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Boundary Control For Cooperative Elliptic Systems Governed By Schrödinger Operator

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Abstract

In this paper, we study the existence of solutions for a cooperative elliptic systems governed by Schrödinger operator defined on $\mathbb{R}^n$, then we discuss the optimal control of boundary type for these systems.

Keywords: Cooperative elliptic systems in $\mathbb{R}^n$, Schrödinger operator, Existence of solution, Boundary control, Optimality conditions.

1 Introduction

We consider the following cooperative elliptic system :

$$\begin{cases} (-\Delta + q)y_1 = ay_1 + by_2 + f_1 \quad \text{in} \quad \mathbb{R}^n \\ (-\Delta + q)y_2 = cy_1 + dy_2 + f_2 \quad \text{in} \quad \mathbb{R}^n \\ y_1 = g_1 \quad \text{as} \quad |x| \to \infty \\ y_2 = g_2 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{1}$$

where :

$$\begin{cases} a, b, c \text{ and } d \text{ are given numbers such that} \quad b, c > 0 \\ \quad \text{(in this case, we say that the system (1) is cooperative)} \end{cases} \tag{2}$$

$$q(x) \text{ is a positive function and tending to } \infty \text{ at infinity.} \tag{3}$$

In [22], Gali et al. proved the existence of optimal control for system like (S) with $q(x) = 0$ and with positive weight function. Also they found the set of inequalities which described the distributed control for systems (S) with $q(x) = 0$ and defined on bounded domain [21]. The case of semilinear cooperative system with $q(x) = 0$ is discussed in [17].

In [16] Fleckinger, obtained the necessary and sufficient conditions for having the maximum principle and the existence of positive solutions for cooperative system (1) which are:

$$\begin{cases} a < \lambda(q), \quad d < \lambda(q) \\ (\lambda(q) - a)(\lambda(q) - d) > bc, \end{cases} \tag{4}$$

where $\lambda(q)$ is defined later.

Here, we shall use the same conditions (4) to prove the existence of the state of our system (1); then using the theory of Lions [30], we study the existence of boundary control for system (1). Our model in the problem is Schrodinger operator.

2 Operator equation.

To prove the existence of the state $y = \{y_1, y_2\}$ of system (1), we state briefly some results introduced in [15] concerning the eigenvalue problem

$$\begin{cases}(-\Delta + q)\phi = \lambda(q)\phi \quad \text{in} \quad \mathbb{R}^n \\ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty, \quad \phi > 0.\end{cases} \tag{5}$$

The associated variational space is $V_q(\mathbb{R}^n)$, the completion of $D(\mathbb{R}^n)$, with respect to the norm :

$$||y||_q = \left(\int_{\mathbb{R}^n} |\Delta y|^2 + q|y|^2 dx \right)^{\frac{1}{2}}$$

Since the imbedding of $V_q(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is compact. Then the operator $(-\Delta + q)$ considered as an operator in $L^2(\mathbb{R}^n)$ is positive self-adjoint with compact inverse. Hence its spectrum consists of an infinite sequence of positive eigenvalue tending to infinity; moreover the smallest one which is called the principle eigenvalue denoted by $\lambda(q)$ is simple and is associated with an eigenfunction which does not change sign in $\mathbb{R}^n$. It is characterized by:

$$\lambda(q) \int_{\mathbb{R}^n} |y|^2 dx \leq \int_{\mathbb{R}^n} |\Delta y|^2 + q|y|^2 dx \quad \forall y \in V_q(\mathbb{R}^n). \tag{6}$$

Now, to study our system (1) we have the embedding

$$V_q(\mathbb{R}^n) \times V_q(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$$

is continuous and compact then, we define a bilinear form

$$\pi : (V_q(\mathbb{R}^n))^2 \times (V_q(\mathbb{R}^n))^2 \to \mathbb{R}$$

by

$$\begin{aligned}\pi((y_1, y_2), (\phi_1, \phi_2)) = \frac{1}{b} \int_{\mathbb{R}^n} [\Delta y_1 \Delta \phi_1 + q y_1 \phi_1] dx + \frac{1}{c} \int_{\mathbb{R}^n} [\Delta y_2 \Delta \phi_2 + q y_2 \phi_2] dx \\ - \int_{\mathbb{R}^n} y_1 \phi_2 dx - \frac{d}{c} \int_{\mathbb{R}^n} y_2 \phi_2 dx - \frac{a}{b} \int_{\mathbb{R}^n} y_1 \phi_1 dx - \int_{\mathbb{R}^n} y_2 \phi_1 dx.\end{aligned} \tag{7}$$

It is easy to check that π is a continuous bilinear form; and then by Lax-Milgram Lemma, we have the following theorem:

