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Ordinary differential equations

Boundedness and stability of solutions to certain second order differential equations

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Abstract

In this paper, the behaviour of solutions of certain second order nonlinear non-autonomous differential equations is considered. By employing the Lyapunov's second method, a suitable complete Lyapunov function is constructed and used to establish sufficient conditions that guarantee existence of solutions that are periodic, uniformly asymptotically stable and uniformly ultimately bounded. Obtained results are not only new but also include many outstanding recent results in the literature.

Keywords: Second order; Nonlinear differential equation; Uniform stability; Uniform ultimate boundedness; Periodic solutions

1 Introduction

The investigation of the problems of qualitative behaviour of solutions such as stability, boundedness, convergence, periodicity, to mention few, is an important subject in the theory of ordinary differential equations. In this regards up till today, Lyapunovs direct (or second) method is the most effective method when dealing with these problems. Of course, when one applies this method, finding a suitable complete Lyapunov function in general is always a big challenge. The major advantage of this method is that the behaviour of solutions of the equation in question can be obtained without any prior knowledge of solutions.

So far, there have been many results about the qualitative behavior of solutions of nonlinear differential equations see for instance the books of Yoshizawa [11, 12] which contain general results on the subject matters other eminent authors that have contributed immensely to the study of stability, boundedness, asymptotic behaviour, existence and uniqueness of solutions of second order ordinary differential equations include Alaba and Ogundare [1], Grigoryan [2], Kroopnick [3], Ogundare and Afuwape [4], Ogunadare and Okecha [5], Tunç [6] - [9], Yoshizawa [10] and the references cited therein.

Meanwhile, in 2011, Kroopnick [3], discussed conditions under which all solutions of the second order differential equation

$$x'' + q(t)b(x) = f(t)$$

are bounded on $\mathbb{R}^+ = [0, \infty)$. The results obtained are the generalizations of the linear case.
In 2013, Grigoryan [2] established criteria for boundedness and stability for the ordinary differential equation of the form

$$\phi''(t) + p(t)\phi'(t) + q(t)\phi(t) = 0, \qquad t \geq t_0.$$

Recently in 2014, Ogundare and Afuwape [4] studied conditions which guarantee boundedness and stability properties of solutions of generalized Lienard equations

$$x'' + f(x)x' + g(x) = p(t, x, x').$$

Furthermore, in [7] Tunç discussed boundedness of solutions to the second order ordinary differential equation

$$x'' + c(t, x, x') + q(t)b(x) = f(t).$$

Finally, Alaba and Ogundare [1] gave conditions for asymptotic behaviour of solutions of certain second order non-autonomous nonlinear ordinary differential equation

$$x'' + a(t)f(x, x')x' + b(t)g(x) = p(t, x, x').$$

Most of these works were done by constructing suitably Lyapunov functions except in [3] where the integral test was used.

However, the problem of stability, boundedness and existence of periodic solutions of second order nonlinear non-autonomous ordinary differential equation

$$[\phi(x(t))x'(t)]' + g(t, x(t), x'(t))x'(t) + \varphi(t)h(x(t)) = p(t, x(t), x'(t)), \tag{1.1}$$

is yet to be considered. Setting $x'(t) = \phi^{-1}(x(t))y(t)$, $\phi(x(t)) \neq 0$, Eq. (1.1) is equivalent to system of first order ordinary differential equations

$$\begin{aligned} x'(t) &= \phi^{-1}(x(t))y(t),\\ y'(t) &= -\phi^{-1}(x(t))y(t)g(t,x(t),\phi^{-1}(x(t))y(t)) - \varphi(t)h(x(t))\\ &\quad + p(t,x(t),\phi^{-1}(x(t))y(t)), \end{aligned} \tag{1.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$; $\phi, h : \mathbb{R} \to \mathbb{R}$; $g, p : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions in their respective argument and the derivatives $\phi'(x(t))$, $g_t(t, x(t), \phi^{-1}(x(t))y(t))$, $g_x((t, x(t), \phi^{-1}(x(t))y(t)))$, $g_y(t, x(t), \phi^{-1}(x(t))y(t))$ and φ' exist and are continuous for all values of $t \geq 0, x$ and y. Motivation for this work comes from the papers in [1], [4] and [7], where stability, boundedness and asymptotic behaviour of solutions of second order ordinary differential equation were proved.

