DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
№ 2, 2019
Electronic Journal,
reg.Эл № ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://diffjournal.spbu.ru
e-mail: jodiff@mail.ru

Boundary value problems

Existence of three positive solutions for nonlinear third order arbitrary two-point boundary value problems

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Abstract

In this paper, we establish the existence criteria for at least three positive solutions of an arbitrary two-point boundary value problem for nonlinear third order ordinary differential equation. The analysis of this paper is based on Leggett-William's fixed point theorem and Krasnoselskii's fixed point theorem. Some new existence and multiplicity results for nonlinear third order ordinary differential equation with arbitrary two-point boundary value conditions are obtained. The results of this paper extend and modify the corresponding results of several authors in literature. Some illustrative examples are given to support the analytic proof.

Keywords: arbitrary two-point boundary value problem; positive solution; Leggett-William's fixed point theorem; Krasnoselskii fixed point theorem;

1 Introducion

Third-order ordinary differential equation is used to explain different physical systems, as for example we mention that the deflection of a curved beam having a constant or a varying cross section, the three-layer beam, the electromagnetic waves or the gravity driven flows, see for instance [12] and references therein. In the last few decades, the existence of positive solutions of two-point, three-point and four-point boundary value problems for third order nonlinear ordinary differential equations has extensively been studied by using various techniques, see for instance [1-4, 6-7, 9-11, 13, 15-16, 18-27] and references therein. But there is a little number of works about the existence of positive solutions for the nonlinear boundary value problem (for short BVP) with arbitrary point boundary conditions. Using the Leggett-William's fixed point theorem [17], Agarwal and O'Regan [5] established the principle for the existence of three positive solutions to a class of second order impulsive differential equations and Anderson [3] developed the principle for existence of at least three positive solutions to a third order three-point boundary value problem. By means of the Krasnoselskii's fixed point theorem [14], Anderson and Davis [2] established the principle for the existence of multiple positive solutions to third order three-point right focal boundary value problem. Recently, in 2009 and 2008, Liu *et al.* [15] and Liu *et al.* [16] studied the existence of three positive solutions of the following nonlinear third order arbitrary two-point boundary value problems by applying both of the Leggett-William's fixed point theorem [17] and Krasnoselskii's fixed point theorem [14]:

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, & a < t < b, \\ x(a) = x''(a) = x(b) = 0, \end{cases} \tag{1.1}$$

where, $f \in C([a,b] \times [0,\infty), [0,\infty))$, and

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, & a < t < b, \\ x(a) = x'(a) = x''(b) = 0, \end{cases} \tag{1.2}$$

where, $f \in C([a,b] \times [0,\infty), [0,\infty))$ respectively.

Inspiring with the above mentioned works, in this paper we establish principles for the existence of three positive solutions to the following nonlinear third order ordinary differential equation:

$$u'''(t) + f(t, u(t)) = 0, \quad t \in [a, b], \tag{1.3}$$

under the following arbitrary two-point boundary conditions:

$$u'(a) = u''(a) = u(b) = 0, \tag{1.4}$$

where, a, b are two arbitrary non-negative constants and $f \in C([a,b] \times [0,\infty), [0,\infty))$, applying both of the Leggett-William's fixed point theorem [17] and Krasnoselskii's fixed point theorem [14].

In this paper, we modify the works of Liu *et al.* [15] and Liu *et al.* [16], in case of boundary conditions. The rest of this paper is furnished as follows:

In preliminary notes section, we provide some basic definitions, lemmas and the Leggett-William's fixed point theorem and Krasnoselskii's fixed point theorem. Results and discussion section is use to state and prove our main results, which provide us the techniques to check the existence of three positive solutions of third order arbitrary non-negative two-point BVP defined by (1.3) and (1.4) under some certain assumptions and here we also discuss some illustrative examples. Finally, we conclude this paper.

2 Prelimaniry Notes

In this section, we recall some basic definitions and state the Leggett-William's fixed point theorem and the Krasnoselskii's fixed point theorem, which are essential to establish main results.

Definition 2.1. Let $(B,\|.\|)$ be a real Banach space and P be a nonempty closed convex subset of B. Then we say that P is a *cone* on B if it is satisfies the following properties:

(i) $\eta c \in P$ for $c \in P, \eta \geq 0$;

(ii) $c, -c \in P$ implies $c = \theta$,

where θ denotes the null element of B.

Definition 2.2. A mapping α is said to be a non-negative continuous concave functional on the cone P if $\alpha : P \to [0,+\infty)$ is continuous and

$$\alpha(\delta v + (1-\delta)w) \geq \delta\alpha(v) + (1-\delta)\alpha(w),\ \forall\ v, w \in P,\ \delta \in [0,1].$$

Definition 2.3. Let $C[a,b]$ denote the Banach space of continuous functions on $[a,b]$ with uniform norm $\|u\| = \sup_{a\leq t\leq b}|u(t)|$, for each $u \in C[a,b]$ and a_1 and b_1 are two constants satisfying $a < a_1 < b_1 < b$. We define the following limits:

$$\underline{f}_0 = \liminf_{u\to 0^+}\left(\frac{1}{u}\right)\min\{f(t,u): t\in[a_1,b_1]\},$$

$$\underline{f}_\infty = \liminf_{u\to +\infty}\left(\frac{1}{u}\right)\min\{f(t,u): t\in[a_1,b_1]\},$$

$$\overline{f}_0 = \limsup_{u\to 0^+}\left(\frac{1}{u}\right)\max\{f(t,u): t\in[a,b]\},$$

$$\overline{f}_\infty = \limsup_{u\to +\infty}\left(\frac{1}{u}\right)\max\{f(t,u): t\in[a,b]\}.$$

Definition 2.4. A solution $u(t)$ of the BVP defined by (1.3) and (1.4) is said to be a *positive solution* if $u(t) > 0$ for all $t \in (a,b)$.

