\limits_t^{\tau} \phi_{j_{q-1}}(t_{q-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_{q-1}\cdot \int\limits_t^{\tau} \phi_l(\theta)d\theta\right)d\tau,$$

where

$$\int\limits_t^s \phi_{j_r}(t_r)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)dt_1\ldots dt_r \stackrel{\mathsf{def}}{=} I_{(j_r\ldots j_1)s,t}. \tag{55}$$

Continuing this process, we get

$$\int\limits_t^s \phi_l(\theta)d\theta \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} = \sum_{(j_1,\ldots,j_q,l)} I_{(lj_q\ldots j_1)s,t}. \tag{56}$$

The equality (54) is proved for the case $p = 1$. Let us assume that the equality (54) is true for $p = 2, 3, \ldots, k-1$, and prove its validity for $p = k$.

From (56) for $j_1 = \ldots = j_q = l$, $q = k-1$ we have

$$(I_1)_{s,t}\,(k-1)!\,(I_{k-1})_{s,t} = k!\,(I_k)_{s,t}\,, \tag{57}$$

where $k \in \mathbf{N}$ and

$$\int\limits_t^s \phi_l(t_k)\ldots \int\limits_t^{t_2} \phi_l(t_1)dt_1\ldots dt_k \stackrel{\mathsf{def}}{=} (I_k)_{s,t}\,, \quad (I_0)_{s,t} \stackrel{\mathsf{def}}{=} 1.$$

Using (57) and the induction hypothesis, we obtain

$$k!\,(I_k)_{s,t} \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} = (I_1)_{s,t}\,(k-1)!\,(I_{k-1})_{s,t} \sum_{(j_1,\ldots,j_q)} I_{(j_q\ldots j_1)s,t} =$$

$$= I_{(l)s,t} \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t} I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t}, \tag{58}$$

where $I_{(j_r\ldots j_1)s,t}$ is defined by (55) and $\boxed{l}$ is the symbol l which does not participate in the following sum with respect to permutations

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})}.$$

By analogy with (56) we obtain

$$\sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} I_{(\boxed{l})s,t} I_{(j_q\ldots j_1\underbrace{l,\ldots,l}_{k-1})s,t} =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k-1})} \Biggl(I_{(\boxed{l}j_q\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\boxed{l}j_{q-1}\ldots j_1\underbrace{l\ldots l}_{k-1})s,t} + \ldots$$

$$\ldots + I_{(j_q\ldots j_1\boxed{l}\underbrace{l\ldots l}_{k-1})s,t} + I_{(j_q\ldots j_1 l\boxed{l}\underbrace{l\ldots l}_{k-2})s,t} + \ldots + I_{(j_q\ldots j_1\underbrace{l\ldots l}_{k-1}\boxed{l})s,t} \Biggr) =$$

$$= \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{k})} I_{(j_q\ldots j_1\underbrace{l\ldots l}_{k})s,t}. \tag{59}$$

Substituting $s = T$ into (58), (59) and combining (58), (59), we conlude that the equality (54) is proved for $p = k$. The equality (54) is proved.

Note that

$$n!\int\limits_t^T \phi_l(t_n)\ldots\int\limits_t^{t_2}\phi_l(t_1)dt_1\ldots dt_n = n!\frac{1}{n!}\left(\int\limits_t^T\phi_l(\tau)d\tau\right)^n = \left(\int\limits_t^T\phi_l(\tau)d\tau\right)^n, \tag{60}$$

where $n \in \mathbf{N}$.

After substituting (60) into (54), we have for $p = n$

$$\left(\int\limits_t^T\phi_l(\tau)d\tau\right)^n \sum_{(j_1,\ldots,j_q)} J_{(j_q\ldots j_1)T,t} = \sum_{(j_1,\ldots,j_q,\underbrace{l,\ldots,l}_{n})} J_{(j_q\ldots j_1\underbrace{l\ldots l}_{n})T,t}. \tag{61}$$

The equality (61) means that

$$J''[\phi_{j_1}\ldots\phi_{j_q}\underbrace{\phi_l\ldots\phi_l}_{n}]_{T,t}^{(\overbrace{0\ldots 0}^{q+n})}=\left(\int\limits_t^T\phi_l(\tau)d\tau\right)^n\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})},\qquad(62)$$

where $n,q=0,1,2\ldots$ and $J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(0\ldots 0)}\stackrel{\sf def}{=}1$ for $q=0$.

The relations (53) and (62) prove Theorem 4 for the case $i_1=\ldots=i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup{\bf N}$.

Remark 1. *Note that the equality* (54) *can be obtained in another way. Let* $D_q=\{(t_1,\ldots,t_q)\in[t,T]^q:\ \exists\ i\ne j\ \textit{such that}\ t_i=t_j\}$ *be the "diagonal set" of* $[t,T]^q$ $(q=2,3,\ldots)$ [42]. *Since the Lebesgue meashure of the set* D_q *is equal to zero* [42], *then* (*see* (19))

$$J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}=\int\limits_{[t,T]^q}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q.\qquad(63)$$

From (63) *we have*

$$J''[\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p})}\cdot J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(\overbrace{0\ldots 0}^{q})}=$$

$$=\int\limits_{[t,T]^q}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)dt_1\ldots dt_q\int\limits_{[t,T]^p}\phi_l(t_1)\ldots\phi_l(t_p)dt_1\ldots dt_p=$$

$$=\int\limits_{[t,T]^{p+q}}\phi_{j_1}(t_1)\ldots\phi_{j_q}(t_q)\phi_l(t_1')\ldots\phi_l(t_p')dt_1'\ldots dt_p'dt_1\ldots dt_q=$$

$$=J''[\phi_{j_1}\ldots\phi_{j_q}\phi_l\ldots\phi_l]_{T,t}^{(\overbrace{0\ldots 0}^{p+q})}.\qquad(64)$$

It is not difficult to see that the equality (64) *is nothing but the equality* (54) *written in another form.*

To complete the proof of Theorem 4, we need to consider the case $i_1,\ldots,i_k=0,1,\ldots,m$ and $j_1,\ldots,j_k\in\{0\}\cup{\bf N}$.

