DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 2, 2018
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru

Integral equations

On a coupled system of Urysohn-Stieltjes integral equations in reflexive Banach space

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Abstract

Urysohn-Stieltjes integral operators and Urysohn-Stieltjes integral equations have been studied by some authors. In this paper we prove the existence of at least one weak solution of a coupled system of Urysohn-Stieltjes integral equations in the reflexive Banach space. We used the O'Regan fixed point theorem and some propositions. As an application, the coupled system of Hammerstien-Stieltjes integral equations is also studied.
Keywords: Weak solution, weakly Riemann-Stieltjes integral, coupled system, weakly relatively compact.

1 Introduction and preliminaries

The Volterra-Stieltjes integral equations and Urysohn-Stieltjes integral equations have been studied by J. Banaś and some other authors (see [1]-[9] and [18]- [20]).

Consider the Urysohn-Stieltjes integral equation

$$x(t) = p(t) + \int_0^1 f(t,s,x(s))\ d_s g(t,s), \quad t \in I = [0,1]. \tag{1}$$

J. Banaś (see [5]) proved the existence of at least one solution $x \in C(I)$ to the equation (1), where $g : I \times I \to R$ is nondecreasing in the second argument on I and the symbol d_s indicates the integration with respect to s.

For the definition, background and properties of the Stieltjes integral we refer to Banaś [1]. However, the coupled system of integral equations have been studied, recently, by some authors (see [13]-[14],[16]).

In this paper, we generalize this results to study the existence of weak solutions $(x,y) \in C[I,E] \times C[I,E]$ for the coupled system of Urysohn-Stieltjes integral equations

$$\begin{aligned} x(t) &= a_1(t) + \int_0^1 f_1(t,s,y(s))\ d_s g_1(t,s),\ t \in I \\ y(t) &= a_2(t) + \int_0^1 f_2(t,s,x(s))\ d_s g_2(t,s),\ t \in I \end{aligned} \tag{2}$$

in reflexive Banach space E under the weak-weak continuity assumption imposed on $f_i : I \times I \times E \to E, \quad i = 1,2$.

As an application, we study the existence of weak solutions $x, y \in C[I,E]$ for the coupled system of Hammerstien-Stieltjes integral equations

$$\begin{aligned} x(t) &= a_1(t) + \int_0^1 k_1(t,s) h_1(s,y(s))\ d_s g_1(t,s),\ t \in I \\ y(t) &= a_2(t) + \int_0^1 k_2(t,s) h_2(s,x(s))\ d_s g_2(t,s),\ t \in I \end{aligned} \tag{3}$$

Throughout this paper, if otherwise is not stated, E denotes a reflexive Banach space with norm $\| \ . \ \|$ and dual E^*. Denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm. Let
$X = C[I,E] \times C[I,E] = \{u(t) = (x(t),y(t)) : x \in C[I,E], y \in C[I,E], t \in I\}$ be a Banach space with the norm defined as

$$\| (x,y) \|_X = \max\{\| x \|_{C[I,E]} + \| y \|_{C[I,E]}\}, \quad \forall (x,y) \in X.$$

Now, we shall present some auxiliary results that will be needed in this work. Let E be a Banach space (need not be reflexive) and let $x : [a, b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [17] and [11]). It is evident that in reflexive Banach spaces, if x is weakly continuous function on $[a, b]$, then x is weakly Riemann integrable (see [17]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of the coupled system (2) in the space $C[I, E] \times C[I, E]$.

Definition 1 *Let $f : I \times E \to E$. Then $f(t, u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that*

$$| \phi(f(t, u) - f(t_0, u_0)) | < \epsilon$$

whenever

$$| t - t_0 | < \delta \text{ and } u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [21]) and some propositions which will be used in the sequel (see [12]).