Lemma 2.1. *Assume that* $0 \leq a < b$. *If* $h(t) \in C[a,b]$, for all $t \in [a,b]$, *then the unique solution of following nonlinear third order arbitrary two-point BVP*

$$\begin{cases} u'''(t) = h(t), & t\in[a,b] \\ u'(a) = u''(a) = u(b) = 0, \end{cases} \tag{2.1}$$

is $u(t) = \int_a^b G(t,s)h(s)ds, \quad t\in[a,b]$,

where, $G(t,s)$ *is the Green's function of homogeneous third order arbitrary two-point BVP*

$$\begin{cases} u'''(t) = 0, & t \in [a, b], \\ u'(a) = u''(a) = u(b) = 0, \end{cases} \tag{2.2}$$

$$i.e., G(t,s) = \begin{cases} \frac{1}{2}(b-s)^2; & a \le t \le s \le b, \\ \frac{1}{2}(b-t)(b+t-2s); & a \le s \le t \le b. \end{cases} \tag{2.3}$$

Proof. The proofs and computations are regular and we omit them. ■

Remark 2.1. By Lemma 2.1, we can convert the BVP defined by (1.3) and (1.4) as the following integral equation

$$u(t) = \int_a^b G(t,s) f(s,u(s))ds, \text{ for all } t \in [a,b], \tag{2.4}$$

where $G(t,s)$ is the Green's function given by (2.3). It is also noted that, the Green's function $G(t,s)$ have the following properties:

(I) $G(t,s)$ is continuous on $[a,b]\times[a,b]$,

(II) $G(a,s) = G(b,s) = G'(a,s) = G'(b,s) = 0, \text{for all } s \in [a,b]$, and

(III) $G(t,s) \ge 0, \text{ for all } t,s \in [a,b]$.

Obviously, $u \in [a,b]\times[a,b]$ is a solution of the BVP (1.3) and (1.4), if and only if it is a solution of the integral equation (2.4). Furthermore, if we consider a cone K on $C[a,b]$ and define an integral operator $A: K \to K$ by

$$Au(t) = \int_a^b G(t,s) f(s,u(s))ds, \text{ for all } u \in K, \tag{2.5}$$

then it is easy to see that the BVP (1.3) and (1.4) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A defined by (2.5).

Lemma 2.2. *If the following condition*

$$f(t_0, 0) > 0, \text{ for some } t_0 \in [a,b], \tag{2.6}$$

is satisfied, then each solution $u \in C[a,b]$ *of the BVP (1.3) and (1.4) satisfies* $\|u\| > 0$.

Proof. The proof is trivial. ■

Definition 2.5. Considering

$$g(t) = \frac{(b-t)(b+t-2a)}{2} \text{ and } h(t) = \frac{(b-t)^2}{2(b-a)^2} \text{ for } t \in [a,b],$$

we define the following:

$$p^{-1}=\int_a^b g(s)ds=\int_a^b \frac{(b-s)(b+s-a)}{2}ds=\mathrm{L}\ =\frac{(b-a)^3}{3}. \tag{2.7}$$

$$\begin{aligned} q^{-1} &= \min_{t\in[a_1,b_1]} h(t)\int_{a_1}^{b_1} g(s)ds \\ &= \min_{t\in[a_1,b_1]} h(t)\int_{a_1}^{b_1} \frac{(b-s)(b+s-a)}{2}ds \\ &= \frac{(b_1-a_1)}{12}\left[6b(b-a)-2\left(b_1^2+b_1a_1+a_1^2\right)+3a(b_1+a_1)\right]\cdot\min\{h(a_1),h(b_1)\}, \end{aligned} \tag{2.8}$$

$$K=\{u\in C[a,b]: u \text{ is concave on } [a,b], u(t)\ge h(t)\|x\|, t\in[a,b]\}. \tag{2.9}$$

$$K_\lambda=\{u\in K:\|u\|<\lambda\}. \tag{2.10}$$

$$\partial K_\lambda=\{u\in K:\|u\|=\lambda\},\ \lambda>0. \tag{2.11}$$

$$\overline{K}_\lambda=\{u\in K:\|u\|\le\lambda\}. \tag{2.12}$$

$$K(\alpha,\lambda,s)=\{u\in K:\lambda\le\alpha(u),\ \|u\|\le s\},\ \ s>\lambda>0, \tag{2.13}$$

where α is a non-negative continuous functional on K. It is easy to prove that K is a cone on $C[a,b]$.

Lemma 2.3. *If* $G(t,s)$ *is a Green's function defined by (2.3) and* $g(t)$ and $h(t)$ *are defined as the Definition 2.5, then*

(I) $0\le h(t)\le\frac{1}{2},\ 0\le g(t)\le g(a)=\frac{(b-a)^2}{2},\ t\in[a,b]$;

(II) $h(t)g(s)\le G(t,s)\le g(s),\ t,s\in[a,b]$;

(III) *For each* $s\in[a,b]$, *the Green's function* $G(t,s)$ *is concave in the first argument on* $[a,b]$.

Proof. From the definition (Definition 2.5) of $h(t)$ and $g(t)$, it is clear that for all $t\in[a,b]$,

$$0\le h(t)\le\frac{1}{2},\ 0\le g(t)\le\frac{(b-a)^2}{2}.$$

This proves (I).