2 Preliminaries

Consider the system of the form

$$X'(t) = F(t, X(t)) \tag{2.1}$$

where $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^n$ is the $n-$dimensional Euclidean space.

Definition 1 *A solution $X(t; t_0, X_0)$ of Eq.* (2.1) *is bounded, if there exists a $\beta > 0$ such that $\|X(t; t_0, X_0)\| < \beta$ for all $t \geq t_0$ where β may depend on each solution.*

Definition 2 *The solutions $X(t; t_0, X_0)$ of Eq.* (2.1) *are uniformly bounded, if for any $\alpha > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\beta(\alpha) > 0$ such that if $\|X_0\| < \alpha$ $\|X(t; t_0, X_0)\| < \beta$ for all $t \geq t_0$.*

Definition 3 *The solutions of Eq.* (2.1) *are uniformly ultimately bounded for bound B if there exists a $B > 0$ and if corresponding to any $\alpha > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T(\alpha) > 0$ such that if $\|X_0\| < \alpha$ implies that $\|X(t; t_0, X_0)\| < B$ for all $t \geq t_0 + T(\alpha)$.*

Definition 4 *(i) A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, strictly increasing with $\phi(0) = 0$, is said to be a function of class $\mathbb{K}$ for such function, we shall write $\phi \in \mathbb{K}$.*
(ii) If in addition to (i) $\phi(r) \to +\infty$ as $r \to \infty$, ϕ is said to be a function of class $\mathbb{K}^*$ and we write $\phi \in \mathbb{K}^*$.

If $F(t, 0) = 0$, and $F \in C(\mathbb{R}^+ \times D, \mathbb{R}^n)$, D is an open set in $\mathbb{R}^n$, in Eq. (2.1) we have the following definitions

Definition 5 [12] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is stable, if for any $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$, there exists a $\delta(t_0, \epsilon) > 0$ such that $\|X_0\| < \delta(t_0, \epsilon)$ implies

$$\|X(t, t_0, X_0)\| < \epsilon \ \text{ for all } t \geq t_0$$

where $X(t, t_0, X_0)$ denotes the solution of Eq. (2.1) through the point (t_0, X_0).

Definition 6 [12] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is uniformly stable, if the δ in Definition 5 is independent of t_0.

Definition 7 [12] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is asymptotically stable, if it is stable, and if there exists a $\delta(t_0) > 0$ such that $\|X_0\| < \delta_0(t_0)$ implies that

$$\|X(t, t_0, X_0)\| \to 0 \text{ as } t \to \infty.$$

Definition 8 [11] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is quasi-equiasymptotically stable, if given any $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$, there exist a $\delta_0(t_0) > 0$ and a $T(t_0, \epsilon) > 0$ such that if $\|X_0\| < \delta_0(t_0)$,

$$\|X(t, t_0, X_0)\| < \epsilon \text{ for all } t \geq t_0 + T(t_0, \epsilon)$$

Definition 9 [12] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is quasi-uniformly asymptotically stable, if the δ_0 and the T in Definition 8 are independent of t_0.

Definition 10 [12] The zero solution $X(t) \equiv 0$ of Eq. (2.1) is uniformly asymptotically stable, if it is uniformly stable and is quasi-uniformly asymptotically stable.

The following lemmas are very important in the proofs of our results.

Lemma 1 *[10] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $R^+, \|X\| < H$ which satisfies the following conditions:*

(i) $V(t, 0) \equiv 0$;

(ii) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, *$a, b$ are continuous and increasing;*

(iii) $V_{(2.1)}(t, X) \leq -c(\|X\|)$ *for all* $(t, X) \in \mathbb{R}^+ \times D$.

Then the trivial solution $X(t) \equiv 0$ of Eq. (2.1) is uniformly asymptotically stable.

