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TWO-DIMENSIONAL REAL SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH QUADRATIC UNPERTURBED PARTS: CLASSIFICATION AND DEGENERATE GENERALIZED NORMAL FORMS [1]

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Abstract

Real two-dimensional autonomous systems of ordinary differential equations whose unperturbed parts are vector homogeneous polynomials of the second order are considered. As to perturbations, they are formal vector power series whose expansions don't contain members of order less than three.

A normalization of the unperturbed part of the system is presented. Namely, nineteen classes of equivalence of vector homogeneous polynomials with respect to any linear invertible transformations have been founded. Each class is represented by a canonical form, i.e. by a polynomial of the special representation having the maximal number of zero coefficients.

Generalized normal forms for five types of systems whose unperturbed parts are degenerate canonical forms are explicitly given. These normal forms can be obtained by almost identical formal transformations.

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Part I

The set of the problem

1 Introduction

This work continues the series of papers ([1] – [6]) devoted to a constructive normalization of real two-dimensional autonomous systems

$$\dot{x}_i = P_i(x) + X_i(x) \qquad (i = 1, 2), \tag{1}$$

where $x = (x_1, x_2)$, $P_i = a_i x_1^2 + 2b_i x_1 x_2 + c_i x_2^2$ is the unperturbed part, $X_i = \sum_{p=2}^{\infty} X_i^{(p+1)}(x)$ is a perturbation of the system and $X_i^{(r)} = \sum_{s=0}^{r} X_i^{(s,r-s)} x_1^s x_2^{r-s}$ – a form of order r $(P = P^{(2)})$.

In [1, p. 1] and [8, § 3] a method of resonance equation is stated. That makes it possible for any system with the fixed unperturbed part to obtain all possible formally equivalent to it generalized normal forms (GNF) in explicit form, on condition that we can overcome some technical computational problems.

As opposed to common definition for resonance normal forms (Poincare normal forms, or formerly normal forms) there are a lot of definitions for GNF. Their concise description is given, for example, in [7], [8].

Sufficiently complete and containing all the proofs the theory of resonance normal forms (i. e. the forms where the matrix of the linear part has nonzero eigenvalues) is given in [9], and in a reduced variant in [10], [11].

This paper pursues two objects.

1) The first goal is to decompose the set of systems

$$\dot{x}_1 = a_1 x_1^2 + 2b_1 x_1 x_2 + c_1 x_2^2, \qquad \dot{x}_2 = a_2 x_1^2 + 2b_2 x_1 x_2 + c_2 x_2^2 \tag{2}$$

unperturbed according to (1) into equivalence classes with respect to linear invertible changes of variables, i. e. to select a finite set of most simple systems: canonical forms (CF) which are mutually linear non-equivalent, and such systems that a system (2) may be reduced to one of CF by a linear invertible change of variables. Here "simplicity" of a CF means that such a form is the most appropriate variant to be the unperturbed part of (1) that later on will be normalized by means of almost identical changes. In fact, reduction to a canonical form means a normalization of quadratic polynomials (P_1, P_2) in (1) or (2).

It is proved in part II that a unperturbed system (2) has 19 canonical forms, being all of them are constructively obtained, i.e. for every CF conditions on coefficients of (2) are given in explicit form and the linear singular change of variables reducing this system to the selected CF is given. It should be noted that 5 CF of 19 have two representations: main and degenerate, being degenerate representation differs from the main one such that one of polynomials in CF, for example P_2, is equal to zero identically.

A similar classification of quadratic canonical forms, including degenerate ones, was previously realized in [2, § 2]. On basis of this classification in [2] – [6] investigations of formal equivalence of system (1) with 11 CF, each taken as the unperturbed part, were carried out and then all the GNF were constructed.

However, as it turned out that given classification was not complete and had some shortcomings, because at that time the conditions (that have to be in a definition of CF) both on the location of nonzero coefficients in P_1, P_2 and polynomial normalizing were not clearly formulated.

These conditions are formulated below in p. 3.3, part II. They do not effect essentially on the following normalization of the system (1), but allows us to extract uniquely so called 'main CF' and linearly equivalent to it additional canonical forms that either have the same number of nonzero elements which may not be located on optimal places or are normalized by other way.

Indicated differences in definition CF led to the situation when in the process of investigation of GNF in some initial systems additional (according to the classification given in [2, § 2]), but not main CF were taken.

In part II all mentioned shortcomings of the previous classification are eliminated, and the classificationf is based on other principles, which allowed one to prove the pairwise linear nonequivalence for obtained CF and to state in explicit form the conditions on the initial system (1) wherein it may be reduced to a definite CF.

2) The second goal of the paper is to normalize all the systems (1) in which every of 5 degenerate CF is sequentially taken as the unperturbed part.

Generally speaking, such a normalization is less effective as compared with normalization the system (1) whose unperturbed part is given by a main canonical form equivalent to a degenerate one. But inasmuch as main canonical forms turn out to be more complex than degenerate ones, up to now due to considerable computational difficulties there is no complete normalization for some of them.

In part III for a system (1) with the unperturbed part selected from 5 degenerate CF and an arbitrary perturbation all GNF that may be obtained by almost identical transformation are written. Examples of normal forms with special structures are also given.

For completeness it should be noted that there is one more method to normalize (1) with a degenerate CF in the unperturbed part. For this purpose we should take some term (or terms) from the perturbation X_2 of (1) that has order greater than two and put it on the place of the absent quadratic polynomial P_2, so that we can justify orders by assigning a weight to each variable.

The above method is described in [1, p. 1] and implemented in [8, § 6]. It is clear that in GNF the new (not quadratic) polynomial P_2 does not change (do not annul even partially), but using the polynomial we can annul some additional terms of the perturbation.

At last, part IV contains results concerning the problem of obtaining all GNF (in explicit form) of systems having one of 19 CF as the unperturbed part.

2 Formal system equivalence

So, consider a two-dimensional real autonomous system (1)

$$\dot{x}_i = P_i(x) + X_i(x) \quad (i = 1, 2),$$

where $P_i = a_i x_1^2 + 2b_i x_1 x_2 + c_i x_2^2$, $X_i = \sum_{p=2}^{\infty} X_i^{(p+1)}(x)$, $X_i^{(p+1)} = \sum_{s=0}^{p+1} X_i^{(s,p-s+1)} x_1^s x_2^{p-s+1}$.

Assume that a formal almost identical change of variables

$$x_i = y_i + h_i(y) \qquad (i = 1, 2), \tag{3}$$

where $y = (y_1, y_2)$, $h_i = \sum_{p=2}^{\infty} h_i^{(p)}(y)$, transforms (1) in the system

$$\dot{y}_i = P_i(y) + Y_i(y) \qquad (i = 1, 2), \tag{4}$$

where $Y_i = \sum_{p=2}^{\infty} Y_i^{(p+1)}(y)$, $Y_i^{(p+1)} = \sum_{s=0}^{p+1} Y_i^{(s,p-s+1)} y_1^s y_2^{p-s+1}$.

By differentiating the change of variables (3) with respect to t along the trajectories of systems (1) and (4), we obtain

$$\sum_{j=1}^{2} \Big(\frac{\partial h_i(y)}{\partial y_j} P_j(y) - \frac{\partial P_i(y)}{\partial y_j} h_j(y) \Big) = X_i(y+h) + P_i(h) - \sum_{j=1}^{2} \frac{\partial h_i(y)}{\partial y_j} Y_j(y) - Y_i(y).$$

Then for any $p \geq 2$ forms $h_i^{(p)}$, $Y_i^{(p+1)}$ satisfy equations

$$\begin{gathered}(a_1 y_1^2 + 2b_1 y_1 y_2 + c_1 y_2^2) \frac{\partial h_i^{(p)}}{\partial y_1} + (a_2 y_1^2 + 2b_2 y_1 y_2 + c_2 y_2^2) \frac{\partial h_i^{(p)}}{\partial y_2} - \\ -2(a_i y_1 + b_i y_2) h_1^{(p)} - 2(b_i y_1 + c_i y_2) h_2^{(p)} = \widehat{Y}_i^{(p+1)} \qquad (i = 1, 2),\end{gathered} \tag{5}$$

where $\widehat{Y}_i^{(p+1)} = \widetilde{Y}_i^{(p+1)}(y) - Y_i^{(p+1)}(y)$, $\widetilde{Y}_i^{(p+1)}(y) = \{X_i(y+h) + P(h) - \sum_{j=1}^{2} Y_j \, \partial h_i / \partial y_j\}^{(p+1)}$ and depend on only $h^{(r)}$ and $Y^{(r+1)}$ with $2 \leq r \leq p-1$.