Obviously, the proof of Theorem 4 will be completed if we prove the following equalities

$$
\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} \times
$$

$$
\times \sum_{(j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_n'}(t_n') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_n'}^{(1)} =
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_n'}(t_n') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') \times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)} \ldots d\mathbf{w}_{t_n'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)}, \qquad (65)
$$

$$
\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} \times
$$

$$
\times \sum_{(j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_n'}(t_n') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(0)} \ldots d\mathbf{w}_{t_n'}^{(0)} =
$$

$$
= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_n')} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_n'}(t_n') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') \times
$$

$$
\times d\mathbf{w}_{t_1'}^{(0)} \ldots d\mathbf{w}_{t_n'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} \qquad (66)
$$

w. p. 1, where $n, q \in \mathbf{N}$, $d\mathbf{w}_{\tau}^{(0)} = d\tau$, $i_1, \ldots, i_q \neq 1$ in (65) and $i_1, \ldots, i_q \neq 0$ in (66),

$$
\sum_{(j_1,\ldots,j_g)}
$$

means the sum with respect to all possible permutations $(j_1, \ldots, j_g)$. At the same time if j_r swapped with j_d in the permutation $(j_1, \ldots, j_g)$, then i_r swapped with i_d in the permutation $(i_1, \ldots, i_g)$.

The equalities (65) and (66) mean that

$$
J''[\phi_{j_1} \ldots \phi_{j_q} \phi_{j_1'} \ldots \phi_{j_n'}]_{T,t}^{(i_1 \ldots i_q 1 \ldots 1)} = J''[\phi_{j_1} \ldots \phi_{j_q}]_{T,t}^{(i_1 \ldots i_q)} \cdot J''[\phi_{j_1'} \ldots \phi_{j_n'}]_{T,t}^{(1 \ldots 1)}, \quad (67)
$$

$$J''[\phi_{j_1}\ldots\phi_{j_q}\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(i_1\ldots i_q 0\ldots 0)} = J''[\phi_{j_1}\ldots\phi_{j_q}]_{T,t}^{(i_1\ldots i_q)} \cdot J''[\phi_{j_1'}\ldots\phi_{j_n'}]_{T,t}^{(0\ldots 0)} \qquad (68)$$

w. p. 1, where $i_1,\ldots,i_q \neq 1$ in (67) and $i_1,\ldots,i_q \neq 0$ in (68).

First, we prove the equality (65). Consider the case $n = 1$. Using the Itô formula, we get w. p. 1

$$\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= J_{(j_1' j_q\ldots j_1)s,t}^{(1 i_q\ldots i_1)} +$$

$$+\int\limits_t^s \phi_{j_q}(\tau)\left(\int\limits_t^\tau \phi_{j_{q-1}}(t_{q-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{q-1}}^{(i_{q-1})}\int\limits_t^\tau \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)}\right)d\mathbf{w}_\tau^{(i_q)} =$$

$$\ldots$$

$$= J_{(j_1' j_q\ldots j_1)s,t}^{(1 i_q\ldots i_1)} + J_{(j_q j_1' j_{q-1}\ldots j_1)s,t}^{(i_q 1 i_{q-1}\ldots i_1)} + \ldots + J_{(j_q\ldots j_1 j_1')s,t}^{(i_q\ldots i_1 1)}, \qquad (69)$$

where

$$\int\limits_t^s \phi_{j_r}(t_r)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_r}^{(i_r)} \stackrel{\mathsf{def}}{=} J_{(j_r\ldots j_1)s,t}^{(i_r\ldots i_1)}, \qquad (70)$$

$i_1,\ldots,i_r = 0,1,\ldots,m.$

From (69) we obtain

$$\int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \sum_{(j_1,\ldots,j_q)} \int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q)} \int\limits_t^s \phi_{j_1'}(\theta)d\mathbf{w}_\theta^{(1)} \int\limits_t^s \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q)} \left(J_{(j_1' j_q\ldots j_1)s,t}^{(1 i_q\ldots i_1)} + J_{(j_q j_1' j_{q-1}\ldots j_1)s,t}^{(i_q 1 i_{q-1}\ldots i_1)} + \ldots + J_{(j_q\ldots j_1 j_1')s,t}^{(i_q\ldots i_1 1)}\right) =$$

$$
= \sum_{(j_1,\ldots,j_q,j_1')} J^{(i_q\ldots i_1 1)}_{(j_q\ldots j_1 j_1')s,t} \qquad (71)
$$

w. p. 1, where $J^{(i_r\ldots i_1)}_{(j_r\ldots j_1)s,t}$ is defined by (70). The equality (65) is proved for the case $n = 1$.

Let us assume that the equality (65) is true for $n = 2, 3, \ldots, k-1$, and prove its validity for $n = k$.

Applying (32), (33), (35)–(37), we obtain w. p. 1

$$
\sum_{(j_1',\ldots,j_k')} \int\limits_t^s \phi_{j_k'}(t_k') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}^{(1)}_{t_1'} \ldots d\mathbf{w}^{(1)}_{t_k'} =
$$

$$
= \int\limits_t^s \phi_{j_k'}(\theta) d\mathbf{w}^{(1)}_{\theta} \sum_{(j_1',\ldots,j_{k-1}')} \int\limits_t^s \phi_{j_{k-1}'}(t_{k-1}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_{k-1}} -
$$

$$
- \sum_{(j_1',\ldots,j_{k-1}')} \int\limits_t^s \phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta) d\theta \int\limits_t^s \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_{k-2}}. \qquad (72)
$$

Substituting $s = T$ in (72) and applying the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, we get w. p. 1

$$
\sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k'}(t_k') \ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}^{(1)}_{t_1'} \ldots d\mathbf{w}^{(1)}_{t_k'} =
$$

$$
= \int\limits_t^T \phi_{j_k'}(\theta) d\mathbf{w}^{(1)}_{\theta} \sum_{(j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-1}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_{k-1}} -
$$

$$
- \sum_{(j_1',\ldots,j_{k-1}')} \mathbf{1}_{\{j_k'=j_{k-1}'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}^{(1)}_{t_1} \ldots d\mathbf{w}^{(1)}_{t_{k-2}}, \qquad (73)
$$

where $\mathbf{1}_A$ is the indicator of the set A.