Lemma 2 *[10, 11] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+$, $\|X\| \geq R$, where R may be large, which satisfies the following conditions:*

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, where $a(r)$ and $b(r)$ are continuous and increasing and $a(r) \to \infty$ as $r \to \infty$;

(ii) $V'_{(2.1)}(t, X) \leq -c(\|X\|)$, where $c(r)$ is positive and continuous,

then solutions of Eq. (2.1) are uniformly ultimately bounded.

Lemma 3 *[10, 11] If there exists a Lyapunov function satisfying the condition of Lemma 2, then Eq. (2.1) has at least a periodic solution of period ω.*

3 Main Results

We shall use the following notations. Let $x(t) = x$, $y(t) = y$, $g(t, x(t), \phi^{-1}(x(t))y(t)) = g(\cdot)$ and $p(t, x(t), \phi^{-1}(x(t))y(t)) = p(\cdot)$. First, we shall consider the case when $p(t, x, x') = 0 = p(\cdot)$ so that equations (1.1) and (1.2) become

$$[\phi(x(t))x'(t)]' + g(t, x(t), x'(t))x'(t) + \varphi(t)h(x(t)) = 0 \tag{3.1}$$

and

$$\begin{aligned} x'(t) &= \phi^{-1}(x(t))y(t), \\ y'(t) &= -\phi^{-1}(x(t))y(t)g(t, x(t), \phi^{-1}(x(t))y(t)) - \varphi(t)h(x(t)), \end{aligned} \tag{3.2}$$

respectively where the functions ϕ, g, h and φ are defined in Section 1. Let $(x(t), y(t))$ be any solution of (3.2), the continuously differentiable function employed in the proof of our results is the function $V = V(t, x, y)$ defined as

$$2V = 2\varphi(t)\phi(x)\int_0^x h(s)ds + (a^2 + b^2\phi^2(x))x^2 + (b\phi(x) + 1)y^2 + 2xyg(\cdot), \tag{3.3}$$

where a and b are positive constants. We have the following result.

Theorem 1 *Further to the basic assumptions of the functions ϕ, φ, g and h, suppose that $a, b, \phi_0, \phi_1, \varphi_0, \varphi_1, A, B$ are positive constants such that* *(i)* $\varphi_0 \leq \varphi(t) \leq \varphi_1$ *for all* $t \geq 0$, $\phi_0 \leq \phi(x) \leq \phi_1$ *for all* x;

(ii) $b \leq \frac{h(x)}{x} \leq B$ $x \neq 0$;

(iii) $a \leq g(\cdot) \leq A$ *for all* $t \geq 0$, x, y; *and*

(iv) $\varphi'(t) \leq 0$ *for all* $t \geq 0$, $xyg_t(\cdot) \leq 0$, $xg_x(\cdot) \leq 0$, $\phi'(x) \geq 0$, $xyg_y(\cdot) \geq 0$ *for all* $t \geq 0, x, y$.

Then the trivial solution of (3.2) *is uniformly asymptotically stable.*

Remark 1 *(i) When* $[\phi'(x)x' + g(\cdot)]x' = ax'$, $\varphi(t)h(x) = b$, $\phi(x) = 1$ *and* $p(\cdot) = 0$ (1.1) *reduces to a linear constant coefficients differential equation and hypotheses (i) to (iv) of Theorem 1 reduce to the corresponding Routh-Hurwitz criterion* $a > 0$, $b > 0$.

(ii) If $\phi(x)x'' = \phi''(t)$, $[\phi'(x)x' + g(\cdot)]x' = p(t)\phi'(t)$ *and* $\varphi(t)h(x) = q(t)\phi(t)$ (3.2) *specializes to linear ordinary differential equation of the second order discussed in [2].*

(iii) Whenever $\phi(x) = 1 = \varphi(t)$, $g(\cdot) = f(x)$, $h(x) = g(x)$ *and* $p(\cdot) = 0$ (3.2) *becomes the second order autonomous differential equation discussed in [4].*

(iv) When $\phi(x) = 1$, $\phi'(x) = 0$, $g(\cdot) = a(t)f(x, y)$ *and* $p(\cdot) = 0$ (3.2) *reduces to the second order non autonomous differential equation studied in [1].*

(v) Thus, the result of Theorem 1 includes and extends the stability results discussed in [1, 2] and [4].