Hence when in (5) forms $h_i^{(p)}$ and $Y_i^{(p+1)}$ are sequentially defined for each $p = 2, 3, \ldots$, forms $\widetilde{Y}_i^{(p+1)}$ are already known.

Equating coefficients in $y_1^s y_2^{p+1-s}$ $(s = 0, 1, \ldots, p+1)$, in equations (5) we obtain the system of $2(p+2)$ equations with $2(p+1)$ unknown quantities:

$$\begin{gathered}a_2(p-s+2) h_1^{(s-2,p-s+2)} + (a_1(s-3) + 2b_2(p-s+1)) h_1^{(s-1,p-s+1)} + \\ +(2b_1(s-1) + c_2(p-s)) h_1^{(s,p-s)} + c_1(s+1) h_1^{(s+1,p-s-1)} - \\ -2b_1 h_2^{(s-1,p-s+1)} - 2c_1 h_2^{(s,p-s)} = \widehat{Y}_1^{(s,p-s+1)}, \\ \\ a_2(p-s+2) h_2^{(s-2,p-s+2)} + (a_1(s-1) + 2b_2(p-s)) h_2^{(s-1,p-s+1)} + \\ +(2b_1 s + c_2(p-s-2)) h_2^{(s,p-s)} + c_1(s+1) h_2^{(s+1,p-s-1)} - \\ -2a_2 h_1^{(s-1,p-s+1)} - 2b_2 h_1^{(s,p-s)} = \widehat{Y}_2^{(s,p-s+1)}.\end{gathered} \tag{6}$$

In what follows we assume that in system (6) coefficients of series $\widehat{Y}_i$ and h_i are equal to zero, if one of upper indices less than zero.

For any $p \geq 2$ consistency conditions for (6) may be written in the form of n_p linear equations connecting coefficients of homogeneous polynomials $Y_i^{(p+1)}$:

$$\sum_{s=0}^{p+1} (c_{\nu 1}^{ps} Y_1^{(s,p-s+1)} + c_{\nu 2}^{ps} Y_2^{(s,p-s+1)}) = \widetilde{c} \qquad (\nu = \overline{1, n_p}, \quad n_p \geq 2), \tag{7}$$

where in each equation $\widetilde{c} = \sum_{s=0}^{p+1} (c_{\nu 1}^{ps} \widetilde{Y}_1^{(s,p-s+1)} + c_{\nu 2}^{ps} \widetilde{Y}_2^{(s,p-s+1)})$.

Definition 1 *We will call equations (7) resonant ones.*

The main goal of the method of the same name that we describe here is to obtain resonant equations in explicit form, i.e. to calculate factors $c^{ps}_{\nu i}$.

However, solving the problem often faces considerable technical obstacles, being their complexity depends on the number of nonzero coefficients in P_1, P_2.

Consequently, we have to simplify the quadratic unperturbed part of system (1) as much as possible, reducing the system to so called canonical form (CF) by a linear invertible change of variables.

In the special case that the system has the linear first approximation, the reducing it to a CF means clearly the reducing the matrix of the linear part to a Jordan form. There is no general accepted definition for CF in this case.

The principles of definition of canonical form will be formulated below, reasoning from demands that arise when solving system (6) and alleviate the problem.

As was mentioned above it is the minimization of the number of nonzero coefficients in P_1 and P_2 that is the important criterion of simplifying system (6).

The question of considerable importance is which coefficients should be annulled in the first place. So, for P_1 it is the best way (if that is possible) to put $c_1 = 0$, for P_2 – $a_2 = 0$. It turns out that it is sufficient to eliminate 3 summands in left-side hands of system (6).

It is also clear that the more residual nonzero coefficients may be normalized by 1 the simpler the solving (6) will be.

In specific cases one of polynomials in (1) (for example P_2) may be transformed to a polynomial that is equal to zero identically. Then consequent simplifications of P_1 lead to arising degenerate CF which are linearly equivalent to main CF and, as mentioned above, have advantages and disadvantages.

In conclusion we remind some definition from [1] that will be necessary later on.

Definition 2 *The coefficients of homogeneous polynomials $Y_i^{(p+1)}$ in (4), entering at least in one of resonant equations (7) will be called resonant coefficients, and the other – nonresonant ones. The coefficients of homogeneous polynomials $h_i^{(p)}$ which remain free when solving (6) will be called resonant ones.*

We correlate the matrix $\Upsilon^p = \{\upsilon^p_{\nu k}\}_{\nu,k=1}^{n_p}$, where $\upsilon^p_{\nu k} = c^{ps_k}_{\nu i_k}$ with an arbitrary set of n_p coefficients $Y_{i_k}^{(s_k,p+1-s_k)}$ of homogeneous polynomials $Y_1^{(p+1)}$, $Y_2^{(p+1)}$, where $k = \overline{1,n_p}$, $s_k \in \{0,\ldots,p+1\}$, $i_k \in \{1,2\}$.

Definition 3 *We call a set of n_p coefficients of homogeneous polynomials $Y_i^{(p+1)}$ resonant if* $\det \Upsilon^p \neq 0$.

Thus for any $p \geq 2$ resonant set is a minimal set of coefficients from $Y_1^{(p+1)}$, $Y_2^{(p+1)}$, such that each of them is at least in one of equations (7), being resonant equations are uniquely decidable with respect to (7). In this case only different resonant coefficients may be in the resonant set, otherwise in Υ^p there will be equal columns or a zero column.

Definition 4 *We will call a system (4) generalized normal form (GNF), if for any $p \geq 2$ all the coefficients of homogeneous polynomials $Y_i^{(p+1)}$ are equal to zero, excepting perhaps coefficients from a resonant set.*

This definition of GNF corresponds to the concept of generalized normal form of the first order that was entered in [12].

Part II

Canonical form of a unperturbed system

3 Linear equivalence of quadratic systems

3.1 Form and characteristic of quadratic systems

Consider a two-dimensional real unperturbed system (2)

$$\dot{x} = P(x) \ \text{ or } \ \dot{x} = A\,q^{[2]}(x) \qquad (P(x) \not\equiv 0, \ \ A \neq \mathbf{0}),$$

where $P = \begin{pmatrix} P_1(x) \\ P_2(x) \end{pmatrix} = \begin{pmatrix} a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2 \\ a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & 2b_1 & c_1 \\ a_2 & 2b_2 & c_2 \end{pmatrix}, \quad q^{[2]}(x) = \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}.$

Definition 5 *For polynomials P_1 and P_2 their common factor with maximal nonzero power l $(l \in \{1,2\})$ will be called common factor P_0 of the polynomials. If there is no common factor for P_1, P_2 we will assume that $l = 0$.*

For the polynomial P consider a function R called resultant:

$$R = \begin{vmatrix} a_1 & 2b_1 & c_1 & 0 \\ 0 & a_1 & 2b_1 & c_1 \\ a_2 & 2b_2 & c_2 & 0 \\ 0 & a_2 & 2b_2 & c_2 \end{vmatrix} = \delta_{ac}^2 - 4\delta_{ab}\delta_{bc}, \tag{8}$$

where $\delta_{ab} = a_1b_2 - a_2b_1, \ \ \delta_{ac} = a_1c_2 - a_2c_1, \ \ \delta_{bc} = b_1c_2 - b_2c_1.$

Assertion 1 *The polynomials P_1, P_2 in (2) have the common factor iff $R = 0$* (see [13, p. 59]).