For $a\le t\le s\le b$, we have

$$\begin{aligned} G(t,s) &= \frac{(b-s)^2}{2}=\frac{(b-s)(b-s)}{2} \\ &\le \frac{(b-s)(b-a)}{2}\le\frac{(b-s)(b+s-2a)}{2}=g(s), \end{aligned}$$

and

$$G(t,s)=\frac{(b-s)^2}{2}=\frac{(b-s)}{(b+s-2a)}\cdot g(s)\ge\frac{(b-t)}{(b+b-2a)}\cdot g(s)$$
$$=\frac{(b-t)}{2(b-a)}\cdot g(s)=\frac{(b-t)(b-a)}{2(b-a)^2}\cdot g(s)\ge\frac{(b-t)(b-t)}{2(b-a)^2}\cdot g(s)=h(t)g(s).$$

Now, for $a\le s\le t\le b$, we obtain

$$G(t,s)=\frac{(b-t)(b+t-2s)}{2}\le\frac{(b-s)(b+s-2a)}{2}=g(s),$$

and

$$G(t,s)=\frac{(b-t)(b+t-2s)}{2}=\frac{(b-t)(b+t-2s)}{(b-s)(b+s-2a)}\cdot g(s)$$
$$\ge\frac{(b-t)(b+t-2t)}{(b-a)(b+b-2a)}\cdot g(s)=\frac{(b-t)(b-t)}{2(b-a)(b-a)}\cdot g(s)=h(t)g(s).$$

Thus (II) holds. To prove (III), we let $\delta\in[0,1]$ and $r,s,t\in[a,b]$ with $t\le r,$ then there arise the following cases:

<u>Case 1.</u> If $s\le t$ and $(\delta t+(1-\delta)r)\ge s,$ then we obtain

$$G((\delta t+(1-\delta)r),s)-\delta G(t,s)-(1-\delta)G(r,s)$$
$$=\frac{1}{2}(b-(\delta t+(1-\delta)r))(b+(\delta t+(1-\delta)r)-2s)$$
$$-\frac{\delta}{2}(b-t)(b+t-2s)-\frac{(1-\delta)}{2}(b-r)(b+r-2s)=\mathrm{L}\ =\frac{\delta(1-\delta)(r-t)^2}{2}\ge 0.$$

<u>Case 2.</u> If $r\le s$ and $(\delta t+(1-\delta)r)\le s,$ then we obtain

$$G((\delta t+(1-\delta)r),s)-\delta G(t,s)-(1-\delta)G(r,s)$$
$$=\frac{1}{2}(b-s)^2-\frac{\delta}{2}(b-s)^2-\frac{(1-\delta)}{2}(b-s)^2=0.$$

<u>Case 3.</u> If $t<s<r$ and $(\delta t+(1-\delta)r)\le s,$ then we obtain

$$G((\delta t+(1-\delta)r),s)-\delta G(t,s)-(1-\delta)G(r,s)$$
$$=\frac{1}{2}(b-s)^2-\frac{\delta}{2}(b-s)^2-\frac{(1-\delta)}{2}(b-r)(b+r-2s)=\mathrm{L}\ =\frac{(1-\delta)(r-s)^2}{2}\ge 0.$$

<u>Case 4.</u> If $t<s<r$ and $(\delta t+(1-\delta)r)>s,$ then we obtain

$$G\left(\left(\delta t+(1-\delta)r\right),s\right)-\delta G(t,s)-(1-\delta)G(r,s)$$

$$=\frac{1}{2}\left(b-\left(\delta t+(1-\delta)r\right)\right)\left(b+\left(\delta t+(1-\delta)r\right)-2s\right)-\frac{\delta}{2}(b-s)^2-\frac{(1-\delta)}{2}(b-r)(b+r-2s)$$

$$\vdots$$

$$=\frac{\delta}{2}\left[(r-s)(s-t)+(r-t)\left[\left(\delta t+(1-\delta)r\right)-s\right]\right]\geq 0.$$

Hence, for the all possible cases

$$G\left(\left(\delta t+(1-\delta)r\right),s\right)-\delta G(t,s)-(1-\delta)G(r,s)\geq 0$$

i.e., $G\left(\left(\delta t+(1-\delta)r\right),s\right)\geq \delta G(t,s)+(1-\delta)G(r,s)$.

Thus, for each $s\in[a,b]$, the Green's function $G(t,s)$ is concave in the first argument on $[a,b]$. Therefore, (III) holds. This completes the proof. ■

Theorem 2.1 (Leggett-William's Fixed Point Theorem [17]). *Let* $A:\overline{K}_c\to\overline{K}_c$ *be a completely continuous operator and* α *be a non-negative continuous concave functional on* K *such that* $\alpha(u)\leq\|u\|$ *for all* $u\in\overline{K}_c$. *Suppose that there exist* $0<d_0<a_0<b_0\leq c$ *such that*

(i) $\{u\in K(\alpha,a_0,b_0):\alpha(u)>a_0\}\neq\Phi$ and $\alpha(Au)>a_0$ *for* $u\in K(\alpha,a_0,b_0)$;

(ii) $\|Au\|<d_0$ *for* $\|u\|\leq d_0$; *(iii)* $\alpha(Au)>a_0$ *for* $u\in K(\alpha,a_0,c)$ *with* $\|Au\|>b_0$.