Theorem 2.1 *For $f_1, f_2 \in L^2(\mathbb{R}^n)$, there exists a unique solution $y = \{y_1, y_2\} \in (V_q(\mathbb{R}^n))^2$ of system* (1) *if conditions (4) are satisfied.*

Proof

We choose m large enough such that $a+m>0$ and $d+m>0$ and define on $V_q(\mathbb{R}^n)$ the equivalent norm

$$||y||^2_{q,m} = \int_{\mathbb{R}^n}[|\Delta y|^2 + (m+q)|y|^2]dx$$

and we write (7) as:

$$\begin{aligned}\pi((y_1,y_2),(\phi_1,\phi_2)) = {} & \frac{1}{b}\int_{\mathbb{R}^n}[\Delta y_1 \Delta\phi_1 + (q+m)y_1\phi_1]dx - \frac{a+m}{b}\int_{\mathbb{R}^n} y_1\phi_1 dx \\ & - \int_{\mathbb{R}^n} y_2\phi_1 dx + \frac{1}{c}\int_{\mathbb{R}^n}[\Delta y_2\Delta\phi_2 + (q+m)y_2\phi_2]dx \\ & - \frac{d+m}{c}\int_{\mathbb{R}^n} y_2\phi_2 dx - \int_{\mathbb{R}^n} y_1\phi_2 dx.\end{aligned}$$

Then

$$\begin{aligned}\pi((y_1,y_2),(y_1,y_2)) = {} & \frac{1}{b}\int_{\mathbb{R}^n}[|\Delta y_1|^2 + (q+m)|y_1|^2]dx - \frac{a+m}{b}\int_{\mathbb{R}^n}|y_1|^2 dx \\ & - \int_{\mathbb{R}^n} y_1y_2 dx + \frac{1}{c}\int_{\mathbb{R}^n}[|\Delta y_2|^2 + (q+m)|y_2|^2]dx \\ & - \frac{d+m}{c}\int_{\mathbb{R}^n}|y_2|^2 dx - \int_{\mathbb{R}^n} y_1y_2 dx.\end{aligned}$$

By Cauchy Schwartz inequality, we have

$$\begin{aligned}\pi((y_1,y_2),(y_1,y_2)) \geq {} & \frac{1}{b}\int_{\mathbb{R}^n}[|\Delta y_1|^2 + (q+m)|y_1|^2]dx - \frac{a+m}{b}\int_{\mathbb{R}^n}|y_1|^2 dx \\ & + \frac{1}{c}\int_{\mathbb{R}^n}[|\Delta y_2|^2 + (q+m)|y_2|^2]dx - \frac{d+m}{c}\int_{\mathbb{R}^n}|y_2|^2 dx \\ & - 2\left(\int_{\mathbb{R}^n}|y_1|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|y_2|^2 dx\right)^{\frac{1}{2}},\end{aligned}$$

from (6), we deduce

$$\begin{aligned}\pi((y_1,y_2),(y_1,y_2)) \geq {} & \frac{1}{b}\left(1 - \frac{a+m}{\lambda(q)+m}\right)||y_1||^2_{q,m} + \frac{1}{c}\left(1 - \frac{d+m}{\lambda(q)+m}\right)||y_2||^2_{q,m} \\ & \frac{2}{\lambda+m}||y_1||_{q,m}||y_2||_{q,m}.\end{aligned}$$

If (5) holds, then

$$\pi((y_1, y_2), (y_1, y_2)) \geq C(||y_1||^2_{q,m} + ||y_2||^2_{q,m}) \tag{8}$$

which prove the coerciveness of the bilinear form π. Then for $f_1, f_2 \in L^2(\mathbb{R}^n)$, system (1) has a unique solution by Lax-Milgram lemma.

3 Formulation of the control problem

The space $L^2(\Gamma) \times L^2(\Gamma)$ is the space of controls. For a control $u = \{u_1, u_2\} \in (L^2(\Gamma)^2$, the state $y(u) = \{y_1(u), y_2(u)\}$ of the system is given by the solution of:

$$\begin{cases} (-\Delta + q)y_1(u) = ay_1(u) + by_2(u) + f_1 \quad \text{in} \quad \mathbb{R}^n \\ (-\Delta + q)y_2(u) = cy_1(u) + dy_2(u) + f_2 \quad \text{in} \quad \mathbb{R}^n \\ y_1 = u_1 \quad \text{as} \quad |x| \to \infty \\ y_2 = u_2 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{9}$$