Theorem 1 *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I, E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

Proposition 1 *In reflexive Banach space, the subset is weakly relatively compact if and only if it is bounded in the norm topology.*

Proposition 2 *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\varphi \in E^*$ with $\| \varphi \| = 1$ and $\| y \| = \varphi(y)$.*

2 Main results

In this section, we present our main result by proving the existence of weak solutions for the coupled system of Urysohn-Stieltjes integral equations (2) in the reflexive Banach space E. Let us first state the following assumptions:

(i) $a_i \in C[I, E],\ i = 1, 2.$

(ii) $f_i : I \times I \times D \to E$, where $D \subset E$ and $i = 1, 2$ satisfy the following condtions:

(1) $f_i(., s, x(s))$is continuous function, $\forall s \in I, x \in D \subset E.$

(2) $f_i(t, ., .)$ is weakly-weakly continuous function, $\forall t \in I.$

(3) The weak closure of the range of $f_i(I \times I \times D)$ are weakly compact in E (or equivalently: there exists a constant M such that $\| f_i(t, s, x) \| \leq M$.

(iii) The functions $g_i : I \times I \to R$ and the functions $t \to g_i(t, 1)$ and $t \to g_i(t, 0)$ $(i = 1, 2)$ are continuous on I, and $\mu = \max\{\sup \mid g_i(t, 1) \mid + \sup \mid g_i(t, 0) \mid \text{ on } I\}$.

(iv) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ are nondecreasing on I.

(v) $g_i(0, s) = 0$ for any $s \in I$.

Remark 1. Observe that assumptions (iv) and (v) imply that the function $s \to g(t, s)$ is nondecreasing on the interval I, for any fixed $t \in I$ (Remark 1 in [6]). Indeed, putting $t_2 = t,\ t_1 = 0$ in (iv) and keeping in mind (v), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on I.

Definition 2 *By a weak solution for the coupled system (2), we mean the pair of functions* $(x, y) \in C[I, E] \times C[I, E]$ *such that*

$$\varphi(x(t)) = \varphi(a_1(t)) + \int_0^1 \varphi(f_1(t, s, y(s)))\ d_s g_1(t, s),\ t \in I$$

$$\varphi(y(t)) = \varphi(a_2(t)) + \int_0^1 \varphi(f_2(t, s, x(s)))\ d_s g_2(t, s),\ t \in I$$

for all $\varphi \in E^*$.

Theorem 2 *Under assumptions (i)-(v), the coupled system of Urysohn-Stieltjes integral equation (2) has at least one weak solution* $(x, y) \in C[I, E] \times C[I, E]$.

Proof. Define an operator A by

$$A(x, y) = (A_1 y, A_2 y)$$

where

$$A_1 y(t) = a_1(t) + \int_0^1 f_1(t, s, y(s))\ d_s g_1(t, s),\ t \in I$$

$$A_2 x(t) = a_2(t) + \int_0^1 f_2(t, s, x(s))\ d_s g_2(t, s),\ t \in I$$

For every $x_i \in C[I, E]$, $f_i(., s, x(s))$ is continuous on I, and $f_i(t, ., x(.))$ are weakly continuous on I, then $\varphi(f_i(t, ., x(.)))$ are continuous for every $\varphi \in E^*$. Hence, in view of bounded variational of g_i it follows, $f_i(t, s, x(s))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g_i(t, s)$. Thus A_i make sense.

Define the sets Q_1 and Q_2 by

$$Q_1 = \{y \in C[I, E] : \| y \| \leq H_1\}, \quad H_1 = \| a_1 \| + M\mu,$$

And

$$Q_2 = \{x \in C[I, E] : \| x \| \leq H_2\}, \quad H_2 = \| a_2 \| + M\mu.$$

Now, define the set Q by

$$Q = \{u = (x, y) \in X :\ \| u \| \leq H_1 + H_2\}$$

Next, let $y \in Q_1$ and $x \in Q_2$. Without loss of generality we may assume that $A_1 y \neq 0, A_2 x(t) \neq 0,\ \ t \in I$. By proposition 2, we have