3.2 Linear transformations of quadratic systems

Simplify a system (2) by a linear invertible change of variables

$$x_1 = r_1y_1 + s_1y_2, \quad x_2 = r_2y_1 + s_2y_2 \quad \text{or} \quad x = Ly, \tag{9}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad L = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}, \quad \delta = \delta_{rs} = \det L \neq 0.$

Let (9) transform system (2) in the system

$$\dot{y} = \widetilde{P}(y) \ \text{ or } \ \dot{y} = \widetilde{A}\, q^{[2]}(y), \tag{10}$$

where $\widetilde{P} = \begin{pmatrix} \widetilde{P}_1 \\ \widetilde{P}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{a}_1 y_1^2 + 2\widetilde{b}_1 y_1 y_2 + \widetilde{c}_1 y_2^2 \\ \widetilde{a}_2 y_1^2 + 2\widetilde{b}_2 y_1 y_2 + \widetilde{c}_2 y_2^2 \end{pmatrix}$, $\widetilde{A} = \begin{pmatrix} \widetilde{a}_1 & 2\widetilde{b}_1 & \widetilde{c}_1 \\ \widetilde{a}_2 & 2\widetilde{b}_2 & \widetilde{c}_2 \end{pmatrix}$.

For (10) analogously to (2) we enter the resultant $\widetilde{R}$ by the formula (8).

By differentiating equations (9) along the trajectories of systems (2) and (10), we obtain $P(Ly) = L\widetilde{P}(y)$ or

$$\widetilde{P}(y) = L^{-1}P(Ly) = L^{-1}Aq^{[2]}(Ly), \tag{11}$$

where $L^{-1} = \delta^{-1}\begin{pmatrix} s_2 & -s_1 \\ -r_2 & r_1 \end{pmatrix}$, $L^{-1}A = \delta^{-1}\begin{pmatrix} \delta_{as} & 2\delta_{bs} & \delta_{cs} \\ -\delta_{ar} & -2\delta_{br} & -\delta_{cr} \end{pmatrix}$.

Hence we have in (11) :

$$\begin{pmatrix} \widetilde{a}_1 y_1^2 + 2\widetilde{b}_1 y_1 y_2 + \widetilde{c}_1 y_2^2 \\ \widetilde{a}_2 y_1^2 + 2\widetilde{b}_2 y_1 y_2 + \widetilde{c}_2 y_2^2 \end{pmatrix} = \delta^{-1}\begin{pmatrix} \delta_{as} & 2\delta_{bs} & \delta_{ds} \\ -\delta_{ar} & -2\delta_{br} & -\delta_{cr} \end{pmatrix}\begin{pmatrix} (r_1y_1 + s_1y_2)^2 \\ (r_1y_1 + s_1y_2)(r_2y_1 + s_2y_2) \\ (r_2y_1 + s_2y_2)^2 \end{pmatrix}.$$

Equating coefficients of $y_1^s y_2^{2-s}$ $(s = \overline{0,2})$, we have

$$\begin{aligned} \delta\widetilde{a}_1 &= s_2P_1(r_1, r_2) - s_1P_2(r_1, r_2), \quad -\delta\widetilde{a}_2 = r_2P_1(r_1, r_2) - r_1P_2(r_1, r_2), \\ \delta\widetilde{b}_1 &= s_2(a_1r_1s_1 + b_1\delta_* + c_1r_2s_2) - s_1(a_2r_1s_1 + b_2\delta_* + c_2r_2s_2), \\ -\delta\widetilde{b}_2 &= r_2(a_1r_1s_1 + b_1\delta_* + c_1r_2s_2) - r_1(a_2r_1s_1 + b_2\delta_* + c_2r_2s_2), \\ \delta\widetilde{c}_1 &= s_2P_1(s_1, s_2) - s_1P_2(s_1, s_2), \quad -\delta\widetilde{c}_2 = r_2P_1(s_1, s_2) - r_1P_2(s_1, s_2). \end{aligned} \tag{12}$$

where $\delta_* = r_1s_2 + r_2s_1$.

Assertion 2 *In the systems (2) and (10) either* $R, \widetilde{R} = 0$, *or* $R\widetilde{R} > 0$, *i. e. the sign of* R *is invariant with respect to any linear invertible change of variables (9).*

Actually it is easy to verify that $\widetilde{R} = \delta^2 R$.

Remark 1 For brevity in what follows we will identify system (2) (and other systems obtained from it) with the matrix A or polynomial P. The equations (9) will be identified with the matrix L.

We set off two standard changes of variables from the changes (9) transforming (2) in the system (10) :

$$\begin{pmatrix} r_1 & 0 \\ 0 & s_2 \end{pmatrix} \text{ – normalizing, } \quad \widetilde{A} = \begin{pmatrix} a_1r_1 & 2b_1s_2 & c_1s_2^2r_1^{-1} \\ a_2r_1^2s_2^{-1} & 2b_2r_1 & c_2s_2 \end{pmatrix}; \tag{13}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ – renumbering, } \quad \widetilde{A} = \begin{pmatrix} c_2 & 2b_2 & a_2 \\ c_1 & 2b_1 & a_1 \end{pmatrix}. \tag{14}$$

3.3 How to define a canonical form

Proposition 1 *Without loss of generality we will further assume that in (2) the polynomial* $P_1(x) \not\equiv 0$, *otherwise* $P_2(x) \not\equiv 0$ *and we can perform (14).*

In system (2) we assign to any element of A or a coefficient of P_1 or P_2 an index equal to the number that is on the place of the element in the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Definition 6 *For the matrix* A *of system (2) the sum of indices of its nonzero elements is said to be matrix index.*

Definition 7 *The system (2) is referred to as a canonical form* (CF^l) *or, that is the same, main* CF^l, *if a linear invertible change of variables (9) does not transform (2) to a system that is more preferable than the initial one according to the following principles of hierarchy:*

1) *The system is nondegenerate, i. e.* $P_1, P_2 \not\equiv 0$, *and, if possible,* $P_1 \equiv P_2$.

2) *The matrix* A *has the minimal number of nonzero elements.*

3) *The index of* A *is minimal.*

4) *The number of elements of* A *with module equal to 1 is maximal.*

5) *The arrangement of nonzero coefficients of* P_1 *is the following:*

  5a) *The order of the first nonzero coefficient in* P_1 *is minimal.*

  5b) *The order of the last nonzero coefficient in* P_1 *is maximal.*

6) *Normalizing nonzero coefficients of the system:*

  6a) *In* P_2 *left nonzero element equals 1.*

  6b) *In* P_1 *the module of the right nonzero coefficient equals 1.*

Remark 2 When defining CF^l the principles $1-4$ are basic. The principles 5 and 6 allows us to select so called main CF^l among existing linear equivalent canonical forms, though other such forms may be also selected as a first approximation in an arbitrary perturbed system when reducing it to GNF.

Such reasonings lead to the concept of additional CF^l.

Definition 8 *The system (2) is said to be additional canonical form* (ACF^l) *if it is linearly equivalent to a main* CF^l, *but the principle 5 and maybe principle 6 do not hold. In this case* ACF^l *obtained from any nonsymmetrical* CF^l_i *by (14) will be denoted as* $CF^{l\mathrm{r}}_i$.

4 Canonical forms of system (2) for $\mathbf{l = 0}$

Consider the system (2) $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2 \\ a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2 \end{pmatrix}$, where homogeneous polynomials $P_1, P_2 \not\equiv 0$ and do not have a common factor.

Then by definition 5 we have $l = 0$, and by the statement 1 the resultant entered in (8) $R = \delta_{ac}^2 - 4\delta_{ab}\delta_{bc} \neq 0$. Hence in particular $a_1^2 + a_2^2 \neq 0$ and $c_1^2 + c_2^2 \neq 0$.

To write all the canonical forms to which system (2) may be transformed by a linear change of variables (9), we have to formulate some conditions.

Consider two cubic polynomials

$$Q_1(t)=t^3-ut^2+vt-1,\quad Q_2(t)=t^3+(v^2-2u)t^2+(u^2-2v)t+1, \tag{15}$$

where parameters $u,\,v$ have the following restrictions:

$$uv\neq 0;1,\quad u\neq v,\quad u^2+v\neq 0,\quad u+v^2\neq 0. \tag{16}$$

Assertion 3 *The roots of $Q_1,\ Q_2$ satisfy one of two conditions:*

$$\begin{array}{l}\exists\ t_1',\,t_1''\in\mathbb{R}:\ Q_1(t_1')=0,\quad Q_1(t_1'')=0,\quad t_1'\neq t_1'';\\ \exists!\ t_1\in\mathbb{R}:\ Q_1(t_1)=0,\quad \forall\ t_2\in\mathbb{R}:\quad Q_2(t_2)=0\ \Rightarrow\ t_2\neq t_1\quad (t_1,t_2\neq 0,u).\end{array} \tag{17}$$

Proof Suppose that conditions (17) are not fulfilled.