Next, we shall state and prove a result that would be useful in the proof of Theorem 1 and the preceding results.

Lemma 4 *Under the hypotheses of Theorem 1, there exist constants* $D_0 = D_0(a, b, \phi_0, \varphi_0) > 0$ *and* $D_1 = D_1(a, b, \phi_1, \varphi_1, A, B) > 0$ *such that*

$$D_0(x^2(t) + y^2(t)) \leq V(t, x(t), y(t)) \leq D_0(x^2(t) + y^2(t)), \tag{3.4}$$

for all $t \geq 0, x$ *and* y. *Furthermore, there exists a constant* $D_2 = D_2(a, b, \varphi_0) > 0$ *such that*

$$\left.\frac{dV}{dt}\right|_{(3.2)} = V'_{(3.2)}(t, x, y) \leq -D_2(x^2(t) + y^2(t)) \tag{3.5}$$

for all $t \geq 0, x$ *and* y.

Proof 1 (Proof) *Let* $(x(t), y(t))$ *be any solution of* (3.2), *if* $x(t) = 0 = y(t)$, *it follows from* (3.3) *that*

$$V(t, 0, 0) = 0. \tag{3.6}$$

From the hypotheses of Theorem 1 we have $\varphi(t) \geq \varphi_0$ *for all* $t \geq 0$, $\phi(x) \geq \phi_0$ *for all* x, $h(x) \geq bx$, $x \neq 0$ *and* $g(\cdot) \geq a$ *for all* $t \geq 0, x$ *and* y *so that the function* V *defined in* (3.3) *becomes*

$$V \geq (ax + y)^2 + b\phi_0(b\phi_0 + \varphi_0)x^2 + b\phi_0 y^2 \geq \delta_0(x^2 + y^2) \tag{3.7}$$

for all $t \geq 0, x$ *and* y, *where*

$$\delta_0 := \frac{1}{2} \min\left\{ \min\{a, 1\} + b\phi_0(b\phi_0 + \varphi_0),\ \min\{a, 1\} + b\phi_0 \right\}.$$

From the inequality (3.7), *we observed that* $V(t, x, y) = 0$ *if and only if* $x^2 + y^2 = 0$ *and* $V(t, x, y) > 0$ *if and only if* $x^2 + y^2 \neq 0$, *it follows that*

$$V(t, x, y) \to +\infty \quad as \quad x^2 + y^2 \to \infty. \tag{3.8}$$

Moreover, applying the upper estimates, defined in Theorem 1, for each of the functions $\varphi(t)$, $\phi(x)$, $h(x)$ *and* $g(\cdot)$, (3.3) *yields*

$$V \leq (a^2 + b^2\phi_1^2 + B\varphi_1\phi_1 + A)x^2 + (1 + A + b\phi_1)y^2 \leq \delta_1(x^2 + y^2) \tag{3.9}$$

for all $t \geq 0, x$ *and* y *where*

$$\delta_1 := \max\{a^2 + b^2\phi_1^2 + B\varphi_1\phi_1 + A,\ 1 + A + b\phi_1\}.$$

Hence, from estimates (3.7) *and* (3.9) *the inequality* (3.4) *is established with* $\delta_0 = D_0$ *and* $\delta_1 = D_1$ *respectively.*

Next, we shall establish the inequality (3.5). *To see this, let* $(x(t), y(t))$ *be any solution of* (3.2), *the derivative of the function* V *defined in* (3.3), *with respect to the independent variable* t *along the solution path of* (3.2) *after simplification is*

$$V'_{(3.2)} = \sum_{i=1}^{3} U_i - U_4 - \left[b\phi(x)\left(\varphi(t)\frac{h(x)}{x} - b\right) + \phi^{-1}(x)\left(g^2(\cdot) - a^2\right)\right] xy \tag{3.10}$$

where:

$$U_1 := \varphi'(t)\phi(x)\int_0^x h(s)ds + xyg_t(\cdot);$$

$$U_2 := y\phi'(x)\left[b^2x^2 + \frac{1}{2}b\phi^{-1}(x)y^2 + \varphi(t)\phi^{-1}(x)\int_0^x h(s)ds\right];$$