Then A has at least three fixed points u_1,u_2,u_3 *in* $\overline{K}_c$ *satisfying*

$$\|u_1\|<d_0,\ a_0<\alpha(u_2),\ \|u_3\|>d_0 \text{ and } \alpha(u_3)<a_0.$$

Theorem 2.2 (Krasnoselskii's Fixed Point Theorem [14]). *Let* $(B,\|.\|)$ *be a Banach space and* $K\subset B$ *is a cone in* B. *Assume that* Ψ_1 and Ψ_2 *are two bounded open subsets of* B *with* $0\in\Psi_1$ *and* $\overline{\Psi}_1\subset\Psi_2$. *Let* $A:K\cap\left(\overline{\Psi}_2\setminus\Psi_1\right)\to K$ *be a completely continuous operator satisfying one of the following two conditions:*

(i) $\|Au\|\leq\|u\|,\forall u\in K\cap\partial\Psi_1$ and $\|Au\|\geq\|u\|,\forall u\in K\cap\partial\Psi_2$

and

(ii) $\|Au\|\leq\|u\|,\forall u\in K\cap\partial\Psi_2$ and $\|Au\|\geq\|u\|,\forall u\in K\cap\partial\Psi_1$.

Then A *has at least one fixed point* $u^*\in K\cap\left(\overline{\Psi}_2\setminus\Psi_1\right)$ *and* $u^*>0$.

3 Results and Discussion

In this section, we state and prove some theorems, which analytically prove the existence of three positive solutions of the BVP defined by (1.3) and (1.4). First, we apply the Theorem 2.1(Leggett-William's Fixed Point Theorem).

Theorem 3.1. *Assume that there exist four positive constants* a_0, b_0, c *and* d_0 *satisfying the following inequalities:*

$$0 < d_0 < a_0, \quad \frac{a_0}{\min\{h(a_1), h(b_1)\}} \le b_0 \le c; \tag{3.1}$$

$$f(t,u) < pd_0, \ t \in [a,b], u \in [0, d_0]; \tag{3.2}$$

$$f(t,u) > qa_0, \ t \in [a_1, b_1], u \in [a_0, b_0]; \tag{3.3}$$

$$f(t,u) < pc, \ t \in [a,b], u \in [0,c], \tag{3.4}$$

where p *and* q *are defined by (2.7) and (2.8) respectively. If the Lemma 2.2 holds, then the BVP defined by (1.3) and (1.4) has at least three positive solutions* $u_1, u_2, u_3 \in \overline{K}_c$ *such that*

$$0 < \|u_1\| < d_0, \quad a_0 < \min\{u_2(a_1), u_2(b_1)\}, \quad \|u_3\| > d_0 \text{ and } \min\{u_3(a_1), u_3(b_1)\} < a_0. \tag{3.5}$$

Proof. First, we define the integral operator $A: K \to C[a,b]$ by

$$Au(t) = \int_a^b G(t,s) f(s, u(s))ds, \quad \text{for all } t \in [a,b], \ u \in C[a,b], \tag{3.6}$$

and set $\alpha(u) = \min\limits_{t \in [a_1, b_1]} |u(t)|, \ u \in K.$

Then, it is clear that α is a non-negative continuous concave functional on K and $\alpha(u) = \min\{u(a_1), u(b_1)\} \le \|u\|, \ u \in K.$

Now, by Lemma 2.3 and (3.6), we obtain

$$\begin{aligned} Au(\delta t + (1-\delta)r) &= \int_a^b G((\delta t + (1-\delta)r), s) f(s, u(s))ds \\ &\ge \int_a^b [\delta G(t,s) + (1-\delta)G(r,s)] f(s,u(s))ds \\ &= \delta Au(t) + (1-\delta)Au(r), \ u \in K, \ t, r \in [a,b], \ \delta \in [0,1], \end{aligned}$$

$$\|Au\| = \sup_{t \in [a,b]} \int_a^b G(t,s) f(s,u(s))ds \le \int_a^b g(s) f(s,u(s))ds, \ u \in K,$$

and

$$\begin{aligned} Au(t) &= \int_a^b G(t,s) f(s,u(s))ds \ge h(t) \int_a^b g(s) f(s,u(s))ds \\ &\ge h(t)\|Au\|, \ t \in [a,b], u \in K. \end{aligned}$$

That is A maps K into itself. By Arzela-Ascoli Theorem [8], it is also easy to prove that A is completely continuous.

Now, we proclaim that $A(\overline{K}_c) \subset K_c$. Suppose that $u \in \overline{K}_c$. Then it is easy to observe that

$$0 \le h(t)\|u\| \le u(t) \le c, \ t \in [a,b]. \tag{3.7}$$

Combining (3.4), (3.7) and Lemma 2.3, we obtain

$$\|Au\| = \sup_{t\in[a,b]} \int_a^b G(t,s)\, f\left(s,u(s)\right)ds \le \int_a^b g(s)\, f\left(s,u(s)\right)ds, < c\cdot p\int_a^b g(s)\,ds = c,$$

which implies that $A\left(\overline{K}_c\right) \subset K_c$.

Similarly, using Lemma 2.3 and (3.2), we can easily prove that $\|Au\| < d_0$ for $\|u\| \le d_0$.