It means that $Q_1(t)$ has the unique real root $t_1=-\tau$ and this root is a root for $Q_2(t)$ as well. Then the polynomials (15) have the form

$$Q_i(t)=(t+\tau)(t^2+\mathfrak{b}_it+\mathfrak{c}_i)=t^3+(\tau+\mathfrak{b}_i)t^2+(\tau\mathfrak{b}_i+\mathfrak{c}_i)t+\tau\mathfrak{c}_i\qquad(i=1,2).$$

So, $\begin{cases}\mathfrak{b}_1=-u-\tau\\ \mathfrak{c}_1=-\tau^{-1}\\ \tau\mathfrak{b}_1+\mathfrak{c}_1=v\end{cases}$ and $\begin{cases}\mathfrak{b}_2=v^2-\tau-2u\\ \mathfrak{c}_2=\tau^{-1}\\ \tau\mathfrak{b}_2+\mathfrak{c}_2=u^2-2v\end{cases}$. From the second system we obtain $(u+\tau)^2=\tau(v+\tau^{-1})^2$. Hence $\tau>0$, otherwise $u=-\tau,\ v=-\tau^{-1}$, that contradicts (16).

Uniqueness condition for the real root of Q_1 implies the inequality $\mathfrak{b}_1^2-4\mathfrak{c}_1\leq 0$, that is equivalent to the impossible inequality $(u+\tau)^2+4\tau^{-1}\leq 0$, as $\tau>0$. $\square$

The list of canonical forms of the system (2) for $l=0$:

$$\mathrm{CF}_3^0=\begin{pmatrix}1&u&0\\0&0&1\end{pmatrix},\ \mathrm{CF}_1^0=\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix},\ \mathrm{CF}_2^0=\begin{pmatrix}0&0&1\\1&0&0\end{pmatrix},\ \mathrm{CF}_7^0=\begin{pmatrix}0&u&1\\1&0&0\end{pmatrix},$$
$$\mathrm{CF}_4^0=\begin{pmatrix}u&0&1\\0&0&1\end{pmatrix},\quad \mathrm{CF}_5^0=\begin{pmatrix}u&0&\sigma\\0&1&0\end{pmatrix},\quad \mathrm{CF}_6^0=\begin{pmatrix}u&0&1\\1&0&0\end{pmatrix},$$
$$\mathrm{CF}_8^0=\begin{pmatrix}u&1&0\\0&1&v\end{pmatrix},\ \mathrm{CF}_9^0=\begin{pmatrix}u&0&\sigma\\0&1&v\end{pmatrix},\ \mathrm{CF}_{10}^0=\begin{pmatrix}1/2&u&-1\\0&1&0\end{pmatrix},$$

where $u,v\neq 0,\ \sigma=\pm 1;$ in CF_3^0 $u\geq 1,$ in CF_4^0 $u>1/4,$ in CF_5^0 $u\neq\pm 1/2$ for $R>0,$ in CF_9^0 $u\neq 1/2$ for $R<0,$ in CF_{10}^0 $0<u<2^{1/2}.$

Remark 3 In $\mathrm{CF}_1^0,\ \mathrm{CF}_2^0,\ \mathrm{CF}_3^0,\ \mathrm{CF}_6^0$ and in CF_7^0 $R=1,$ in CF_4^0 $R=u^2,$ in CF_5^0 $R=u\sigma,$ in CF_8^0 $R=u^2v^2-uv,$ in CF_9^0 $R=u^2v^2+u\sigma,$ in CF_{10}^0 $R=-1/2.$

Theorem 1 *In case of $l=0$ the canonical forms CF_1^0 — CF_{10}^0 are pairwise linearly nonequivalent and system (2) is reduced to one of them by a linear invertible change of variables (9).*

Proof

1) $R = \delta_{ac}^2 - 4\delta_{ab}\delta_{bc} > 0$.

We demonstrate at first that there is a change (9) that reduces (2) to a system (10) of the form

$$\begin{pmatrix} \widetilde{a}_1 & 0 & \widetilde{c}_1 \\ \widetilde{a}_2 & 0 & \widetilde{c}_2 \end{pmatrix}. \tag{18}$$

1a) $\delta_{ab} \neq 0$. Then for $r_1 = t_1^* t^*,\ s_1 = t_2^* t^*,\ r_2, s_2 = t^*,$ when $t_1^* = (-\delta_{ac} + R^{1/2})(2\delta_{ab})^{-1}$, $t_2^* = (-\delta_{ac} - R^{1/2})(2\delta_{ab})^{-1},\ t^* = R^{1/2}(\delta_{ab})^{-1}\ (\delta = t^{*3})$, we obtain the system (18) in which $\widetilde{a}_i = (-1)^i (t_{3-i}^* P_2(t_1^*, 1) - P_1(t_1^*, 1)),\ \widetilde{c}_i = (-1)^i (t_{3-i}^* P_2(t_2^*, 1) - P_1(t_2^*, 1))\quad (i = 1, 2)$.

1b) $\delta_{ab} = 0,\ \delta_{bc} \neq 0\ \ (\delta_{ac} = 0)$.

1b1) $a_2 \neq 0$. Then for $r_1 = 1,\ s_1 = -b_2,\ r_2 = 0,\ s_2 = a_2$ we obtain the system (18), where $\widetilde{a}_1 = a_1 + b_2,\ \widetilde{c}_1 = a_1 b_2^2 - 2b_1 a_2 b_2 + a_2^2 c_1 - b_2^3 + a_2 b_2 c_2,\ \widetilde{a}_2 = 1,\ \widetilde{c}_2 = a_2 c_2 - b_2^2$.

1b2) $a_2 = 0\ \ (b_2 = 0, a_1, c_2 \neq 0)$. Then for $r_1 = a_1^{-1},\ s_1 = -(a_1 c_2)^{-1} b_1,\ r_2 = 0,$ $s_2 = c_2^{-1}$ we obtain (18) with $\widetilde{a}_1 = 1,\ \widetilde{c}_1 = (a_1 c_1 - b_1^2 + b_1 c_2) c_2^{-2},\ \widetilde{a}_2 = 0,\ \widetilde{c}_2 = 1$.

1c) $b_1, b_2 = 0\ \ (\delta_{ac} \neq 0)$. Then in the system (18) $\widetilde{a}_i = a_i,\ \widetilde{c}_i = c_i\quad (i = 1, 2)$.

Now a change of variables (9) reduces (18) to a system with coefficients

$$\begin{gathered}
\widetilde{\delta}\breve{a}_1 = s_2(\widetilde{a}_1 r_1^2 + \widetilde{c}_1 r_2^2) - s_1(\widetilde{a}_2 r_1^2 + \widetilde{c}_2 r_2^2),\quad -\widetilde{\delta}\breve{a}_2 = r_2(\widetilde{a}_1 r_1^2 + \widetilde{c}_1 r_2^2) - r_1(\widetilde{a}_2 r_1^2 + \widetilde{c}_2 r_2^2),\\
\widetilde{\delta}\breve{b}_1 = s_2(\widetilde{a}_1 r_1 s_1 + \widetilde{c}_1 r_2 s_2) - s_1(\widetilde{a}_2 r_1 s_1 + \widetilde{c}_2 r_2 s_2),\\
-\widetilde{\delta}\breve{b}_2 = r_2(\widetilde{a}_1 r_1 s_1 + \widetilde{c}_1 r_2 s_2) - r_1(\widetilde{a}_2 r_1 s_1 + \widetilde{c}_2 r_2 s_2),\\
\widetilde{\delta}\breve{c}_1 = s_2(\widetilde{a}_1 s_1^2 + \widetilde{c}_1 s_2^2) - s_1(\widetilde{a}_2 s_1^2 + \widetilde{c}_2 s_2^2),\quad -\widetilde{\delta}\breve{c}_2 = r_2(\widetilde{a}_1 s_1^2 + \widetilde{c}_1 s_2^2) - r_1(\widetilde{a}_2 s_1^2 + \widetilde{c}_2 s_2^2).
\end{gathered} \tag{19}$$

For brevity we will further omit the symbol $\sim$ over coefficients of the system (18).