$U_3 := xg_x(\cdot)\phi^{-4}(x)\phi'(x)y^4;$ *and*

$$U_4 := \left[\varphi(t)g(\cdot)\frac{h(x)}{x} + yg_y(\cdot)\varphi(t)\phi^{-1}(x)\frac{h(x)}{x}\right]x^2 + \left[xg_y(\cdot)\phi^{-1}(x) + bg(\cdot)\right]y^2.$$

Now since $\varphi'(t) \leq 0$ *for all* $t \geq 0$, $\phi(x) \geq \phi_0$ *for all* x, $h(x) \geq bx$ *for* $x \neq 0$, *and* $xyg_t(\cdot) \leq 0$ *for all* $t \geq 0, x$ *and* y, *it follows that*

$$U_1 = \varphi'(t)\phi(x)\int_0^x h(s)ds + xyg_t(\cdot) \leq 0$$

for all $t \geq 0, x$ *and* y.
Also, from the hypotheses of Theorem 1 $\phi'(x) \geq 0$ *for all* x, $\phi(x) \leq \phi_1$ *for all* x, $\varphi(t) \geq \varphi_0$ *for all* $t \geq 0$, *and* $h(x) \geq bx$ *for all* $x \neq 0$, *so that*

$$\phi'(x)\left[b^2x^2 + \frac{1}{2}b\phi^{-1}(x)y^2 + \varphi(t)\phi^{-1}(x)\int_0^x h(s)ds\right] \geq 0$$

for all $t \geq 0, x$ *and* y. *Recall* $y < 1 + y^2$ *it follows that*

$$U_2 := y\phi'(x)\left[b^2x^2 + \frac{1}{2}b\phi^{-1}(x)y^2 + \varphi(t)\phi^{-1}(x)\int_0^x h(s)ds\right] \leq 0$$

for all $t \geq 0, x$ *and* y.
Moreover, $xg_x(\cdot) \leq 0$ *for all* $t \geq 0, x, y$, $\phi(x) \leq \phi_1$ *for all* x *and* $\phi'(x) \geq 0$ *for all* x, *we have*

$$U_3 = xg_x(\cdot)\phi^{-4}(x)\phi'(x)y^4 \leq 0$$

for all $t \geq 0, x$ *and* y.
Finally, since $xyg_y(\cdot) \geq 0$ *for all* $t \geq 0, x$ *and* y *we find that*

$$U_4 \geq ab(\varphi_0x^2 + y^2)$$

for all $t \geq 0, x$ *and* y. *Using estimates* U_i $(i = 1, 2, 3, 4)$ *in* (3.10) *we obtain*

$$V'_{(3.2)} \leq -\frac{1}{2}ab(\varphi_0x^2 + y^2) - \sum_{i=5}^{6} U_i \tag{3.11}$$

where

$$U_5 := \frac{ab}{4}\left[\varphi_0x^2 + 4a^{-1}\phi(x)\left(\varphi(t)\frac{h(x)}{x} - b\right)xy + y^2\right]$$

and

$$U_6 := \frac{ab}{4}\left[\varphi_0x^2 + 4a^{-1}b^{-1}\phi^{-1}(x)\left(g^2(\cdot) - a^2\right)xy + y^2\right].$$

Employing the inequalities

$$16a^{-2}\phi^2(x)\left[\varphi(t)\frac{h(x)}{x} - b\right]^2 < \varphi_0 \quad \text{and} \quad 16a^{-2}b^{-2}\phi^{-2}(x)\left[g^2(\cdot) - a^2\right]^2 < \varphi_0$$

in U_5 and U_6 respectively we obtain,

$$U_5 \geq \frac{ab}{4}\left[\sqrt{\varphi_0}|x| - |y|\right]^2 \geq 0 \quad and \quad U_6 \geq \frac{ab}{4}\left[\sqrt{\varphi_0}|x| - |y|\right]^2 \geq 0, \tag{3.12}$$

for all $t \geq 0, x, y$. Using the inequalities (3.12) *in* (3.11)*, there exists a constant $\delta_2 > 0$ such that*

$$V'_{(3.2)} \leq -\delta_2(x^2 + y^2) \tag{3.13}$$

for all $t \geq 0, x$ and y where

$$\delta_2 := \frac{ab}{2}\min\{\varphi_0, 1\}.$$

From the inequality (3.13) *estimate* (3.5) *is established with $\delta_2 \equiv D_2$ respectively. This completes the proof of Lemma 4.*