Now, if we put $u_0(t) = \dfrac{3}{4}a_0 + \dfrac{1}{4}b_0,\ \ t\in[a,b],$ then (3.1) assure that

$$u_0 \in \left\{x \in K\left(\alpha, a_0, b_0\right) : \alpha(u) > a_0\right\} \ne \varnothing.$$

For any $u \in K\left(\alpha, a_0, b_0\right)$, from Lemma 2.3 and (3.3), we get

$$\begin{aligned}\alpha(Au) &= \min_{t\in[a_1,b_1]} \int_a^b G(t,s)\, f\left(s,u(s)\right)ds \ge \min_{t\in[a_1,b_1]} \int_a^b h(t)g(s)\, f\left(s,u(s)\right)ds\\ &\ge \min\left\{h(a_1),h(b_1)\right\}\int_{a_1}^{b_1} g(s)\, f\left(s,u(s)\right)ds\\ &> a_0 q\cdot \min\left\{h(a_1),h(b_1)\right\}\int_{a_1}^{b_1} g(s)\,ds = a_0.\end{aligned}$$

Again, for any $u \in K\left(\alpha, a_0, c\right)$ and $\|Au\| > b_0$, Lemma 2.3, (3.1) and (3.6) assure that

$$\|Au\| = \max_{t\in[a,b]} \int_a^b G(t,s)\, f\left(s,u(s)\right)ds \le \int_a^b g(s)\, f\left(s,u(s)\right)ds$$

and

$$\begin{aligned}\alpha(Au) &= \min_{t\in[a_1,b_1]} \int_a^b G(t,s)\, f\left(s,u(s)\right)ds \ge \min_{t\in[a_1,b_1]} \int_a^b h(t)g(s)\, f\left(s,u(s)\right)ds\\ &\ge \min\left\{h(a_1),h(b_1)\right\}\int_{a_1}^{b_1} g(s)\, f\left(s,u(s)\right)ds \ge \min\left\{h(a_1),h(b_1)\right\}\|Au\|\\ &> \min\left\{h(a_1),h(b_1)\right\} b_0 \ge a_0.\end{aligned}$$

Hence, all the assumptions of the Theorem 2.1(Leggett-Williams Fixed Point Theorem) are satisfied by the integral operator A defined by (3.6). Therefore, according to the Theorem 2.1, we can say that the integral operator A has at least three fixed points $u_1, u_2, u_3 \in \overline{K}_c$ with

$$0 < \|u_1\| < d_0,\quad a_0 < \alpha(u_2),\quad \|u_3\| > d_0 \text{ and } \alpha(u_3) < a_0. \tag{3.8}$$

Thus Lemma 2.2, Remark 2.1 and (3.8) confirm that the BVP defined by (1.3) and (1.4) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ satisfying (3.5). This completes the proof. ■

Next, we establish other principles for existence of three positive solutions to the BVP (1.3) and (1.4) applying Theorem 2.1.

Theorem 3.2. *Assume that there exist four positive constants* a_0, b_0, c_0 *and* d_0 *satisfying the inequalities* (3.2), (3.3) and

$$0 < d_0 < a_0,\quad \frac{a_0}{\min\left\{h(a_1),h(b_1)\right\}} \le b_0, \tag{3.9}$$

$$f(t,u) < pu,\ t \in [a,b],\ u \in [c_0, +\infty], \tag{3.10}$$

where p and q are defined by (2.7) and (2.8) respectively. If the Lemma 2.2 holds, then there exists $c > \max\{c_0, b_0\}$ *such that the BVP defined by (1.3) and (1.4) has at least three positive solutions* $u_1, u_2, u_3 \in \overline{K}_c$ *satisfying* (3.5).

Proof. To prove this theorem, it is enough to show that there exists $c > \max\{c_0, b_0\}$ satisfying (3.4), i.e., $f(t,u) < pc,\ t \in [a,b],\ u \in [0,c]$.

Now, if f is bounded on $[a,b]\times[0,+\infty)$, the it is clear that there exists $c > \max\{c_0, b_0\}$ such that $f(t,u) < pc,\ t \in [a,b],\ u \in [0,+\infty)$.

And, if f is unbounded on $[a,b]\times[0,+\infty)$, then the continuity of f and (3.10) imply that there exist $c > \max\{c_0, b_0\}$ and $u_0 \in (c_0, c)$ such that

$$f(t,u) \le f(t,u_0) < pu_0 < pc,\ \ t \in [a,b],\ u \in [0,c].$$

Hence, (3.4) is satisfied and the proof follows from Theorem 3.1. This completes the proof. ■

Theorem 3.3. *Assume that there exist two positive constants* a_0 *and* b_0 *with*

$$0 < \frac{a_0}{\min\{h(a_1), h(b_1)\}} \le b_0$$

satisfying the inequalities (3.3) *and*

$$\max\{\overline{f}_0, \overline{f}_\infty\} < p. \tag{3.11}$$

where p and q are defined by (2.7) and (2.8) respectively and $\overline{f}_0$ *and* $\overline{f}_\infty$ *are defined as the Definition 2.3. If the Lemma 2.2 holds, then there exist two constants* d_0 *and* c *with* $0 < d_0 < a_0$ *and* $c > b_0$ *such that the BVP defined by (1.3) and (1.4) has at least three positive solutions* $u_1, u_2, u_3 \in \overline{K}_c$ *satisfying* (3.5).

Proof. For any $0 < d_0 < a_0$ and $c_0 > b_0$ from (3.11) we obtain

$$\frac{1}{s}\max\{f(t,u) : t \in [a,b]\} < \frac{\overline{f}_0 + p}{2},\ u \in (0, d_0],$$
$$\frac{1}{s}\max\{f(t,u) : t \in [a,b]\} < \frac{\overline{f}_\infty + p}{2},\ u \in [c_0, +\infty),$$

which provide that

$$f(t,u) < pd_0,\quad t \in [a,b],\ u \in [0,d_0],\ f(t,u) < ps,\qquad t \in [a,b],\ u \in [c_0, +\infty).$$

Hence, rest of the proof follows from Theorem 3.2. This completes the proof. ■

In our next theorems we use Theorem 2.2 (Krasnoselskii's fixed point theorem) to establish the principles for existence of three positive solutions to the BVP (1.3) and (1.4).