$\mathbf{1_1}$) $a_1 c_2 = 0$. Then $a_2 c_1 \neq 0$.

$\mathbf{1_1^1}$) $a_1, c_2 = 0$. Then (19) for $r_1 = (a_2^2 c_1)^{-1/3},\ s_1, r_2 = 0,\ s_2 = a_2 r_1^2$ is CF_2^0.

$\mathbf{1_1^2}$) $a_1 = 0,\ c_2 \neq 0$. Then (19) for $r_1 = 0,\ s_1 = c_1 r_2^2,\ r_2 = (a_2 c_1^2)^{-1/3},\ s_2 = 0$ is CF_6^0 with $u = c_2 (a_2 c_1^2)^{-1/3} \neq 0$. $\mathbf{1_1^3}$) $a_1 \neq 0,\ c_2 = 0$. Then (19) for $r_1 = (a_2^2 c_1)^{-1/3},\ s_1, r_2 = 0,$ $r_2 = a_2 r_1^2$ is CF_6^0 with $u = a_1 (a_2^2 c_1)^{-1/3} \neq 0$.

$\mathbf{1_2}$) $a_1 c_2 \neq 0$.

$\mathbf{1_2^1}$) $a_2, c_1 = 0$. Then (19) for $r_1 = a_1^{-1},\ s_1, r_2 = 0,\ s_2 = c_2^{-1}$ is CF_1^0.

$\mathbf{1_2^2}$) $a_2 = 0,\ c_1 \neq 0$.

$\mathbf{1_2^2a}$) $0 \neq a_1 c_1 c_2^{-2} \leq 1/4$. Then (19) for $r_1 = a_1^{-1},\ s_1 = (2a_1)^{-1}(1 + (1 - 4a_1 c_1 c_2^{-2})^{1/2}),$ $r_2 = 0,\ s_2 = c_2^{-1}$ is CF_3^0 with $u = 1 + (1 - 4a_1 c_1 c_2^{-2})^{1/2} \geq 1$.

$\mathbf{1_2^2b}$) $a_1 c_1 c_2^{-2} > 1/4$. Then (19) for $r_1 = c_1 c_2^{-2},\ s_1, r_2 = 0,\ s_2 = c_2^{-1}$ is CF_4^0 with $u = a_1 c_1 c_2^{-2} > 1/4$.

$\mathbf{1_2^3}$) $a_2 \neq 0,\ c_1 = 0$.

$\mathbf{1_2^3a}$) $0 \neq a_2 c_2 a_1^{-2} \leq 1/4$. Then system (19) for $r_1 = 0,\ s_1 = a_1^{-1},\ r_2 = c_2^{-1},\ s_2 =$ $(2c_2)^{-1}(1 + (1 - 4a_2 c_2 a_1^{-2})^{1/2})$ is CF_3^0 with $u = 1 + (1 - 4a_2 c_2 a_1^{-2})^{1/2} \geq 1$.

$\mathbf{1_2^3b}$) $a_2 c_2 a_1^{-2} > 1/4$. Then (19) for $r_1 = 0,\ s_1 = a_1^{-1},\ r_2 = a_2 a_1^{-2},\ s_2 = 0$ is CF_4^0 with $u = a_2 c_2 a_1^{-2} > 1/4$.

$\mathbf{1_2^4}$) $a_2 c_1 \neq 0$.

$\mathbf{1_2^4a})$ $a_1c_1^{1/3} = a_2^{1/3}c_2$. Then the system (19) for $r_1 = a_1(2a_1^2 - 2a_2c_2)^{-1}$, $s_1 = a_1|2a_1^4 - 2a_2^2c_2^2|^{-1/2}\mathrm{sign}\,(a_2c_2)$, $r_2 = a_1^2(2a_1^2c_2 - 2a_2c_2^2)^{-1}$, $s_2 = -a_1^2c_2^{-1}|2a_1^4 - 2a_2^2c_2^2|^{-1/2}\mathrm{sign}\,(a_2c_2)$ – CF_5^0 with $u = (a_1^2 + a_2c_2)(2a_1^2 - 2a_2c_2)^{-1} \neq 0; \pm 1/2$, $\sigma = \mathrm{sign}\,u$. At the same time $a_1^2 \pm a_2c_2 \neq 0$, else in (2) $P_2 = a_1^{-1}a_2P_1$ and $l = 3$.

$\mathbf{1_2^4b})$ $a_1^2 + a_2c_2 = 0$. Then (19) for $r_1 = a_1(a_1^3 + a_2^2c_1)^{-2/3}$, $s_1 = (a_1^3 + a_2^2c_1)^{-1/3}$, $s_2 = 0$, $r_2 = a_2(a_1^3 + a_2^2c_1)^{-2/3}$ – CF_7^0 with $u = 2a_1(a_1^3 + a_2^2c_1)^{-1/3} \neq 0$, being $a_1^3 + a_2^2c_1 = -a_2\delta_{ac} \neq 0$.

$\mathbf{1_2^4c})$ $c_2^2 + a_1c_1 = 0$. Then (19) for $r_1 = c_1(c_2^3 + c_1^2a_2)^{-2/3}$, $s_1 = 0$, $r_2 = c_2(c_2^3 + c_1^2a_2)^{-2/3}$, $s_2 = (c_2^3 + c_1^2a_2)^{-1/3}$ – CF_7^0 with $u = 2c_2(c_2^3 + c_1^2a_2)^{-1/3} \neq 0$, being $c_2^3 + c_1^2a_2 = -c_1\delta_{ac} \neq 0$.

$\mathbf{1_2^4d})$ $a_1c_1^{1/3} \neq a_2^{1/3}c_2$, $a_1^2 + a_2c_2 \neq 0$, $c_2^2 + a_1c_1 \neq 0$. Then (19) for $r_1 = (a_2^2c_1)^{-1/3}$, $s_1, r_2 = 0$, $s_2 = (a_2c_1^2)^{-1/3}$ has the form

$$\mathrm{F}_1^0 = \begin{pmatrix} u & 0 & 1 \\ 1 & 0 & v \end{pmatrix}$$

with $u = a_1(a_2^2c_1)^{-1/3}$, $v = c_2(a_2c_1^2)^{-1/3}$ satisfying (16) and $R = (uv - 1)^2 > 0$.

Thus, it is proved that under above mentioned conditions for u and v F_1^0 with $R > 0$ and index 8 cannot be reduced to the system having more than two zeroes. Reduce F_1^0 to CF_8^0 with index 6, and if we will not succeed, then to CF_9^0 with index 7.

An arbitrary change (9) brings F_1^0 to the system with $\widetilde{R} > 0$ and coefficients

$$\begin{aligned}
&\widetilde{a}_1 = -((s_1 - us_2)r_1^2 + (vs_1 - s_2)r_2^2)\delta^{-1}, && \widetilde{a}_2 = (r_1^3 - ur_1^2r_2 + vr_1r_2^2 - r_2^3)\delta^{-1},\\
&\widetilde{b}_1 = -((s_1 - us_2)r_1s_1 + (vs_1 - s_2)r_2s_2)\delta^{-1}, && \widetilde{b}_2 = ((r_1 - ur_2)r_1s_1 + (vr_1 - r_2)r_2s_2)\delta^{-1},\\
&\widetilde{c}_1 = -(s_1^3 - us_1^2s_2 + vs_1s_2^2 - s_2^3)\delta^{-1}, && \widetilde{c}_2 = ((r_1 - ur_2)s_1^2 + (vr_1 - r_2)s_2^2)\delta^{-1}.
\end{aligned}$$

Coefficients $\widetilde{a}_2, \widetilde{c}_1$ in this system may be taken as zeros, if $Q_1(t)$ from (15) has two different real roots, i. e. the condition (17_1) holds.