The observation equation is given by $z(u) = \{z_1(u), z_2(u)\} = y(u) = \{y_1(u), y_2(u)\}$. For given $z_d = \{z_{d1}, z_{d2}\}$ in $(L^2(\mathbb{R}^n))^2$; the cost function is given by:

$$J(v) = \int_{\mathbb{R}^n} (y_1(v) - z_{d1})^2 + (y_2(v) - z_{d2})^2 dx + (Nv, v)_{(L^2(\Gamma))^2} \tag{10}$$

where $N \in L((L^2(\Gamma))^2, (L^2(\Gamma))^2)$ is hermitian positive definite operator:

$$(Nu, u) \geq \eta ||u||^2_{(L^2(\mathbb{R}^n))^2}. \tag{11}$$

The control problem then is to find

$$\begin{cases} u = \{u_1, u_2\} \in U_{ad} \quad \text{such that} \\ J(u) \leq J(v) \end{cases}$$

where U_{ad} is a closed convex subset of$(L^2(\Gamma))^2$.

Under the given consideration, we may apply the Theorem 2.4 of Lions [30] to obtain the following result:

Theorem 3.1 *Assume that (8) and (11) hold. If the cost function is given by (10), then there exists an optimal control $u = \{u_1, u_2\}$; Moreover it is charac-*

terized by the following equations and inequalities:

$$\begin{cases} (-\Delta + q)p_1(u) - ap_2(u) - cp_2(u) = y_1(u) - z_{1d} \quad in \quad \mathbb{R}^n \\ (-\Delta + q)p_2(u) - bp_1(u) - dp_2(u) = y_2(u) - z_{2d} \quad in \quad \mathbb{R}^n \\ p_1(u) = 0 \quad p_2(u) = 0 \quad on \quad \Gamma \end{cases}$$

$$\int_\Gamma \frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2) d\Gamma + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0 \quad \forall v \in U_{ad},$$

together with (9), where $p(u) = \{p_1(u), p_2(u)\}$ is the adjoint state.

Proof

The control u is characterized by

$$J'(u)(v - u) \geq 0 \quad \forall u \in U_{ad}$$

which is equivalent to

$$(y(u) - z_d, y(v) - y(u))_{L^2(\mathbb{R}^n)^2} + (Nu, v - u)_{L^2(\Gamma)^2} \geq 0$$

i.e.,

$$(y_1(u) - z_{1d}, y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)^2} + (y_2(u) - z_{2d}, y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)^2} + (Nu, v - u)_{L^2(\Gamma)^2} \geq 0 \tag{12}$$

Since $(A^*P, Y) = (P, AY)$, where

$$A(\phi = \{\phi_1, \phi_2\}) \to A\phi = \{(-\Delta + q)\phi_1 - a\phi_1 - b\phi_2, (-\Delta + q)\phi_2 - c\phi_1 - d\phi_2\}$$

$$\text{for} \quad \phi \in (V_q'(\mathbb{R}^n))^2.$$

Then

$$\begin{aligned} (P, AY) &= (p_1, (-\Delta + q)y_1 - ay_1 - by_2) + (p_2, (-\Delta + q)y_2 - cy_1 - dy_2) \\ &= (p_1, (-\Delta + q)y_1) - a(p_1, y_1) - b(p_1, y_2) + (p_2, (-\Delta + q)y_2) - c(p_2, y_1) \\ &- d(p_2, y_2) \\ &= ((-\Delta + q)p_1, y_1) - a(p_1, y_1) - c(p_2, y_1) + ((-\Delta + q)p_2, y_2) - d(p_2, y_2) \\ &- b(p_1, y_2) \\ &= ((-\Delta + q)p_1 - ap_1 - cp_2, y_1) + ((-\Delta + q)p_2 - bp_1 - dp_2, y_2) \\ &= (A^*P, Y), \end{aligned}$$

where

$$A^*(P = \{p_1, p_2\}) \to \{(-\Delta + q)p_1 - ap_1 - cp_2, (-\Delta + q)p_2 - bp_1 - dp_2\}$$

where A^* is the adjoint for A, P is the adjoint state. Then $A^*P = Y(u) - Z_d$ can be written as

$$(-\Delta + q)p_1 - ap_1 - cp_2 = y_1(u) - z_{1d}$$

$$(-\Delta + q)p_2 - bp_1 - dp_2 = y_2(u) - z_{2d}$$

$$p_1(u) = p_2(u) = 0.$$

So (12) is equivalent to

$$((-\Delta + q)p_1 - ap_1 - cp_2, y_1(v) - y_1(u)) + ((-\Delta + q)p_2 - bp_1 - dp_2, y_2(v) - y_2(u))$$
$$+(Nu, v - u)_{(L^2(\Gamma))^2} \geq 0$$