Theorem 3.4. *Assume that there exist four positive constants* a_0, b_0, c *and* d_0 *with* $d_0 < a_0 < b_0 < c$ *satisfying (3.2) and following inequalities:*

$$f(t,u) > qa_0,\ t \in [a_1, b_1],\ u \in \left[\min\{h(a_1),\ h(b_1)\} a_0,\ a_0\right]; \tag{3.12}$$

$$f(t,u) < pb_0,\ t \in [a,b],\ u \in [0, b_0]; \tag{3.13}$$

$$f(t,u) > qc,\ t \in [a_1, b_1],\ u \in \left[\min\{h(a_1),\ h(b_1)\} c,\ c\right], \tag{3.14}$$

where p and q are defined by (2.7) and (2.8) respectively. Then the BVP defined by (1.3) and (1.4) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ *such that*

$$d_0 < \|u_1\| < a_0 < \|u_2\| < b_0 < \|u_3\| < c. \tag{3.15}$$

Proof. First, we define the integral operator $A: K \to C[a,b]$ by (3.6). Then it is obvious that $A: K \to K$ is completely continuous operator, which is proved elaborately in our Theorem 3.1. Now, we proclaim that (3.2) and (3.12) imply that there exist two positive constants d_1 and a_2 with $d_0 < d_1 < a_2 < a_0$ satisfying

$$f(t,u) < pd_1,\ t \in [a,b],\ u \in [0, d_1]; \tag{3.16}$$

$$f(t,u) > qa_2,\ t \in [a_1, b_1],\ u \in \left[\min\{h(a_1),\ h(b_1)\} a_2, a_2\right]. \tag{3.17}$$

If we put

$$\psi(s) = \min\left\{ f(t,s) : t \in [a_1, b_1],\ s \in \left[\min\{h(a_1),\ h(b_1)\} u,\ u\right]\right\},\quad u \ge 0,$$
$$\varphi(s) = \max\left\{ f(t,s) : t \in [a,b],\ s \in [0,\ u]\right\},\quad u \ge 0,$$

then it is clear that (3.2) and (3.12) are equivalent to $\dfrac{\varphi(d_0)}{d_0} < p$ and $\dfrac{\psi(a_0)}{a_0} > q$, respectively and we obtain $\dfrac{\psi(d_0)}{d_0} \le \dfrac{\varphi(d_0)}{d_0} < p < q < \dfrac{\psi(a_0)}{a_0}$.

Now, the continuity of ψ yields that there exists some $a_2 \in (d_0, a_0)$ satisfying $\dfrac{\psi(a_2)}{a_2} = q$, which means that (3.17) holds.

Again, since

$$\frac{\varphi(d_0)}{d_0} < p < q = \frac{\psi(a_2)}{a_2} \le \frac{\varphi(a_2)}{a_2},$$

then from the continuity of φ we yield that there exists some $d_1 \in (d_0, a_2)$ satisfying $\dfrac{\varphi(d_1)}{a_2} = p$, which infers that (3.16) holds.

Now, we assert that the BVP (1.3) and (1.4) has at least one positive solution $u_1 \in K$ with $d_0 < d_1 \le \|u_1\| \le a_2 < a_0$. Let $u \in \partial K_{d_1}$, then it is easy to verify that $\|u\| = d_1$ and

$$0 \le h(t) d_1 \le u(t) \le d_1,\ t \in [a,b].$$

Hence from (3.16), we have

$$f\big(t, u(t)\big) < p d_1, \quad t \in [a,b]. \tag{3.18}$$

In view of Lemma 2.3, (2.7), (3.6) and (3.18), we obtain that

$$\|Au\| = \sup_{t\in[a,b]} \int_a^b G(t,s) f\big(s,u(s)\big) ds \le \int_a^b g(s) f\big(s,u(s)\big) ds, \le p d_1 \int_a^b g(s) ds = d_1,$$

which gives us

$$\|Au\| \le \|u\|, \quad u \in \partial K_{d_1}. \tag{3.19}$$

Again, let $u \in \partial K_{a_2}$, then we have $\|u\| = a_2$ and

$$\min\{h(a_1), h(b_1)\} a_2 \le h(t)\|u\| \le u(t) \le a_2,\ t \in [a_1, b_1].$$

From (3.17), we get

$$f\big(t, u(t)\big) > q a_2,\ t \in [a_1, b_1]. \tag{3.20}$$

In view of Lemma 2.3, (2.8), (3.6) and (3.19), we obtain that

$$\begin{aligned} Au(t) &= \int_a^b G(t,s) f\big(s,u(s)\big) ds \\ &\ge h(t) \int_a^b g(s) f\big(s,u(s)\big) ds \ge \min\{h(a_1), h(b_1)\} \int_{a_1}^{b_1} g(s) f\big(s,u(s)\big) ds \\ &\ge \min\{h(a_1), h(b_1)\} q a_2 \int_{a_1}^{b_1} g(s) ds = a_2, \qquad t \in [a_1, b_1], \end{aligned}$$

which gives us

$$\|Au\| \ge \|u\|, \quad u \in \partial K_{a_2}. \tag{3.21}$$

Hence, according to Theorem 2.2, (3.19) and (3.21), we confirm that the BVP (1.3) and (1.4) has at least one solution $u_1 \in K$ with $d_0 < d_1 \le \|u_1\| \le a_2 < a_0$, and we have

$$u_1(t) \ge h(t)\|u_1\| \ge d_1 h(t) > 0,\ t \in (a, b).$$

This ensures that the solution $u_1 \in K$ of the BVP (1.3) and (1.4) is positive.