In this case, assuming in (9) $r_1 = t_1'r_2$, $s_1 = t_1''s_2$, being the change remains invertible, we have $\widetilde{a}_2 = 0$ and $\widetilde{c}_1 = 0$, so there remains to fulfill a normalization.

Thus, F_1^0 with $u = u_*$, $v = v_*$, satisfying (16) and (17_1), for selected r_1, s_1 and $r_2 = (t_1' - t_1'')(2(t_1'^2t_1'' + v_*t_1' - u_*t_1't_1'' - 1))^{-1}$, $s_2 = (t_1'' - t_1')(2(t_1't_1''^2 - u_*t_1't_1'' + v_*t_1'' - 1))^{-1}$ may be reduced to CF_8^0 with $u = (u_*t_1'^2 - t_1'^2t_1'' - v_*t_1'' + 1)(2(t_1'^2t_1'' + v_*t_1' - u_*t_1't_1'' - 1))^{-1}$, $v = (u_*t_1''^2 - t_1't_1''^2 - v_*t_1' + 1)(2(t_1't_1''^2 - u_*t_1't_1'' + v_*t_1'' - 1))^{-1}$, at that $uv \neq 0$, else $R = 0$.

Let the condition (17_2) fulfill instead of (17_1). Then $\widetilde{a}_2^2 + \widetilde{c}_1^2 \neq 0$.

Taking in (9) $r_1 = (tv - 1)(u - t)^{-1}t^{-1}r_2$, $s_1 = ts_2$, we obtain the system

$$\begin{pmatrix} (tv-1)t^{-1}r_2 & 0 & t(t-u)r_2^{-1}s_2^2 \\ Q_2(t)t^{-2}(t-u)^{-2}r_2^2s_2^{-1} & 2(uv-1)(t-u)^{-1}r_2 & (tu+v)s_2 \end{pmatrix},$$

at that the change has to be invertible, i. e. $(1 - tv)(t - u)^{-1}t^{-1} \neq t$ or $Q_1(t) \neq 0$.

In accordance with (17_2) we have $Q_1(t_2) \neq 0$. Hence $\widetilde{a}_2 = 0$ for $t = t_2$, i. e. F_1^0 with $u = u_*$, $v = v_*$ under conditions (16), (17_2), by the change (9) with $t = t_2$ for selected r_1, s_1 and $r_2 = (u_* - t_2)(2u_*v_* - 2)^{-1}$, $s_2 = |2t_2(u_*v_* - 1)|^{-1/2}$ is reduced to CF_9^0 with $\sigma = -\mathrm{sign}\,(t_2(u_*v_* - 1))$, $u = (u_* - t_2)(t_2v_* - 1)(2t_2(u_*v_* - 1))^{-1}$, $v = (t_2u_* + v_*)|2t_2(u_*v_* - 1)|^{-1/2}$. In this case $uv \neq 0$, otherwise, if $t_2v_* - 1 = 0$, then $Q_2(t_2) = t_2(t - u)^2 \neq 0$, and if $t_2u_* + v_* = 0$, then $Q_2(-u_*^{-1}v_*) = (1 - u_*v_*)(1 - u_*^{-3}v_*^3) \neq 0$.

2) $R=\delta_{ac}^2-4\delta_{ab}\delta_{bc}<0$. Then $\delta_{ab}\delta_{bc}>0$.

At first we will prove that there is a change (9) that brings (2) to a system (10) of the form

$$\begin{pmatrix} \widetilde{a}_1 & 2\widetilde{b}_1 & \widetilde{c}_1 \\ 0 & 2\widetilde{b}_2 & \widetilde{c}_2 \end{pmatrix} \qquad (\widetilde{a}_1\widetilde{b}_2\neq 0). \tag{20}$$

2a) $a_2\neq 0$. Then for $r_1=r_*,\ s_1=1,\ r_2=-1,\ s_2=0$, where r_* is a real root of the polynomial $a_2r_1^3+(2b_2-a_1)r_1^2+(c_2-2b_1)r_1-c_1$, we obtain system (20), where $\widetilde{a}_1=a_2r_*^2+2b_2r_*+c_2,\ \widetilde{b}_1=-a_2r_*-b_2,\ \widetilde{c}_1=a_2,\ \widetilde{b}_2=b_1+a_1r_*-b_2r_*-a_2r_*^2,\ \widetilde{c}_1=a_2r_*-a_1$.

2b) $a_2=0$. Then in (20) $\widetilde{a}_i=a_i,\ \widetilde{b}_i=b_i,\ \widetilde{c}_i=c_i\quad(i=1,2)$.

An arbitrary change (9) with $r_2=0$ brings system (20) to the system

$$\begin{pmatrix} \widetilde{a}_1r_1 & 2(\widetilde{a}_1-\widetilde{b}_2)s_1+2\widetilde{b}_1s_2 & ((\widetilde{a}_1-2\widetilde{b}_2)s_1^2+(2\widetilde{b}_1-\widetilde{c}_2)s_1s_2+\widetilde{c}_1s_2^2)r_1^{-1} \\ 0 & 2\widetilde{b}_2r_1 & 2\widetilde{b}_2s_1+\widetilde{c}_2s_2 \end{pmatrix}. \tag{21}$$

We further omit $\widetilde{\ }$ over the coefficients in (20) and mark coefficients in (21) by $\breve{\ }$. $\mathbf{2_1}$) $(a_1-b_2)c_2-2b_1b_2=0$ (i. e. we may take $\breve{b}_1,\breve{c}_2=0$). Then (21) for $r_1=(2b_2)^{-1},\ s_1=-|a_1|^{1/2}c_2(-2|b_2|R)^{-1/2}\mathrm{sign}\,b_2,\ r_2=0,\ s_2=|2a_1b_2|^{1/2}(-R)^{-1/2}$ is a CF_5^0 with $\sigma=-\mathrm{sign}\,u$, $u=a_1(2b_2)^{-1}\neq 0$.

$\mathbf{2_2}$) $(a_1-b_2)c_2-2b_1b_2\neq 0$. Let $d_*=(2b_1-c_2)^2+4c_1(2b_2-a_1)$.

$\mathbf{2_2^1}$) $d_*\geq 0$, i. e. we may take $\breve{c}_1=0$. In this case $b_*=2c_1(a_1-b_2)+b_1(c_2-2b_1+d_*^{1/2}\mathrm{sign}\,(c_2-2b_1))\neq 0$, else for $r_1=a_1^{-1},\ s_1=2c_1,\ r_2=0,\ s_2=(c_2-2b_1+d_*^{1/2}\mathrm{sign}\,(c_2-2b_1))$ in system (21) $\breve{R}=\breve{c}_2^2\geq 0$, that is impossible. Therefore for $r_1=(2b_2)^{-1},\ s_1=c_1b_*^{-1}$, $r_2=0,\ s_2=(2b_*)^{-1}(c_2-2b_1+d_*^{1/2}\mathrm{sign}\,(c_2-2b_1))$ (21) is a CF_8^0 with $u=a_1(2b_2)^{-1},\ v=(2b_*)^{-1}(4c_1b_2+c_2^2-2b_1c_2+c_2d_*^{1/2}\mathrm{sign}\,(c_2-2b_1))$ and $0<uv<1$, because $R=uv(uv-1)<0$.

$\mathbf{2_2^2}$) $d_*<0$.

$\mathbf{2_2^2a}$) $a_1\neq b_2$. Then $c_*=2b_2((a_1-b_2)(a_1c_1+b_1c_2-c_1b_2)-a_1b_1^2)\neq 0$, else for $r_1=(2b_2)^{-1}$, $s_1=b_1,\ r_2=0,\ s_2=b_2-a_1$ in system (21) $\breve{c}_1=0$, that is impossible for $d_*<0$.