Proof 2 (Proof of Theorem 1) *Let $(x(t), y(t))$ be any solution of system of first order Eq.* (3.2)*. From Eq.* (3.6)*, the inequalities* (3.7)*,* (3.9) *and* (3.13) *all hypotheses of Lemma 1 hold hence, by Lemma 1 the trivial solution $X(t) \equiv 0$ of* (3.2) *is uniformly asymptotically stable where $X = (x, y) \in \mathbb{R}^2$.*

Next, if $p(\cdot) \neq 0$ in Eq. (1.1) (in particular Eq. (1.2)) we have the following result.

Theorem 2 *In addition to the assumptions of Theorem 1, suppose that*

$$|p(\cdot)| \leq M, \qquad 0 < M < \infty, \tag{3.14}$$

then the solutions of (1.2) *are uniformly ultimately bounded.*

Remark 2 *(i) If $\phi(x) = 1$, $g(\cdot) = f(x)$ and $\varphi(t) = 1$. Equation* (1.1) *reduces to that discussed in [4].*

(ii) Whenever $\phi(x) = 1$, $x'(t) = 1$ and $p(\cdot) = f(t)$, (1.1) *specializes to that studied in [7].*

(iii) When $\phi(x) = 1$ and $g(\cdot) = a(t)f(x, x')$, then (1.1) *reduces to the second order non-autonomous non linear ordinary differential equation discussed in [1].*

(iv) The boundedness result presented in Theorem 2 includes and extends the results in [1], [4] and [7].

Proof 3 (Proof of Theorem 2) *Let $(x(t), y(t))$ be any solution of Eq. (1.2). The Lyapunov's function defined in Eq. (3.3) gives rise to the inequalities (3.7), (3.9) and the expression in (3.8) hold for Eq. (1.2). Furthermore, the derivative of the function V defined by Eq. (3.3) with respect to the independent variable t along the solution path of Eq. (1.2) is*

$$V'_{(1.2)} = V'_{(3.2)} + \left[x\left(yg_y(\cdot) + g(\cdot)\right) + y\left(b\phi(x) + 1\right)\right]p(\cdot). \tag{3.15}$$

From the inequality (3.13) $V'_{(3.2)} \le -\delta_2(x^2 + y^2)$ for all $t \ge 0, x, y$ and in view of hypotheses Theorem 1 we find Eq. (3.15) to be

$$V'_{(1.2)} \le -\delta_2(x^2 + y^2) + K_0(|x| + |y|)|p(\cdot)| \tag{3.16}$$

for all $t \ge 0, x$ and y where

$$K_0 := \max\left\{A,\ b\phi_1 + 1\right\}.$$

Now by estimate (3.14) and noting the fact that $(|x| + |y|)^2 \le 2(x^2 + y^2)$, it follows from the inequality (3.16) that

$$V'_{(1.2)} \le -\delta_3(x^2 + y^2) \tag{3.17}$$

for all $t \ge 0, x$ and y where $\delta_3 := \dfrac{\delta_2}{2} > 0$ provided that

$$(x^2 + y^2)^{1/2} = \|X\| \ge K_1 = 2^{3/2}K_0M\delta_2^{-1} > 0$$

for all $X = (x, y) \in \mathbb{R}^2$, where K_1 may be large. Now from the inequalities (3.7), (3.9) and (3.17) the hypotheses of Lemma 2 hold. Hence, by Lemma 2 the solutions of Eq. (1.2) are uniformly ultimately bounded.

Next, we shall state and prove a result on the periodic solutions of the system of Eq. (1.2).