Using (73) and the induction hypothesis, we obtain w. p. 1

$$\sum_{(j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_k'}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_k}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\sum_{(j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_{k-1}'}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-1}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\int\limits_t^T\phi_{j_k'}(\theta)d\mathbf{w}_{\theta}^{(1)}\times$$

$$\times\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}-$$

$$-\sum_{(j_1',\ldots,j_{k-1}')}\mathbf{1}_{\{j_k'=j_{k-1}'\}}\int\limits_t^T\phi_{j_{k-2}'}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1'}(t_1)d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{74}$$

Further, applying the induction hypothesis, we have w. p. 1

$$\sum_{(j_1',\ldots,j_{k-1}')} \mathbf{1}_{\{j_k'=j_{k-1}'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-2}}^{(1)} \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \Biggl(\sum_{(j_1',\ldots,j_{k-2}')} \mathbf{1}_{\{j_k'=j_{k-1}'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-2}}^{(1)} +$$

$$+ \sum_{(j_1',\ldots,j_{k-3}',j_{k-1}')} \mathbf{1}_{\{j_k'=j_{k-2}'\}} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-2}) \int\limits_t^{t_{k-2}} \phi_{j_{k-3}'}(t_{k-3}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) \times$$

$$\times d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-3}}^{(1)} d\mathbf{w}_{t_{k-2}}^{(1)} + \ldots$$

$$\ldots + \sum_{(j_2',\ldots,j_{k-1}')} \mathbf{1}_{\{j_k'=j_1'\}} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_3} \phi_{j_2'}(t_2) \int\limits_t^{t_2} \phi_{j_{k-1}'}(t_1) \times$$

$$\times d\mathbf{w}_{t_1}^{(1)} d\mathbf{w}_{t_2}^{(1)} \ldots d\mathbf{w}_{t_{k-2}}^{(1)} \Biggr) \times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \Biggl(\mathbf{1}_{\{j_k'=j_{k-1}'\}} \sum_{(j_1',\ldots,j_{k-2}')} \int\limits_t^T \phi_{j_{k-2}'}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1) d\mathbf{w}_{t_1}^{(1)} \ldots d\mathbf{w}_{t_{k-2}}^{(1)} +$$

$$+\mathbf{1}_{\{j'_k=j'_{k-2}\}}\sum_{(j'_1,\ldots,j'_{k-3},j'_{k-1})}\int\limits_t^T\phi_{j'_{k-1}}(t_{k-2})\int\limits_t^{t_{k-2}}\phi_{j'_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j'_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(1)}\ldots d\mathbf{w}_{t_{k-3}}^{(1)}d\mathbf{w}_{t_{k-2}}^{(1)}+\ldots$$

$$\ldots+\mathbf{1}_{\{j'_k=j'_1\}}\sum_{(j'_2,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j'_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_3}\phi_{j'_2}(t_2)\int\limits_t^{t_2}\phi_{j'_{k-1}}(t_1)\times$$

$$\left.\times d\mathbf{w}_{t_1}^{(1)}d\mathbf{w}_{t_2}^{(1)}\ldots d\mathbf{w}_{t_{k-2}}^{(1)}\right)\times$$

$$\times\sum_{(j_1,\ldots,j_q)}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$=\mathbf{1}_{\{j'_k=j'_{k-1}\}}\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-2})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\mathbf{1}_{\{j'_k=j'_{k-2}\}}\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-3},j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-2})\times$$

$$\times\int\limits_t^{t'_{k-2}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$\ldots$$

$$+\mathbf{1}_{\{j'_k=j'_1\}}\sum_{(j_1,\ldots,j_q,j'_2,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)\int\limits_t^{t'_2}\phi_{j'_{k-1}}(t'_1)d\mathbf{w}_{t'_1}^{(1)}d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\stackrel{\mathsf{def}}{=}$$

$$\stackrel{\sf def}{=} S_4(T). \qquad (75)$$

By analogy with (34) we obtain w. p. 1

$$\int\limits_t^T \phi_l(\tau)\phi_{j_r}(\tau)d\tau \int\limits_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})} =$$

$$= \int\limits_t^T \phi_l(\tau)\phi_{j_r}(\tau) \int\limits_t^{\tau} \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})}d\tau + \ldots$$

$$\ldots + \int\limits_t^T \phi_{j_{r-1}}(t_{r-1})\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_l(\tau)\phi_{j_r}(\tau)d\tau d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{r-1}}^{(i_{r-1})}, \qquad (76)$$

where $i_1,\ldots,i_{r-1} = 0, 1,\ldots,m$.

Using iteratively the Itô formula, as well as (76) and combinatorial reasoning, we get w. p. 1 (see Remark 2 below for details)

$$\int\limits_t^T \phi_{j'_k}(\theta)d\mathbf{w}_{\theta}^{(1)}\times$$

$$\times \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_k)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j'_k}(t'_k)\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)\times$$

$$\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_k}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} +$$

$$
+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-1}')}\Biggl(\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}'}(t_{k-2}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\int\limits_t^{t_{k-1}'}\phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}'}(t_{k-3}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots
$$

$$
\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-1}'}(t_{k-1}')\ldots\int\limits_t^{t_3'}\phi_{j_2'}(t_2')\int\limits_t^{t_2'}\phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\mathbf{w}_{\theta}^{(0)}\times
$$

$$
\times d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\Biggr)=
$$

$$
=\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_k'}(t_k')\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')\times
$$

$$
\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+
$$

$$
+\sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')}\Biggl\{\int\limits_t^T\phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j_{k-2}'}(t_{k-2}')\ldots
$$

$$
\ldots\int\limits_t^{t_2'}\phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_{\theta}^{(0)}+\ldots
$$

$$\ldots+\int\limits_t^T \phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-2}}(t'_{k-2})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\int\limits_t^{t'_1}\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta)d\mathbf{w}_\theta^{(0)}\times$$

$$\left.\times d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}+$$

$$+\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-3},j'_{k-1})}\left\{\int\limits_t^T\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\times\right.$$

$$\times\int\limits_t^{t'_{k-1}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_\theta^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\int\limits_t^{t'_{k-1}}\phi_{j'_{k-3}}(t'_{k-3})\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)\times$$

$$\left.\times\int\limits_t^{t'_1}\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta)d\mathbf{w}_\theta^{(0)}d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-3}}^{(1)}d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}+\ldots$$