For $r_1=(2b_2)^{-1},\ s_1=b_1|c_*|^{-1/2},\ r_2=0,\ s_2=(b_2-a_1)|c_*|^{-1/2}$ system (21) is CF_9^0 with $u=a_1(2b_2)^{-1}\neq 0;1/2,\ v=(2b_1b_2-(a_1-b_2)c_2)|c_*|^{-1/2}\neq 0,\ \sigma=\mathrm{sign}\,c_*$. At that $\breve{R}=u(uv^2+\sigma)=R(a_1-b_2)^2b_2^{-2}|4c_*|^{-1}<0$, hence $|u|v^2<-\mathrm{sign}\,(uc_*)$. It means that $\sigma=-\mathrm{sign}\,u$ and $|u|v^2<1$.

$\mathbf{2_2^2b}$) $a_1=b_2$ ($\breve{b}_1\neq 0$). Then (21) for $r_1=(2a_1)^{-1}$, $s_1=-c_2(-2R)^{-1/2}\mathrm{sign}\,(a_1b_1)$, $r_2=0$, $s_2=2^{1/2}a_1(-R)^{-1/2}\mathrm{sign}\,(a_1b_1)$ is a CF_{10}^0 with $u=2^{3/2}|a_1b_1|(-R)^{-1/2}$, at that $0<u<\sqrt{2}$, because $u^2-2=-2a_1^2(4b_1^2-4b_1c_2+c_2^2+4a_1c_1)R^{-1}=-2a_1^2d_*R^{-1}<0$. $\square$

Remark 4 CF_3^0 with $u=u_*<1$ by the change (9) with $r_1=1,\ s_1=1-u_*,\ r_2=0$, $s_2=1$ is also may be reduced to a CF_3^0, with $u=2-u_*>1$.

CF_4^0 with $u=u_*\leq 1/4$ is not a canonical one according to principle 3. Using (9) with $r_1=u_*^{-1},\ s_1=(1+(1-4u_*)^{1/2})(2u_*)^{-1},\ r_2=0,\ s_2=1$ we reduce it to a CF_3^0 with $u=1+(1-4u_*)^{1/2}\geq 1$.

CF_5^0 with $|u|=1/2,\ \sigma=\mathrm{sign}\,u\ \ (R=1/2)$ is not a canonical form in accordance with principle 2. In case of $u=1/2$ CF_5^0 is reduced to a CF_1^0 by the change (9) with $r_1,s_1=1$, $r_2=2^{-1/2},\ s_2=-2^{-1/2}$. For $u=-1/2$ CF_5^0 is reduced to a CF_2^0 by the change (9) with $r_1,s_1=-1,\ -s_2,r_2=2^{-1/2}$.

CF_9^0 with $u = u_* = 1/2,\ \sigma = -1\ \ (R = (v^2-2)/4 < 0)$ is not a canonical one according to principle 2. Applying (9) with $r_1 = 1,\ s_1 = -2^{1/2}v(2-v^2)^{-1/2},\ r_2 = 0,\ s_2 = 2^{1/2}(2-v^2)^{-1/2}$ we reduce it to a CF_5^0 with $u = 1/2,\ \sigma = -1\ \ (R = -1/2)$.

CF_{10}^0 with $u = u_* \notin (0, 2^{1/2})$ is not a canonical form because of principles 2 or 3. For $|u_*| = 2^{1/2}$ by the change (9) with $r_1 = 1,\ s_1 = 0,\ r_2 = 2^{-1/2}\mathrm{sign}\, u_*,\ s_2 = \mathrm{sign}\, u_*$ CF_{10}^0 may be reduced to a CF_9^0 with $u = 1,\ \sigma = -1,\ v = 2^{-1/2}\ (R = -1/2)$. For $|u_*| > 2^{1/2}$ using (9) with $r_1 = (u_* + (u_*^2-2)^{1/2})u_*^{-1},\ s_1 = (u_* - (u_*^2-2)^{1/2})u_*^{-1}$, $r_2, s_2 = u_*^{-1}$ CF_{10}^0 is reduced to a CF_8^0 with $u = (u_*^2 - 2 + u_*^2(u_*^2-2)^{1/2})u_*^{-1}(u_*^2-2)^{-1/2} \neq 0$, $v = (-u_*^2 + 2 + u_*^2(u_*^2-2)^{1/2})u_*^{-1}(u_*^2-2)^{-1/2} \neq 0$, being $uv = 2u_*^{-2} < 1$. For $-2^{1/2} < u_* < 0$ CF_{10}^0 by (9) with $r_1 = 1,\ s_1, r_2 = 0,\ s_2 = -1$ is also brought to a CF_{10}^0, but with $u = -u_*$, i.e. $0 < u < 2^{1/2}$.

Remark 5 Forms $\mathrm{F}_2^0 = \begin{pmatrix} u_* & v_* & 1 \\ 1 & 0 & 0 \end{pmatrix}$ with $u_*v_* \neq 0,\ \ R = 1$, and $\mathrm{F}_3^0 = \begin{pmatrix} u_* & v_* & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with $u_*v_* \neq 0,\ \ R = u_*^2$ which are not given in the list are noncanonical ones according to principle 2 or 3.

Form F_2^0 for $v_*^3 - 4u_*v_* - 8 = 0,\ \ v_*^4 + 32v_* < 0$ by change (9) with $r_1 = 4v_*^{-2},\ \ s_1 = 0$, $r_2 = -2v_*^{-1},\ \ s_2 = 2v_*^{-1}$ is brought to a CF_4^0 with $u = -8v_*^{-3} > 1/4$; for $v_*^3 - 4u_*v_* - 8 = 0$, $v_*^4 + 32v_* \geq 0$ by the change (9) with $r_1 = -v_*/2,\ \ s_1 = 4v_*^{-1}(v_*^2 \pm (v_*^4+32v_*)^{1/2})(16 + v_*^3 \pm v_*(v_*^4+32v_*)^{1/2})^{-1},\ \ r_2 = v_*^2/4,\ \ s_2 = 32v_*^{-1}(16 + v_*^3 \pm v_*(v_*^4+32v_*)^{1/2})^{-1}$ is reduced to a CF_3^0 with $u = 16v_*^{-2}(\mp(v_*^4+32v_*)^{1/2} - v_*^2)(16 + v_*^3 \pm v_*(v_*^4+32v_*)^{1/2})^{-1} \neq 0$. For $v_*^3 - 4u_*v_* - 8 \neq 0$, if $4u_* = v_*^2$, then F_2^0 is reduced to a CF_7^0 by the change (9) with $r_1 = 0,\ \ s_1 = 1,\ \ r_2 = 1$, $s_2 = -v_*/2$ with $u = v_*/2 \neq 0$; if $4u_* \neq v_*^2$ then by the change (9) with $r_1 = 0,\ \ s_1 = 4(v_*^3 - 4u_*v_* - 8)^{-2/3},\ \ r_2 = -2(v_*^3 - 4u_*v_* - 8)^{-1/3},\ \ s_2 = -2v_*(v_*^3 - 4u_*v_* - 8)^{-2/3}$ F_2^0 is reduced to a F_1^0 with $u = -v_*(v_*^3 - 4u_*v_* - 8)^{-1/3},\ \ v = (4u_* - v_*^2)(v_*^3 - 4u_*v_* - 8)^{-2/3}$, being $uv \neq 0; 1$.

Form F_3^0 for $v_*^2 - 2v_* - 4u_* + 1 < 0$ by the change (9) with $r_1 = (2v_* - v_*^2 + 4u_*)(4u_*)^{-1}$, $s_1 = -(2u_*)^{-1}v_*,\ \ r_2 = 0,\ \ s_2 = 1$ is brought to a CF_4^0 with $u = (2v_* - v_*^2 + 4u_*)/4 > 1/4$. But for $v_*^2 - 2v_* - 4u_* + 1 \geq 0$, if $v_*^2 - 2v_* - 4u_* \neq 0$, by the change (9) with $r_1 = u_*^{-1}$, $s_1 = (1 - v_* + (v_*^2 - 2v_* - 4u_* + 1)^{1/2})(2u_*)^{-1},\ \ r_2 = 0,\ \ s_2 = 1$ F_3^0 is reduced to a CF_3^0 with $u = 1 + (v_*^2 - 2v_* - 4u_* + 1)^{1/2} \geq 1$; if $v_*^2 - 2v_* - 4u_* = 0\ \ (v_* \neq 2)$, then by the change (9) with $r_1 = u_*^{-1},\ \ s_1 = -v_*(2u_*)^{-1},\ \ r_2 = 0,\ \ s_2 = 1$ F_3^0 is reduced to a CF_1^0.