$$\begin{aligned}
\| A_1 y(t) \| &= \varphi(A_1 y(t)) \\
&\leq | \varphi(a_1(t)) | + | \varphi(\int_0^1 f_1(t, s, y(s))\ d_s g_1(t, s)) | \\
&\leq \| a_1 \| + \int_0^1 | \varphi(f_1(t, s, y(s))) |\ \ d_s(\bigvee_{z=0}^{s} g_1(t, z)) \\
&\leq \| a_1 \| + \int_0^1 \| f_1(t, s, y(s)) \|\ \ d_s(\bigvee_{z=0}^{s} g_1(t, z))
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \parallel a_1 \parallel + M \int_0^1 \ d_s g(t,s) \\
&\leq \ \parallel a_1 \parallel + M[\ g(t,1) - g(t,0)\ ] \\
&\leq \ \parallel a_1 \parallel + M[\ \sup_{t\in I} \mid g(t,1) \mid + \sup_{t\in I} \mid g(t,0) \mid\ ] \\
&\leq \ \parallel a_1 \parallel + M\mu
\end{aligned}
$$

Then

$$\parallel A_1 y(t) \parallel \leq \parallel a_1 \parallel + M\mu = H_1$$

Similarly we can prove that

$$\parallel A_2 x(t) \parallel \leq \parallel a_2 \parallel + M\mu = H_2.$$

Therefore, for any $u \in Q$

$$
\begin{aligned}
\parallel Au(t) \parallel \ &= \ \parallel A(x,y)(t) \parallel = \parallel (A_1 y(t), A_2 x(t)) \parallel \\
&\leq \ \parallel A_1 y(t) \parallel + \parallel A_2 x(t) \parallel \\
&\leq \ \parallel a_1 \parallel + M\mu + \parallel a_2 \parallel + M\mu = H_1 + H_2.
\end{aligned}
$$

i.e., $\forall\ u \in Q \Rightarrow Au \in Q \Rightarrow AQ \subset Q.$ Thus $A : Q \to Q.$

Also, we can prove that $A_1 : C[I,E] \to C[I,E]$ and for $t_1, t_2 \in I,\ \ t_1 < t_2$ (without loss of generality, we may assume that $Ax(t_2) - Ax(t_1) \neq 0$) and there exists $\varphi \in E^*$, such that

$$
\begin{aligned}
\parallel A_1 y(t_2) \ - \ & A_1 y(t_1) \parallel \leq \ \varphi(A_1 y(t_2) - A_1 y(t_1)) \\
\leq \ & \mid \varphi(a_1(t_2) - a_1(t_1)) \mid + \mid \int_0^1 \varphi(f_1(t_2, s, y(s)))\ d_s g_1(t_2, s) \\
- \ & \int_0^1 \varphi(f_1(t_1, s, y(s)))\ d_s g_1(t_1, s) \mid \\
\leq \ & \parallel a_1(t_2) - a_1(t_1) \parallel + \mid \int_0^1 \varphi(f_1(t_2, s, y(s)))\ d_s g_1(t_2, s) \\
- \ & \int_0^1 \varphi(f_1(t_1, s, y(s)))\ d_s g_1(t_2, s) \mid + \mid \int_0^1 \varphi(f_1(t_1, s, y(s)))\ d_s g_1(t_2, s) \\
- \ & \int_0^1 \varphi(f_1(t_1, s, y(s)))\ d_s g_1(t_1, s) \mid \\
\leq \ & \parallel a_1(t_2) - a_1(t_1) \parallel
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \int_0^1 \mid \varphi(f_1(t_2,s,y(s)) - f_1(t_1,s,y(s))) \mid \ d_s(\bigvee_{z=0}^{s} g_1(t_2,z)) \\
&+ \quad \int_0^1 \mid \varphi(f_1(t_1,s,y(s))) \mid \ d_s(\bigvee_{z=0}^{s} [g_1(t_2,z) - g_1(t_1,z))]) \\
&\leq \quad \| a_1(t_2) - a_1(t_1) \| + \| f_1(t_2,s,y(s)) - f_1(t_1,s,y(s)) \| \int_0^1 \ d_s g_1(t_2,s) \\
&+ \quad \int_0^1 \| f_1(t_1,s,y(s)) \| \ d_s[g_1(t_2,s) - g_1(t_1,s))] \\
&\leq \quad \| a_1(t_2) - a_1(t_1) \| \\
&+ \quad \| f_1(t_2,s,y(s)) - f_1(t_1,s,y(s)) \| [g_1(t_2,1) - g_1(t_2,0)] \\
&+ \quad M[(g_1(t_2,1) - g_1(t_1,0)) - (g_1(t_2,0) - g_1(t_1,0))] \\
&\leq \quad \| a_1(t_2) - a_1(t_1) \| \\
&+ \quad \| f_1(t_2,s,y(s)) - f_1(t_1,s,y(s)) \| [g_1(t_2,1) - g_1(t_2,0)] \\
&+ \quad M[\mid g_1(t_2,1) - g_1(t_1,0) \mid + \mid g_1(t_2,0) - g_1(t_1,0) \mid]
\end{aligned}
$$