$$\ldots+\sum_{(j_1,\ldots,j_q,j'_2,\ldots,j'_{k-1})}\left\{\int\limits_t^T\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)\int\limits_t^{\theta}\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\ldots\right.$$

$$\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}d\mathbf{w}_\theta^{(0)}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_{k-1}}(t'_{k-1})\ldots\int\limits_t^{t'_3}\phi_{j'_2}(t'_2)\int\limits_t^{t'_2}\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)d\mathbf{w}_\theta^{(0)}\times$$

$$\left.\times d\mathbf{w}_{t'_2}^{(1)}\ldots d\mathbf{w}_{t'_{k-1}}^{(1)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\right\}=$$

$$= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-1}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-2}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-2}'}(t_{k-2}')\ldots$$

$$\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-2}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+$$

$$+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_{k-2}'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_{k-3}',j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}')\times$$

$$\times \int\limits_t^{t_{k-1}'} \phi_{j_{k-3}'}(t_{k-3}')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1') d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_{k-3}'}^{(1)} d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+\ldots$$

$$\ldots+\int\limits_t^T \phi_{j_k'}(\theta)\phi_{j_1'}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j_2',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_{k-1}'}(t_{k-1}')\ldots$$

$$\ldots \int\limits_t^{t_3'} \phi_{j_2'}(t_2') d\mathbf{w}_{t_2'}^{(1)}\ldots d\mathbf{w}_{t_{k-1}'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}=$$

$$= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(1)}\ldots d\mathbf{w}_{t_k'}^{(1)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}+S_4(T). \tag{77}$$

From (74), (75), and (77) we conclude that the equality (65) is proved for $n=k$. The equality (65) is proved.

Remark 2. *It should be noted that the sums with respect to permutations*

$$\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})}$$

in (77)*, containing the expressions* $\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta),\ldots,\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)$*, should be understood in a special way. Let us explain this rule on the following sum*

$$\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})}\int\limits_t^T\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\int\limits_t^{t_1}\phi_{j'_k}(\theta)\phi_{j'_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j'_{k-2}}(t'_{k-2})\ldots$$

$$\ldots\int\limits_t^{t'_2}\phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(1)}\ldots d\mathbf{w}_{t'_{k-2}}^{(1)}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \tag{78}$$

More precisely, permutations $\left(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1}\right)$ *when summing in* (78) *are performed in such a way that if* j_r^* *swapped with* j_d^* *in the permutation* $\left(j_{q+k-1}^*,\ldots,j_1^*\right)=\left(j_q,\ldots,j_1,j'_{k-1},j'_{k-2},\ldots,j'_1\right)$*, then* i_r^* *swapped with* i_d^* *in the permutation*

$$\left(i_{q+k-1}^*,\ldots,i_1^*\right)=\left(i_q,\ldots,i_1,0,\underbrace{1,\ldots,1}_{k-2}\right).$$

Moreover, $\bar\phi_{j_r^*}$ *swapped with* $\bar\phi_{j_d^*}$ *in the permutation*

$$\left(\bar\phi_{j_{q+k-1}^*},\ldots,\bar\phi_{j_1^*}\right)=\left(\phi_{j_q},\ldots,\phi_{j_1},\ \phi_{j'_k}\cdot\phi_{j'_{k-1}},\ \phi_{j'_{k-2}},\ldots,\phi_{j'_1}\right).$$

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})}$$

in (77) *that contain the expressions* $\phi_{j'_k}(\theta)\phi_{j'_{k-2}}(\theta),\ldots,\phi_{j'_k}(\theta)\phi_{j'_1}(\theta)$*.*

Let us prove the equality (66). Consider the case $n=1$. By analogy with (69) and (71) we obtain

$$\int\limits_t^s\phi_{j'_1}(\theta)d\mathbf{w}_{\theta}^{(0)}\sum_{(j_1,\ldots,j_q)}\int\limits_t^s\phi_{j_q}(t_q)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots\mathbf{w}_{t_q}^{(i_q)}=\sum_{(j_1,\ldots,j_q,j'_1)}J_{(j_q\ldots j_1j'_1)s,t}^{(i_q\ldots i_1 0)}$$

w. p. 1, where $J^{(i_r\ldots i_1)}_{(j_r\ldots j_1)s,t}$ is defined by (70). The equality (66) is proved for the case $n=1$.

Let us assume that the equality (66) is true for $n=2,3,\ldots,k-1$, and prove its validity for $n=k$.

In complete analogy with (56) we get

$$\int\limits_t^s \phi_{j_k'}(\theta)d\theta \int\limits_t^s \phi_{j_{k-1}'}(t_{k-1})\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)dt_1\ldots dt_{k-1} =$$

$$= J^{(0\ldots 0)}_{(j_k'j_{k-1}'\ldots j_1')s,t} + J^{(0\ldots 0)}_{(j_{k-1}'j_k'j_{k-2}'\ldots j_1')s,t} + \ldots + J^{(0\ldots 0)}_{(j_{k-1}'\ldots j_1'j_k')s,t}. \tag{79}$$

Applying (79), we have

$$\sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}^{(0)}_{t_1'}\ldots d\mathbf{w}^{(0)}_{t_k'} =$$

$$= \sum_{(j_1',\ldots,j_{k-1}')} \left(J^{(0\ldots 0)}_{(j_k'j_{k-1}'\ldots j_1')s,t} + J^{(0\ldots 0)}_{(j_{k-1}'j_k'j_{k-2}'\ldots j_1')s,t} + \ldots + J^{(0\ldots 0)}_{(j_{k-1}'\ldots j_1'j_k')s,t} \right) =$$

$$= \int\limits_t^T \phi_{j_k'}(\theta)d\theta \sum_{(j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-1})\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1)d\mathbf{w}^{(0)}_{t_1}\ldots d\mathbf{w}^{(0)}_{t_{k-1}}. \tag{80}$$