In similar way if we use (3.13) and (3.14), then in view of Theorem 2.2, we obtain that there exist four positive constants a_3, b_2, b_3, c_1 with $a_0 < a_3 < b_2 < b_0 < b_3 < c_1 < c$ such the BVP (1.3) and (1.4) has at least two positive solutions $u_2 \in K$ and $u_3 \in K$ with

$$a_0 < a_3 \le \|u_2\| \le b_2 < b_0 \text{ and } b_0 < b_3 \le \|u_3\| \le c_1 < c$$

respectively. This completes the proof. ■

Theorem 3.5. *Assume that there exist two positive constants a_0 and b_0 with $a_0 < b_0$ satisfying the inequalities (3.12) and (3.13), respectively. If the function f satisfies*

$$\overline{f}_0 < p \text{ and } \underline{f}_\infty > \frac{q}{\min\{h(a_1), h(b_1)\}}, \tag{3.22}$$

where p and q are defined by (2.7) and (2.8) respectively, then there exists two positive constants d_0 and c with $d_0 < a_0$ and $b_0 < c$ such that the BVP defined by (1.3) and (1.4) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ satisfying (3.15).

Proof. From (3.22), we obtain that there exist $d_0 \in (0, a_0)$ and c with $\min\{h(a_1), h(b_1)\} c > b_0$ satisfying

$$\frac{1}{u}\max\{f(t,u): t \in [a,b]\} < \frac{\overline{f}_0 + p}{2}, u \in (0, d_0],$$
$$\frac{1}{u}\min\{f(t,u): t \in [a_1,b_1]\} > \frac{q}{\min\{h(a_1), h(b_1)\}}, \ u \in \left[\min\{h(a_1), h(b_1)\}c, +\infty\right),$$

which provide that

$$f(t,u) < pd_0, \quad t \in [a,b], u \in [0, d_0],$$
$$f(t,u) > \frac{q}{\min\{h(a_1), h(b_1)\}}u, \quad t \in [a_1,b_1], \ u \in \left[\min\{h(a_1), h(b_1)\}c, c\right].$$

Hence, rest of the proof follows from Theorem 3.4. This completes the proof. ■

Now, if we combine the arguments of Theorem 3.1-Theorem 3.5, then we obtain some consequences about the existence of positive solutions of the BVP (1.3) and (1.4). For conciseness, here we state these consequences without proof.

Theorem 3.6. *Suppose that there exist four positive constants a_0, b_0, c and d_0 with $d_0 < a_0 < b_0 < c$ satisfying (3.4) and following inequalities:*

$$f(t, u) > qd_0, \ t \in [a_1,b_1], u \in \left[\min\{h(a_1), h(b_1)\}d_0, \ d_0\right]; \tag{3.23}$$

$$f(t, u) < pa_0, \ t \in [a,b], u \in [0, a_0]; \tag{3.24}$$

$$f(t, u) > qb_0, \ t \in [a_1,b_1], u \in \left[\min\{h(a_1), h(b_1)\}b_0, b_0\right], \tag{3.25}$$

where p and q are defined by (2.7) and (2.8) respectively. Then the BVP defined by (1.3) and (1.4) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ *satisfying* (3.15).

Theorem 3.7. *Suppose that there exist three positive constants* d_0, a_0 *and* b_0 *with* $d_0 < a_0 < b_0$ *satisfying (3.23), (3.24), and (3.25) respectively and if the function* f *satisfies (3.10) for some positive constant* c_0. *Then there exists* $c > \max\{c_0, b_0\}$ *such that the BVP defined by (1.3) and (1.4) has at least three positive solutions* $u_1, u_2, u_3 \in \overline{K}_c$ *satisfying* (3.15).

Theorem 3.8. *Suppose that there exist two positive constants* a_0 *and* b_0 *with* $a_0 < b_0$ *satisfying (3.23) and (3.24) respectively. If the function* f *satisfies*

$$\overline{f}_\infty < p, \text{ and } \underline{f}_0 > \frac{q}{\min\{h(a_1), h(b_1)\}}, \tag{3.26}$$

then there exists constants d_0 *and* c *with* $0 < d_0 < a_0$ *and* $b_0 < c$ *such that the BVP defined by (1.3) and (1.4) has at least three positive solutions* $u_1, u_2, u_3 \in \overline{K}_c$ *satisfying* (3.15).

Now, we provide some examples to illustrate our main results.

Example 3.1. Consider the following nonlinear third order two-point boundary value problem:

$$\begin{cases} u'''(t) + f(t,u) = 0, \ t \in [0,3], \\ u'(0) = u''(0) = u(3) = 0, \end{cases} \tag{3.27}$$

with

$$f(t,u) = \begin{cases} \dfrac{1-\frac{tu}{3}}{18+t^2}; & t \in [0,3], u \in (-\infty, d_0] \\ \dfrac{1-\frac{t}{3}}{18+t^2} + \dfrac{3}{17}(t^2+u^2)(u-1)^3; & t \in [0,3], u \in (d_0, b_0) \\ \dfrac{1-\frac{t}{3}}{18+t^2} + \dfrac{507}{17}(t^2+90) + \dfrac{(u-98)}{17}; & t \in [0,3], u \in (b_0, +\infty). \end{cases}$$

Now, if we let $a_1 = 1, b_1 = 2, d_0 = 1, a_0 = 7, b_0 = 100, c = 114318$, then we get

$$p = \frac{1}{9}, \ h(a_1) = \frac{2}{9}, h(b_1) = \frac{1}{18}, q = 6 \text{ and}$$

$$f(t,u) \le \frac{1}{18} < \frac{1}{9} \cdot 1 = p \cdot d_0, \ t \in [0, 3], u \in [0, d_0],$$

$$f(t,u) > \frac{3}{17} u(u-1)^3 > q \cdot a_0, \ t \in [0, 3], u \in [a_0, b_0],$$

$$f(t,u) \le 1 + \frac{50193}{17} + \frac{114220}{17} < p \cdot c, \ t \in [0, 3], u \in [0, c].$$

That is all the assumptions of Theorem 3.1 are satisfied. Therefore, according to the Theorem 3.1, we can say that the BVP defined by (3.27) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ satisfying (3.5).