Similarly we can show that

$$
\begin{aligned}
\| A_2x(t_2) \quad - \quad & A_2x(t_1) \| \leq \| a_2(t_2) - a_2(t_1) \| \\
+ \quad & \| f_2(t_2,s,x(s)) - f_2(t_1,s,x(s)) \| [g_2(t_2,1) - g_2(t_2,0)] \\
+ \quad & M[\mid g_2(t_2,1) - g_2(t_1,0) \mid + \mid g_2(t_2,0) - g_2(t_1,0) \mid]
\end{aligned}
$$

Now, we obtain

$$
\begin{aligned}
A(x,y)(t_2) - A(x,y)(t_1) \quad &= \quad (A_1y(t_2), A_2x(t_2)) - (A_1y(t_1), A_2x(t_1)) \\
&= \quad ((A_1y(t_2) - A_1y(t_1)), (A_2x(t_2) - A_2x(t_1)))
\end{aligned}
$$

and

$$
\begin{aligned}
\| A(x,y)(t_2) \quad - \quad & A(x,y)(t_1) \| \leq \| A_1y(t_2) - A_1y(t_1) \| + \| A_2x(t_2) - A_2x(t_1) \| \\
\leq \quad & \| a_1(t_2) - a_1(t_1) \| \\
+ \quad & \| f_1(t_2,s,y(s)) - f_1(t_1,s,y(s)) \| [g_1(t_2,1) - g_1(t_2,0)] \\
+ \quad & M[\mid g_1(t_2,1) - g_1(t_1,0) \mid + \mid g_1(t_2,0) - g_1(t_1,0) \mid] \\
\leq \quad & \| a_2(t_2) - a_2(t_1) \| \\
+ \quad & \| f_2(t_2,s,x(s)) - f_2(t_1,s,x(s)) \| [g_2(t_2,1) - g_2(t_2,0)] \\
+ \quad & M[\mid g_2(t_2,1) - g_2(t_1,0) \mid + \mid g_2(t_2,0) - g_2(t_1,0) \mid]
\end{aligned}
$$

Note that Q is nonempty, uniformly bounded and strongly equi-continuous subset of X, by the uniform boundedness of AQ. Thus, according to propositions

1, AQ is relatively weakly compact.

It remains to prove that A is weakly sequentially continuous.
Let $\{y_n(t)\}$ and $\{x_n(t)\}$ be sequence in $C[I,E]$ weakly convergent to $y(t)$ and $x(t)$ respectively ($\forall\ t \in I$), since $f_1(t,s,.)$ and $f_2(t,s,.)$ are weakly continuous. Then $f_1(t,s,y_n(s))$ and $f_2(t,s,x_n(s))$ converge weakly to $f_1(t,s,y(s))$ and $f_2(t,s,x(s))$ respectively. Furthermore, ($\forall\ \varphi \in E^*$) $\varphi(f_1(t,s,y_n(s)))$ and $\varphi(f_2(t,s,x_n(s)))$ converge strongly to $\varphi(f_1(t,s,y(s)))$ and $\varphi(f_2(t,s,x(s)))$ respectively. Using assumption (3) and applying Lebesgue dominated convergence theorem, we get

$$\varphi(\int_0^1 f_1(t,s,y_n(s))\ d_s g_1(t,s)) = \int_0^1 \varphi(f_1(t,s,y_n(s)))\ d_s g_1(t,s)$$

$$\to \int_0^1 \varphi(f_1(t,s,y(s)))\ d_s g_1(t,s),\ \forall \varphi \in E^*,\ t \in I.$$