$$(p_1(u), (-\Delta + q)(y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} - (\frac{\partial p_1(u)}{\partial \nu_A}, y_1(v) - y_1(u))_{L^2(\Gamma)} + (p_1(u),$$
$$\frac{\partial}{\partial \nu_A}(y_1(v) - y_1(u))_{L^2(\Gamma)} - a(p_1(u), y_1(v) - y_1(u)) - b(p_1(u), y_2(v) - y_2(u)) +$$
$$(p_2(u), (-\Delta + q)(y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} - (\frac{\partial p_2(u)}{\partial \nu_A}, y_2(v) - y_2(u))_{L^2(\Gamma)} + (p_2(u),$$
$$\frac{\partial}{\partial \nu_A}(y_2(v) - y_2(u)))_{L^2(\Gamma)} - c(p_2(u), y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)} - d(p_2(u), y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)}$$
$$+(Nu, v - u)_{(L^2(\Gamma))^2} \geq 0.$$

From (9), we obtain

$$(p_1(u), a(y_1(v) - y_1(u)) + b(y_2(v) - y_2(u)) + f_1 - f_1 - a(y_1(v) - y_1(u)))_{L^2(\mathbb{R}^n)} +$$
$$(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1)_{L^2(\Gamma)} + (0, \frac{\partial}{\partial \nu_A}(y_1(v) - y_1(u))_{L^2(\Gamma)} - c(p_2(u), y_1(v) - y_1(u))_{L^2(\mathbb{R}^n)}$$
$$(p_2(u), c(y_1(v) - y_1(u)) + d(y_2(v) - y_2(u)) + f_2 - f_2 - c(y_1 - y_1(u)))_{L^2(\mathbb{R}^n)} +$$
$$(\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2)_{L^2(\Gamma)} + (0, \frac{\partial}{\partial \nu_A}(y_2(v) - y_2(u))_{L^2(\Gamma)} - d(p_2(u), y_2(v) - y_2(u))_{L^2(\mathbb{R}^n)} +$$
$$(Nu, v - u)_{(L^2(\Gamma))^2} \geq 0.$$

Then we have

$$(\frac{\partial p_1(u)}{\partial \nu_A}, v_1 - u_1)_{L^2(\Gamma)} + (\frac{\partial p_2(u)}{\partial \nu_A}, v_2 - u_2)_{L^2(\Gamma)} + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0.$$

i.e.,

$$\int_\Gamma (\frac{\partial p_1(u)}{\partial \nu_A}(v_1 - u_1) + \frac{\partial p_2(u)}{\partial \nu_A}(v_2 - u_2)) d\Gamma + (Nu, v - u)_{(L^2(\Gamma))^2} \geq 0 \quad \forall u \in U_{ad}, v \in U_{ad}.$$

Which completes the proof of the theorem.

Remark 3.2 *To study the optimal control for the scalar case*

$$\begin{cases} (-\Delta + q)y &= ay + f \quad in \quad \mathbb{R}^n \\ y(x) &= g \quad in \quad \Gamma, \end{cases} \tag{13}$$

we define a bilinear form $\pi : V_q(\mathbb{R}^n) \times V_q(\mathbb{R}^n) \to \mathbb{R}$ *by*

$$\pi(y, \phi) = \int_{\mathbb{R}^n} (\nabla y \nabla \phi + q y \phi) dx - a \int_{\mathbb{R}^n} y \phi dx$$

As in theorem (1), we can prove π *is coercive if* $a < \lambda(q)$ *and then there exists a unique solution of* (13) *for* $f \in L^2(\mathbb{R}^n)$*. Therefore, the state of the system is given by the solution of*

$$\begin{cases} (-\Delta + q)y(u) = ay(u) + f + u \quad in \quad \mathbb{R}^n \\ y(u) = u \quad in \quad \Gamma, \end{cases} \tag{14}$$

where u *is given in the space* $L^2(\Gamma)$ *of controls. For given* z_d *in* $L^2(\mathbb{R}^n)$*, the cost function is given by*

$$J(v) = \int_{\mathbb{R}^n} |y(v) - z_d|^2 dx + \int_{\Gamma} (Nv) v d\Gamma$$

where N *is a given hermitian positive definite operator. Then we have the following characterization of optimal control for this system :*

$$\begin{cases} (-\Delta + q)p(u) - ap(u) = y_1(u) - z_d \quad in \quad \mathbb{R}^n \\ p(u) = 0 \quad in \quad \Gamma, \end{cases}$$

$$\int_{\Gamma} \frac{\partial p(u)}{\partial \nu_A} (v - u) d\Gamma + (Nu, v - u)_{L^2(\Gamma)} \geq 0, \quad \forall \quad v \in U_{ad}$$

together with (14), where $p(u)$ *is the adjoint state.*

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