Using (80) and the induction hypothesis, we obtain w. p. 1

$$\sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k'}(t_k)\ldots \int\limits_t^{t_2} \phi_{j_1'}(t_1)d\mathbf{w}^{(0)}_{t_1}\ldots d\mathbf{w}^{(0)}_{t_k}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}^{(i_1)}_{t_1}\ldots d\mathbf{w}^{(i_q)}_{t_q} =$$

$$= \int\limits_t^T \phi_{j_k'}(\theta)d\theta \sum_{(j_1',\ldots,j_{k-1}')} \int\limits_t^T \phi_{j_{k-1}'}(t_{k-1}')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}^{(0)}_{t_1'}\ldots d\mathbf{w}^{(0)}_{t_{k-1}'}\times$$

$$\times \sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \int\limits_t^T \phi_{j'_k}(\theta)d\theta \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(0)}\ldots d\mathbf{w}_{t'_{k-1}}^{(0)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= \sum_{(j_1,\ldots,j_q,j'_1,\ldots,j'_{k-1})} \int\limits_t^T \phi_{j'_k}(\theta)d\theta \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(0)}\ldots d\mathbf{w}_{t'_{k-1}}^{(0)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}. \qquad (81)$$

An iterative application of the Itô formula leads to the following equality

$$\int\limits_t^T \phi_{j'_k}(\theta)d\theta \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)\times$$

$$\times \int\limits_t^{t_1} \phi_{j'_{k-1}}(t'_{k-1})\ldots \int\limits_t^{t'_2} \phi_{j'_1}(t'_1)d\mathbf{w}_{t'_1}^{(0)}\ldots d\mathbf{w}_{t'_{k-1}}^{(0)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)} =$$

$$= J_{(j'_k j_q\ldots j_1 j'_{k-1}\ldots j'_1)T,t}^{(0 i_q\ldots i_1 0\ldots 0)} + J_{(j_q j'_k j_{q-1}\ldots j_1 j'_{k-1}\ldots j'_1)T,t}^{(i_q 0 i_{q-1}\ldots i_1 0\ldots 0)} + \ldots J_{(j_q\ldots j_1 j'_k j'_{k-1}\ldots j'_1)T,t}^{(i_q\ldots i_1 0\ldots 0)} +$$

$$+ J_{(j_q\ldots j_1 j'_{k-1} j'_k j'_{k-2}\ldots j'_1)T,t}^{(i_q\ldots i_1 0\ldots 0)} + \ldots + J_{(j_q\ldots j_1 j'_{k-1}\ldots j'_1 j'_k)T,t}^{(i_q\ldots i_1 0\ldots 0)} \qquad (82)$$

w. p. 1.

Combining (81) and (82) we finally obtain w. p. 1

$$\sum_{(j_1,\ldots,j_q)} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}\times$$

$$\times \sum_{(j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)} =$$

$$= \sum_{(j_1,\ldots,j_q,j_1',\ldots,j_k')} \int\limits_t^T \phi_{j_q}(t_q)\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k'}(t_k')\ldots \int\limits_t^{t_2'} \phi_{j_1'}(t_1')\times$$

$$\times d\mathbf{w}_{t_1'}^{(0)}\ldots d\mathbf{w}_{t_k'}^{(0)} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_q}^{(i_q)}.$$

The equality (66) is proved for $n = k$. The equality (66) is proved. Theorem 4 is proved.

Let us consider the following theorem.

Theorem 5. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_2([t,T])$. Then the following representation*

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=~i_{g_{2s}}\neq 0\}} \mathbf{1}_{\{j_{g_{2s-1}}=~j_{g_{2s}}\}} \prod_{l=1}^{k-2r} \zeta_{j_{q_l}}^{(i_{q_l})} \qquad (83)$$

is valid w. p. 1, where $i_1,\ldots,i_k = 0,1,\ldots,m$, $J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}$ is defined by (19), $[x]$ is an integer part of a real number x, $\prod\limits_{\emptyset} \stackrel{\rm def}{=} 1$, $\sum\limits_{\emptyset} \stackrel{\rm def}{=} 0$; another notations are the same as in Theorems 1, 2.

Remark 3. *It should be noted that the formulas* (29), (64), (67), (68) *follow from* (83). *It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula* (83) *equal to* 0 *or* 1.

Proof. The proof of Theorem 5 is carried out by induction using the following recurrence relation

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)} = J''[\phi_{j_k}]_{T,t}^{(i_k)} \cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})} -$$

$$-\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\ne 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})}\quad \text{w. p. 1.}\quad (84)$$

Let us prove the recurrence relation (84). Using iteratively the Itô formula, the orthonormality of $\{\phi_j(x)\}_{j=0}^{\infty}$, as well as (76) and combinatorial reasoning, we obtain w. p. 1 (see Remark 4 below for details)

$$J''[\phi_{j_k}]_{T,t}^{(i_k)}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-1})}=$$

$$=\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_{k-1})}\int\limits_t^T\phi_{j_k}(\theta)d\mathbf{w}_{\theta}^{(i_k)}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=\sum_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\sum_{(j_1,\ldots,j_{k-1})}\Biggl(\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}+$$

$$+\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-3}}(t_{k-3})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}+\ldots$$

$$\ldots+\mathbf{1}_{\{i_k=i_1\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_k}(\theta)\phi_{j_1}(\theta)\times$$

$$\times d\mathbf{w}_{\theta}^{(0)} d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\Biggr) =$$

$$= \sum_{(j_1,\ldots,j_k)} \int\limits_t^T \phi_{j_k}(t_k) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} +$$

$$+ \sum_{(j_1,\ldots,j_{k-2})} \mathbf{1}_{\{i_k=i_{k-1}\neq 0\}} \Biggl\{ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta) \int\limits_t^{\theta} \phi_{j_{k-2}}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})} d\mathbf{w}_{\theta}^{(0)} + \ldots$$

$$\ldots + \int\limits_t^T \phi_{j_{k-2}}(t_{k-2}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta) d\mathbf{w}_{\theta}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})} \Biggr\} +$$

$$+ \sum_{(j_1,\ldots,j_{k-3},j_{k-1})} \mathbf{1}_{\{i_k=i_{k-2}\neq 0\}} \Biggl\{ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta) \int\limits_t^{\theta} \phi_{j_{k-1}}(t_{k-1}) \int\limits_t^{t_{k-1}} \phi_{j_{k-3}}(t_{k-3}) \ldots$$

$$\ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})} d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} d\mathbf{w}_{\theta}^{(0)} + \ldots$$

$$\ldots + \int\limits_t^T \phi_{j_{k-1}}(t_{k-1}) \int\limits_t^{t_{k-1}} \phi_{j_{k-3}}(t_{k-3}) \ldots \int\limits_t^{t_2} \phi_{j_1}(t_1) \int\limits_t^{t_1} \phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta) \times$$