Theorem 3 *If all the assumptions of Theorem 2 hold, then Eq. (1.2) has at least a periodic solution of period ω.*

Proof 4 (Proof) *Let $(x(t), y(t))$ be any solution of Eq. (1.2). Since the Lyapunov's function defined in (3.3) satisfies the assumptions of Theorem 2, which in turn satisfy the assumptions of Lemma 2. Thus by Lemma 2 and Lemma 3 equation (1.2) has at least a periodic solution of period ω. This completes the proof of Theorem 3.*

Finally, if the forcing term $p(\cdot)$ is replaced with the function $p(t)$, $p \in C[\mathbb{R}^+, \mathbb{R}]$ in Eq. (1.1) we have

$$[\phi(x(t))x'(t)]' + g(t, x(t), x'(t))x'(t) + \varphi(t)h(x(t)) = p(t), \tag{3.18}$$

and Eq. (3.18) is equivalent to system of first order ordinary differential equations

$$\begin{aligned} x'(t) &= \phi^{-1}(x(t))y(t), \\ y'(t) &= -\phi^{-1}(x(t))y(t)g(t, x(t), \phi^{-1}(x(t))y(t)) - \varphi(t)h(x(t)) + p(t). \end{aligned} \tag{3.19}$$

We have the following result.

Theorem 4 *Further to the assumptions of Theorem 1, suppose that*

$$\int_0^x |p(\mu)|d\mu \le N, \qquad 0 \le N < \infty, \tag{3.20}$$

then there exists a constant $D_4 = D_4(x_0, y_0, a, b, \phi_0, \phi_1, \varphi_0, \varphi_1) > 0$ *such that any solution* $(x(t), y(t))$ *of Eq.* (3.19) *determined by* $x(0) = x_0, y(0) = y_0$, *for* $t = 0$, *satisfies*

$$|x(t)| \le D_4, \quad |y(t)| \le D_4, \ \forall \ \ t > 0. \tag{3.21}$$

Proof 5 (Proof) *Let* $(x(t), y(t))$ *be any solution of Eq.*(3.19). *In view of the Lyapunov's function defined in Eq.* (3.3), *estimate* (3.7) *holds for Eq.* (3.19). *The derivative of the function* V *with respect to* t *along the solution path of Eq.* (3.19) *is*

$$V'_{(3.19)} = V'_{(3.2)} + \left[x\biggl(yg_y(\cdot) + g(\cdot)\biggr) + y\biggl(b\phi(x) + 1\biggr)\right]p(t). \tag{3.22}$$

Now from the inequality (3.13), $V'_{(3.2)} \le 0$ *for all* $t \ge 0, x, y$ *and by the assumptions of Theorem 1, Eq.* (3.22) *yields*

$$V'_{(3.19)} \le K_0(|x| + |y|)|p(t)|$$

for all $t \ge 0, x$ *and* y. *From the inequality* (3.7), *the fact that* $|x| < 1 + x^2$ *and* $|y| < 1 + y^2$ *we find that*

$$V'_{(3.19)} - K_2|p(t)|V \le 2K_0|p(t)| \tag{3.23}$$

for all $t \ge 0, x$ *and* y, *where* $K_2 = k_0\delta_0^{-1}$. *Solving this first order differential inequality using the integrating factor*

$$\exp\left[- K_2 \int_0^x |p(s)|ds \right],$$

together with estimate (3.20), *Eq.* (3.23) *becomes*

$$V(t,x,y) \leq (V(0)+1)e^{K_2 N} - 1 \tag{3.24}$$

where $V(0) = V(0, x_0, y_0)$ *for* $t = 0$. *Engaging estimate* (3.7) *in the right hand side of the inequality* (3.24) *we obtain*

$$|x(t)| \leq K_3, \quad |y(t)| \leq K_3$$

for all $t > 0$, *where*

$$K_3 := \left[(V(0)+1)e^{K_2 N} - 1\right]^{1/2} \delta_0^{-1/2}.$$

This establish the inequalities (3.21) *with* $K_3 = D_4$. *The proof of Theorem 4 is established.*

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