Moreover, we have

$$\varphi(\int_0^1 f_2(t,s,x_n(s))\ d_s g_2(t,s)) = \int_0^1 \varphi(f_2(t,s,x_n(s)))\ d_s g_2(t,s)$$

$$\to \int_0^1 \varphi(f_2(t,s,x(s)))\ d_s g_2(t,s),\ \forall \varphi \in E^*,\ t \in I.$$

Thus, A is weakly sequentially continuous on Q.

Since all conditions of Theorem 1 are satisfied, then the operator A has at least one fixed point $(x,y) = u \in Q$ and the coupled system of Urysohn-Stieltjes integral equations (2) has at least one weak solution.∎

3 Hammerstien-Stieltjes coupled system

This section, as an application, deals with the existence of weak continuous solution for the coupled system of Hammerstien-Stieltjes integral equations (3), consider the following the assumption:

$(ii)^*$ Let $h_i : I \times E \to E$ and $k_i : I \times I \to R_+$ are such that h_i, k_i satisfy the following assumptions:

$(1)^*$ $h_i(s, x(s))$ are weakly-weakly continuous functions.

$(2)^*$ There exists a constant M such that $\| h_i(s,x) \| \leq M$.

$(3)^*$ $k_i(t,s)$ is continuous function such that $K = \sup_t \mid k_i(t,s) \mid$, where K is positive constant.

Definition 3 *By a weak solution for the coupled system (3), we mean the pair of functions* $(x,y) \in C[I,E] \times C[I,E]$ *such that*

$$\begin{aligned} \varphi(x(t)) &= \varphi(a_1(t)) + \int_0^1 k_1(t,s)\varphi(h_1(s,y(s)))\; d_s g_1(t,s),\ t \in I \\ \varphi(y(t)) &= \varphi(a_2(t)) + \int_0^1 k_2(t,s)\varphi(h_2(s,x(s)))\; d_s g_2(t,s),\ t \in I \end{aligned}$$

for all $\varphi \in E^*$.

Now, to prove the existence of a weak solutions of (3), we have the following theorem

Theorem 3 *Let the assumptions (i),(iii)-(v)and* $(ii)^*$ *be satisfied. Then the coupled system of Hammerstien-Stieltjes integral equations (3) has at least one weak solution* $(x,y) \in X$.

Proof. Let

$$f_i(t,s,x(s)) = k_i(t,s)h_i(s,x(s)).$$

Then from the assumption $(ii)^*$, we find that the assumptions of Theorem 2 are satisfied and result follows.

Example : Consider the functions $g_i : I \times I \to R$ defined by the formula

$$\begin{aligned} g_1(t,s) &= \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0,1], \quad s \in I, \\ 0, & \text{for } t = 0, \quad s \in I. \end{cases} \\ g_2(t,s) &= t(t+s-1), \quad t \in I. \end{aligned}$$

It can be easily seen that the functions $g_1(t,s)$ and $g_2(t,s)$ satisfy assumptions (iii)-(v) given in Theorem 2. In this case, the coupled system of Urysohn-

Stieltjes integral equations (2) has the form

$$\begin{aligned} x(t) &= a_1(t) + \int_0^1 \frac{t}{t+s} f_1(t,s,y(s)\ ds, \quad t \in I \\ y(t) &= a_2(t) + \int_0^1 t f_2(t,s,x(s))\ ds, \quad t \in I \end{aligned} \tag{4}$$

Therefore, the coupled system (4) has at least one weak solution $u = (x, y) \in X$.

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