$$\times d\mathbf{w}_{\theta}^{(0)} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})} d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} \Biggr\} + \ldots$$

$$\ldots + \sum_{(j_2,\ldots,j_{k-1})} \mathbf{1}_{\{i_k=i_1\neq 0\}} \Biggl\{ \int\limits_t^T \phi_{j_k}(\theta)\phi_{j_1}(\theta) \int\limits_t^{\theta} \phi_{j_{k-1}}(t_{k-1}) \ldots \int\limits_t^{t_3} \phi_{j_2}(t_2) \times$$

$$\times d\mathbf{w}_{t_2}^{(i_2)} \ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})} d\mathbf{w}_{\theta}^{(0)} + \ldots$$

$$\ldots+\int\limits_t^T \phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_k}(\theta)\phi_{j_1}(\theta)d\mathbf{w}_{\theta}^{(0)}d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}\Biggr\}=$$

$$=\sum\limits_{(j_1,\ldots,j_k)}\int\limits_t^T\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}+$$

$$+\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)d\theta\sum\limits_{(j_1,\ldots,j_{k-2})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}+$$

$$+\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta)d\theta\sum\limits_{(j_1,\ldots,j_{k-3},j_{k-1})}\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\int\limits_t^{t_{k-1}}\phi_{j_{k-3}}(t_{k-3})\ldots$$

$$\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-3}}^{(i_{k-3})}d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}+\ldots$$

$$\ldots+\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_1}(\theta)d\theta\sum\limits_{(j_2,\ldots,j_{k-1})}\mathbf{1}_{\{i_k=i_1\ne 0\}}\int\limits_t^T\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_3}\phi_{j_2}(t_2)\times$$

$$\times d\mathbf{w}_{t_2}^{(i_2)}\ldots d\mathbf{w}_{t_{k-1}}^{(i_{k-1})}=$$

$$=J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\mathbf{1}_{\{j_k=j_{k-1}\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-2}}]_{T,t}^{(i_1\ldots i_{k-2})}+$$

$$+\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\mathbf{1}_{\{j_k=j_{k-2}\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{k-3}}\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{k-3}i_{k-1})}+\ldots$$

$$\ldots+\mathbf{1}_{\{i_k=i_1\ne 0\}}\mathbf{1}_{\{j_k=j_1\}}\cdot J''[\phi_{j_2}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_2\ldots i_{k-1})}=$$

$$=J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}+$$

$$+\sum_{l=1}^{k-1}\mathbf{1}_{\{i_l=i_k\ne 0\}}\mathbf{1}_{\{j_l=j_k\}}\cdot J''[\phi_{j_1}\ldots\phi_{j_{l-1}}\phi_{j_{l+1}}\ldots\phi_{j_{k-1}}]_{T,t}^{(i_1\ldots i_{l-1}i_{l+1}\ldots i_{k-1})}. \quad (85)$$

The equality (84) is proved. Theorem 5 is proved.

Remark 4. *It should be noted that the sums with respect to permutations*

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (85), *containing the expressions*

$$\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\ne 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta),$$

should be understood in a special way. Let us explain this rule on following sum

$$\sum_{(j_1,\ldots,j_{k-1})}\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\int\limits_t^T\phi_{j_k}(\theta)\phi_{j_{k-1}}(\theta)\int\limits_t^{\theta}\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{k-2}}^{(i_{k-2})}d\mathbf{w}_{\theta}^{(0)}. \quad (86)$$

More precisely, permutations $(j_1,\ldots,j_{k-1})$ *when summing in* (86) *are performed in such a way that if* j_r *swapped with* j_d *in the permutation* $(j_1,\ldots,j_{k-1})$, *then* i_r *swapped with* i_d *in the permutation* $(i_1,\ldots,i_{k-2},i_{k-1})$ (*note that* $i_{k-1}=0$). *Moreover,* $\bar{\phi}_{j_r}$ *swapped with* $\bar{\phi}_{j_d}$ *in the permutation*

$$\left(\bar{\phi}_{j_1},\ldots,\bar{\phi}_{j_{k-1}}\right)=\left(\phi_{j_1},\ldots,\phi_{j_{k-2}},\ \mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\cdot\phi_{j_k}\cdot\phi_{j_{k-1}}\right),$$

where $\bar{\phi}_{j_{k-1}}(\tau)=\mathbf{1}_{\{i_k=i_{k-1}\ne 0\}}\phi_{j_k}(\tau)\phi_{j_{k-1}}(\tau)$.

A similar rule should be applied to all other sums with respect to permutations

$$\sum_{(j_1,\ldots,j_{k-1})}$$

in (85) *that contain the expressions*

$$\mathbf{1}_{\{i_k=i_{k-2}\ne 0\}}\phi_{j_k}(\theta)\phi_{j_{k-2}}(\theta),\ldots,\mathbf{1}_{\{i_k=i_1\ne 0\}}\phi_{j_k}(\theta)\phi_{j_1}(\theta).$$

3 Main Results

3.1 Generalizations of Theorem 2 to the Case of an Arbitrary Complete Orthonormal Systems of Functions in the Space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$

Theorems 3–5 imply the following two theorems on expansion of iterated Itô stochastic integrals (2).