Example 3.2. Consider the following nonlinear third order two-point boundary value problem:

$$\begin{cases} u'''(t) + f(t,u) = 0, \ t \in [1,4], \\ u'(1) = u''(1) = u(4) = 0, \end{cases} \tag{3.28}$$

with

$$f(t,u) = \begin{cases} \dfrac{1-u}{18 + t^2 (u-1)^2}; & t \in [1,4], u \in (-\infty, d_0], \\ (1040 + 17t^2)(u-1); & t \in [1,4], u \in (d_0, a_0], \\ 1541 + 123t^2 + \dfrac{u-20}{18+tu}; & t \in [1,4], u \in (a_0, b_0], \\ 1541 + 123t^2 + \dfrac{1560}{18+1600t} + 229t^2 (u-1600)^2; & t \in [1,4], u \in (b_0, +\infty). \end{cases}$$

Now, if we let $a_1 = 2, b_1 = 3, d_0 = 1, a_0 = 20, b_0 = 129600, c = 158800,$ then we get

$p = \dfrac{1}{9}, \ h(a_1) = \dfrac{2}{27}, h(b_1) = \dfrac{1}{18}, q = \dfrac{216}{49}$ and

$$f(t,u) \le \frac{1}{18} < \frac{1}{9} \cdot 1 = p \cdot d_0, \ t \in [1, 4], u \in [0, d_0],$$

$$f(t,u) \ge \frac{1108}{9} > q \cdot a_0, \ t \in [2, 3], u \in \left[\min\{h(a_1), h(b_1)\} a_0, a_0\right],$$

$$f(t,u) < 3509 + \frac{129580}{18} < p \cdot b_0, \ t \in [1, 4], u \in [0, b_0],$$

$$f(t,u) > 3509.243 + 229\left(\frac{c}{18} - 1600\right)^2 > q \cdot c, \ t \in [1, 4], u \in \left[\min\{h(a_1), h(b_1)\} c, c\right].$$

That is all the assumptions of Theorem 3.4 are satisfied. Therefore, according to the Theorem 3.4, we can say that the BVP defined by (3.28) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K}_c$ satisfying (3.15).

Example 3.3. Consider the following nonlinear third order two-point boundary value problem:

$$\begin{cases} u'''(t) + f(t,u) = 0, \ t \in [0,5], \\ u'(0) = u''(0) = u(5) = 0, \end{cases} \tag{3.29}$$

with

$$f(t,u)=\begin{cases}\dfrac{201+(1+t)\sqrt{2-u}}{51}; & t\in[0,5],\ u\in(-\infty,\ d_0],\\ \dfrac{201}{51}+\dfrac{(u-200)}{50}; & t\in[0,5],\ u\in(d_0,a_0],\\ \dfrac{59187}{2550}+\dfrac{(24+t^2)(u-205)}{8}; & t\in[0,5],\ u\in(a_0,b_0],\\ \dfrac{59187}{2550}+\dfrac{55115(24+t^2)}{8}+\dfrac{\left(1+\dfrac{t}{4}\right)(u-55320)}{125}; & t\in[0,5],\ u\in(b_0,+\infty).\end{cases}$$

Now, if we let $a_1=1,\ b_1=2,\ d_0=2,\ a_0=205,\ b_0=55320,\ c=176812581$, then we get

$$p=\frac{3}{125},\ h(a_1)=\frac{8}{25},\ h(b_1)=\frac{9}{50},\ q=\frac{25}{51}\ \text{and}$$

$$f(t,u)>\frac{201}{51}>\frac{25}{51}\cdot 2=q\cdot d_0,\ t\in[1,\ 2],\ u\in\left[\min\{h(a_1),\ h(b_1)\}d_0,\ d_0\right],$$

$$f(t,u)\le\frac{201}{51}+\frac{5}{50}<p\cdot a_0,\ t\in[0,\ 5],\ u\in[0,\ a_0],$$

$$f(t,u)\ge 121934>q\cdot b_0,\ t\in[1,\ 2],\ u\in\left[\min\{h(a_1),\ h(b_1)\}b_0,\ b_0\right],$$

$$f(t,u)<337607+\frac{2.25}{125}(c-55320)<p\cdot c,\quad t\in[0,\ 5],\ u\in[0,\ c].$$

That is all the conditions of Theorem 3.5 are satisfied. Therefore, according to the Theorem 3.1, we can say that the BVP defined by (3.29) has at least three positive solutions $u_1,u_2,u_3\in\overline{K_c}$ satisfying (3.15).

Conclusion

In this study, we have established general principles for checking the existence of three positive solutions of nonlinear third order arbitrary two-point boundary value problem defined by (1.3) and (1.4) applying Leggett-William's fixed point theorem and Krasnoselskii's fixed point theorem. By using any one of our Theorem 3.1 to Theorem 3.8, one may check the existence of three positive solutions of BVP (1.3) and (1.4). The results of this paper have modified of the corresponding results of Liu *et al.* [15] and Liu *et al.* [16]. Finally, we have verified our results by some particular examples.

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