Theorem 6 [10], [15]. *Suppose that the condition* $(\star)$ *is fulfilled for the multi-index* $(i_1 \ldots i_k)$ *(see Sect.* 2.2*) and the condition* (27) *is also fulfilled. Furthermore, let* $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \times$$

$$\times \prod_{l=1}^{k} \left(\mathbf{1}_{\{m_l=0\}} + \mathbf{1}_{\{m_l>0\}} \begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right) \ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if} \quad i_l \neq 0 \\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}} \ldots \left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if} \quad i_l = 0 \end{cases} \right) \tag{87}$$

converging in the mean-square sense is valid, where $H_n(x)$ *is the Hermite polynomial* (28)*,* $\mathbf{1}_A$ *is the indicator of the set* A*,* $i_1, \ldots, i_k = 0, 1, \ldots, m$*;* $n_{1,l} + n_{2,l} + \ldots + n_{d_l,l} = m_l$*;* $n_{1,l}, n_{2,l}, \ldots, n_{d_l,l} = 1, \ldots, m_l$*;* $d_l = 1, \ldots, m_l$*;* $l = 1, \ldots, k$*;* $m_1 + \ldots + m_k = k$*; the numbers* $m_1, \ldots, m_k$*,* $g_1, \ldots, g_k$ *depend on* $(i_1, \ldots, i_k)$ *and the numbers* $n_{1,l}, \ldots, n_{d_l,l}$*,* $h_{1,l}, \ldots, h_{d_l,l}$*,* d_l *depend on* $\{j_1, \ldots, j_k\}$*; moreover,* $\{j_{g_1}, \ldots, j_{g_k}\} = \{j_1, \ldots, j_k\}$*;*

$$\zeta_j^{(i)} = \int_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)} \quad (i = 0, 1, \ldots, m; \quad j = 0, 1, 2, \ldots)$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*) and* $d\mathbf{w}_\tau^{(0)} = d\tau$*; another notations as in Theorems* 1, 2.

Theorem 7 [10], [15]. *Suppose that* $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr) \qquad (88)$$

converging in the mean-square sense is valid, where $[x]$ *is an integer part of a real number* x; *another notations are the same as in Theorems* 1, 2, 5.

3.2 Modifications of Theorems 6, 7 for the Case of an Arbitrary Complete Orthonormal Systems of Functions in the Space $L_2([t,T])$ and $\Phi(t_1,\ldots,t_k) \in L_2([t,T])$.

Replacing the function $K(t_1,\ldots,t_k)$ of the form (4) in Theorems 6, 7 by the function $\Phi(t_1,\ldots,t_k) \in L_2([t,T])$, we get the following two theorems.

Theorem 8 [10], [15]. *Suppose that the condition* $(\star)$ *is fulfilled for the multi-index* $(i_1\ldots i_k)$ *(see Sect.* 2.2*) and the condition* (27) *is also fulfilled. Furthermore, let* $\Phi(t_1,\ldots,t_k) \in L_2([t,T])$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$. *Then the following expansion*

$$J''[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\prod_{l=1}^{k}\left(\mathbf{1}_{\{m_l=0\}}+\mathbf{1}_{\{m_l>0\}}\begin{cases} H_{n_{1,l}}\left(\zeta_{j_{h_{1,l}}}^{(i_l)}\right)\ldots H_{n_{d_l,l}}\left(\zeta_{j_{h_{d_l,l}}}^{(i_l)}\right), & \text{if}\quad i_l\neq 0\\ \\ \left(\zeta_{j_{h_{1,l}}}^{(0)}\right)^{n_{1,l}}\ldots\left(\zeta_{j_{h_{d_l,l}}}^{(0)}\right)^{n_{d_l,l}}, & \text{if}\quad i_l=0\end{cases}\right) \qquad (89)$$

converging in the mean-square sense is valid, where the sum of iterated Itô stochastic integrals $J''[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (19),

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} \Phi(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k \tag{90}$$

is the Fourier coefficient, $H_n(x)$ *is the Hermite polynomial* (28), $\mathbf{1}_A$ *is the indicator of the set* A, $i_1,\ldots,i_k = 0,1,\ldots,m;$ $n_{1,l}+n_{2,l}+\ldots+n_{d_l,l}=m_l;$ $n_{1,l},n_{2,l},\ldots,n_{d_l,l}=1,\ldots,m_l;$ $d_l=1,\ldots,m_l;$ $l=1,\ldots,k;$ $m_1+\ldots+m_k=k;$ *the numbers* $m_1,\ldots,m_k,$ $g_1,\ldots,g_k$ *depend on* $(i_1,\ldots,i_k)$ *and the numbers* $n_{1,l},\ldots,n_{d_l,l},$ $h_{1,l},\ldots,h_{d_l,l},$ d_l *depend on* $\{j_1,\ldots,j_k\}$*; moreover,* $\{j_{g_1},\ldots,j_{g_k}\}=\{j_1,\ldots,j_k\};$

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau)d\mathbf{w}_\tau^{(i)} \quad (i=0,1,\ldots,m;\quad j=0,1,2,\ldots)$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*) and* $d\mathbf{w}_\tau^{(0)}=d\tau$.

Theorem 9 [10], [15]. *Suppose that* $\Phi(t_1,\ldots,t_k)\in L_2([t,T]^k)$ *and* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary complete orthonormal system of functions in the space* $L_2([t,T])$*. Then the following expansion*

$$J''[\Phi]_{T,t}^{(i_1\ldots i_k)} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + \sum_{r=1}^{[k/2]}(-1)^r\times$$

$$\times \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r}\mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}}\prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr)$$

converging in the mean-square sense is valid, where the sum of iterated Itô stochastic integrals $J''[\Phi]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (19)*, the Fourier coefficient* $C_{j_k\ldots j_1}$ *has the form* (90)*; another notations are the same as in Theorems* 5, 8.

4 Comparison with Other Results and Conclusions

Before starting this section, we recall that the sum of iterated Itô stochastic integrals (19), which plays a central role in the proofs of Theorems 6–9, is equal w. p. 1 to the multiple Wiener stochastic integral with respect to the components of a multidimensional Wiener process (see the proof in [10], Sect. 1.11).

It should be noted that an analogue of Theorem 8 (more precisely, the expansion like (89) for the case $i_1, \ldots, i_k = 1, \ldots, m$) was obtained in [39]. The mentioned expansion is formulated in [39] using the multiple Wiener stochastic integral and the Wick product. Also note that the proof in [39] is different from the proof given in this article. Let us describe these differences.

In [39], the author interprets the multiple Wiener stochastic integral from a finite-dimensional kernel $K_{p,\ldots,p}(t_1, \ldots, t_k)$ of the form (6) as a linear operator and proves that this operator is bounded. We note that the proof from [39] is essentially based on Theorem 3.1 from [40].

In our proof of Theorems 6–9 we use the sum of iterated Itô stochastic integrals (19) several times and do not explicitly use the multiple Wiener stochastic integral. Moreover, our proof of Theorems 6–9 is based on the Itô formula and does not use Theorem 3.1 from [40]. The methodology of our proof is a direct development of the approach we used to prove Theorem 5.1 in [6] (2006).

Note that the results of [39], as well as the results of this article, are based on our idea [1] (2006) on the expansion of the kernel (4) (or $\Phi(t_1, \ldots, t_k) \in L_2([t, T]^k)$) into a generalized multiple Fourier series (see [1], Chapter 5, Theorem 5.1, pp. 235-245 or [10], Chapter 1 for details).

We also note a number of works [40], [42]-[46] in which the properties of multiple Wiener stochastic integrals were studied using measure theory, in particular, the formulas for the product of such integrals were obtained.

First of all, let us compare Theorem 5 with Proposition 5.1 from [43]. An analogue of the right-hand side of (83) for nonrandom $x_1, \ldots, x_k$ is constructed in [43] using diagrams (see the formula (5.1) in [43]). This means that the application of the formula (5.1) from [43], unlike the formula (83), is difficult when performing algebraic transformations.

Further, we note that the formula (5.1) from [43] was applied to the representation of the multiple Wiener stochastic integral somewhat differently than the formula (83). Namely, using Proposition 5.1 [43]. Let us expain this difference in more detail.

Proposition 5.1 from [43] in our degree of generality and in our notations can be written as

$$J''[\phi_{j_1}\ldots\phi_{j_k}]_{T,t}^{(i_1\ldots i_k)}=$$

$$=J''\Biggl[\underbrace{\phi_{j_1}\ldots\phi_{j_1}}_{m_1}\underbrace{\phi_{j_2}\ldots\phi_{j_2}}_{m_2}\ldots\underbrace{\phi_{j_p}\ldots\phi_{j_p}}_{m_p}\Biggr]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1}\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2}\ldots\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}=$$

$$=J''[\phi_{j_1}\ldots\phi_{j_1}]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})}\cdot J''[\phi_{j_2}\ldots\phi_{j_2}]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})}\cdot\ldots\cdot J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})} \tag{91}$$

w. p. 1, where

$$J''[\phi_{j_1}\ldots\phi_{j_1}]_{T,t}^{(\overbrace{i_1\ldots i_{m_1}}^{m_1})},\ J''[\phi_{j_2}\ldots\phi_{j_2}]_{T,t}^{(\overbrace{i_{m_1+1}\ldots i_{m_2}}^{m_2})},\ldots,\ J''\left[\phi_{j_p}\ldots\phi_{j_p}\right]_{T,t}^{(\overbrace{i_{m_1+\ldots+m_{p-1}+1}\ldots i_k}^{m_p})}$$

are defined by the right-hand side of the formula (5.1) from [43], $m_1+\ldots+m_p=k$, $m_1,\ldots,m_p>0$, $j_q\neq j_d$ $(q\neq d,\ q,d=1,\ldots,p)$, $i_1,\ldots,i_k=1,\ldots,m.$

This actually means that in [43] an analogue of the formula (83) is constructed for the special case $j_1=\ldots=j_k$. Moreover, the specified analogue is based on the formula (5.1) [43] obtained using diagrams.

Comparing the formulas (83) and (91) (or (5.1) from [43]), it is easy to understand that the transition from (83) and (91) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (83) equal to 0 or 1. The reverse transition from the formula (91) to the formula (83) is not obvious. Note that the formula (83) (not the formula (91)) is convenient for the numerical integration of Itô stochastic differential equations (see [10], Chapter 5 for details).

Let us turn to the comparison of Theorem 5 with another interesting work [46] (2019). As it turned out, a version of Theorem 5 was obtained in terms of Wick polynomials and for the case of vector valued random measures in [46] (see Theorem 7.2, p. 69). However, much earlier the formula (83) (Theorem 5) is obtained in our monograph [8] (2009) as part of the formula (5.30) (see [8],

p. 220). Moreover, particular cases of the formula (83) were obtained even earlier in our works [6] (2006) and [7] (2007). More precisely, particular cases $k = 1, \ldots, 5$ of the formula (83) were obtained in [6] (2006) as parts of the formulas on the pages 243-244 and partiular cases $k = 1, \ldots, 7$ of the formula (83) were obtained in [7] (2007) as parts of the formulas on the pages 208-218.

We also note that we have found an explicit expression for the Wick polynomial of degree k of the arguments $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$ (see the formula (83)), which is very convenient for the numerical simulation of iterated Itô stochastic integrals (2) [13]. Note that the representation of the Wick polynomial of the arguments $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$ in terms of the product of Hermite polynomials is less convenient for the numerical simulation of iterated Itô stochastic integrals (2). For example, the expression for $J''[\phi_{j_1}\phi_{j_2}\phi_{j_3}\phi_{j_4}]_{T,t}^{(i_1 i_2 i_3 i_4)}$ in terms of the product of Hermite polynomials, even under the condition $i_1 = i_2 = i_3 = i_4$, already contains 15 different expressions (see [10], Sect. 1.10). At the same time, all these 15 expressions are contained in one formula (83) provided that $k = 4$ and $i_1 = i_2 = i_3 = i_4$. It is very convenient, since in computer simulation using the formula (83), in addition to modeling of random variables $\zeta_{j_1}^{(i_1)}, \ldots, \zeta_{j_k}^{(i_k)}$, it remains only to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (83) equal to 0 or 1.

It should be noted that in [44] (Theorem 6.1) a diagram formula was obtained for the product of two multiple Wiener stochastic integrals with respect to vector valued random measures. The formula (65) can be derived from the diagram formula [44]. Although the proof of the diagram formula [44] is much more complicated than our proof of the formula (65).

To conclude this article, we say a few words about expansions (87) and (88). The transition from the expansion (88) to the expansion (87) is obvious. It is only necessary to set the values of the corresponding indicators of the form $\mathbf{1}_A$ from the formula (88) equal to 0 or 1. The reverse transition from the formula (87) to the formula (88) is also possible but not obvious. However, Theorems 4 and 5 provide a transition from (87) to (88) and vice versa. Note that the expansion (87) is interesting from the point of view of studying the structure of the expansion of iterated Itô stochastic integrals. On the other hand, the expansion (88) is exceptionally convenient for applications [13], [14]. For example, in [13], [14], approximations of iterated Itô stochastic integrals of multiplicities 1 to 6 in the Python programming language were successfully implemented using (88) ($k = 1, \ldots, 6$) and Legendre polynomials.

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