Remark 6 Forms $\mathrm{F}_4^0 = \begin{pmatrix} u_* & 0 & 1 \\ 1 & v_* & 0 \end{pmatrix}$ with $u_*v_* \neq 0,\ R = 1 + u_*v_*^2$ and $\mathrm{F}_5^0 = \begin{pmatrix} 0 & u & 1 \\ 1 & v & 0 \end{pmatrix}$ with $uv \neq 0,\ R = 1 - uv \neq 0$ are noncanonical in accordance with principles 2, 3.

Form F_4^0 with $u_* = u,\ \ v_* = -2u$ for $R > 0$ by the change (9) with $r_1 = (R + R^{1/2})(2u^3)^{-1}s_2^2,\ \ s_1 = -(1 + R^{1/2})(2u^2)^{-1}s_2,\ \ r_2 = (R + R^{1/2})(1 + R^{1/2})(4u^4)^{-1}s_2^2$ is brought to the system with coefficients $\widetilde{a}_1, \widetilde{b}_1, \widetilde{b}_2, \widetilde{c}_2 = 0,\ \ \widetilde{a}_2 = -R^{3/2}(R^{1/2} + 1 + 2u^3)(2u^6)^{-1}s_2^3 \neq 0$, else $\widetilde{P}_2 \equiv 0$, that is impossible. For $s_2 = -2^{1/3}u^2R^{-1/2}(R^{1/2} + 1 + 2u^3)^{-1/3}$ the obtained system is a CF_2^0.

Form F_4^0 with $u_* = u,\ \ v_* = v \neq -2u$ for $R > 0$ by the change (9) with $r_1 = 2(R + R^{1/2})(uv^2)^{-1}s_2^2,\ \ s_1 = (1 + R^{1/2})(uv)^{-1}s_2,\ \ r_2 = (R + R^{1/2})(1 + R^{1/2})(uv)^{-2}s_2^2$ is brought to the system having $\widetilde{a}_2 = s_*(uv)^{-3}s_2^3$, where $s_* = R((uv^2 - 2u^2v + 4)(R^{1/2} + 1) + 2uv^2),\ \widetilde{b}_2, \widetilde{c}_2 = 0$, being $s_* \neq 0$, else $\widetilde{P}_2 \equiv 0$, that is impossible. For $s_2 = uvs_*^{-1/3}$ the obtained system is a F_2^0 with $u_* = u(2u + v)(R + R^{1/2})s_*^{-2/3} \neq 0,\ \ v_* = (2u + v)(1 + R^{1/2})s_*^{-1/3} \neq 0$.

Form F_4^0 with $u_* = u$, $v_* = -2u$ for $R < 0$ by the change (9) with $s_1 = |2(u^2 - ut^2 + t)|^{-1/2}$, $r_1 = (u + t^2)(2(u^2 - ut^2 + t))^{-1}$, $r_2 = (1 - 2tu)(2(u^2 - ut^2 + t))^{-1}$, $s_2 = t|2(u^2 - ut^2 + t)|^{-1/2}$, where t is a real root of the cubic polynomial $t^3 + 6u^2t^2 - 3ut + 2u^3 + 1$, is reduced to a F_6^0 with $u = -1/2$, $\sigma = \mathrm{sign}(u^2 - ut^2 + t) = 1$.

Form F_4^0 with $u_* = u$, $v_* = v \neq -2u$ for $R < 0$ by (9) with $r_1 = (u+t^2)(vt^2+2t-uv)^{-1}$, $s_1 = |vt^2 + 2t - uv|^{-1/2}$, $r_2 = (1 + tv)(vt^2 + 2t - uv)^{-1}$, $s_2 = t|vt^2 + 2t - uv|^{-1/2}$, where t is a real root of the cubic polynomial $t^3 + (v - u)vt^2 + (u + 2v)t - u^2v + 1$, is reduced to the system

$$\begin{pmatrix} u_* & v_* & \sigma \\ 0 & 1 & 0 \end{pmatrix} \tag{22}$$

with $u_* = (ut^2 - t + u^2 + uv)(vt^2 + 2t - uv)^{-1} < 0$, $v_* = (2u + v)|vt^2 + 2t - uv|^{-1/2} \neq 0$, $\sigma = \mathrm{sign}\,(vt^2 + 2t - uv)$.

1) $v_*^2 + 4\sigma_*(1 - u_*) \geq 0$. If $4\sigma_*u_* - 2\sigma_* - v_*^2 - |v_*|(v_*^2 + 4\sigma_*(1 - u_*))^{1/2} \neq 0$ then system (22) by (9) with $r_1 = 1$, $s_1 = 2\sigma_*(4\sigma_*u_* - 2\sigma_* - v_*^2 - |v_*|(v_*^2 + 4\sigma_*(1 - u_*))^{1/2})^{-1}$, $r_2 = 0$, $s_2 = (-v_* - (v_*^2 + 4\sigma_*(1 - u_*))^{1/2}\mathrm{sign}\, v_*)(4\sigma_*u_* - 2\sigma_* - v_*^2 - |v_*|(v_*^2 + 4\sigma_*(1 - u_*))^{1/2})^{-1}$ is reduced to a CF_8^0 with $u = u_*$, $v = 2\sigma_*(4\sigma_*u_* - 2\sigma_* - v_*^2 - |v_*|(v_*^2 + 4\sigma_*(1 - u_*))^{1/2})^{-1}$. If $4\sigma_*u_* - 2\sigma_* - v_*^2 - |v_*|(v_*^2 + 4\sigma_*(1 - u_*))^{1/2} = 0$ (for example $\sigma_* = 1, u_* = 1, v_* = \pm 1$), then system (22) by (9) with $r_1 = 1$, $s_1 = u_*^{-1}$, $r_2 = (-v_* - (v_*^2 + 4\sigma_*(1 - u_*))^{1/2}\mathrm{sign}\, v_*)(2\sigma_*)^{-1}$, $r_2 = 0$ is reduced to a CF_3^0 with $u = u_*^{-1}$.

2) $v_*^2 + 4\sigma_*(1 - u_*) < 0$. Using (9) with $r_1 = 1$, $s_1 = v_*|u_*v_*^2 - \sigma_*(2u_* - 1)^2|^{-1/2}$, $r_2 = 0$, $s_2 = (1 - 2u_*)|u_*v_*^2 - \sigma(2u_* - 1)^2|^{-1/2}$ we bring (22) to a CF_9^0 with $u = u_*$, $\sigma = -\mathrm{sign}\,(u_*v_*^2 - \sigma_*(2u_* - 1)^2)$, $v = v_*|u_*v_*^2 - \sigma_*(2u_* - 1)^2|^{-1/2}$, being $u_*v_*^2 \neq \sigma_*(2u_* - 1)^2$, else $v_*^2 + 4\sigma_*(1 - u_*) \geq 0$.

Form F_5^0 for $u \neq v$ by (9) with $r_1 = v(v - u^2)s_*^{-2/3}$, $s_1 = us_*^{-1/3}$, $r_2 = u(u - v^2)s_*^{-2/3}$, $s_2 = -vs_*^{-1/3}$, where $s_* = (uv - 1)(v - u)(u^2 + uv + v^2) \neq 0$, is reduced to a F_4^0 with $u_* = uv(1 - uv)s_*^{-2/3}$, $v_* = (u^3 - 2uv + v^3)s_*^{-2/3}$. In the case $u = v = u_*$ by (9) with $r_1, r_2 = -1/2$, $s_1 = |2u_* - 2|^{-1/2}$, $s_2 = -s_1$ a form F_5^0 is brought to a CF_5^0 with $u = -(u_* + 1)/2$, $\sigma = \mathrm{sign}\,(C_* - 1)$.

In Theorem 1 all the linear invertible changes of variables (9) are given in explicit form. Therefore the conditions that guarantee the reducing system (2) to an appropriate CF_i^1 may be written by using coefficients of (2). For $R = \delta_{ac}^2 - 4\delta_{ab}\delta_{bc} > 0$ assume

$$\widetilde{a}_1 = \left[\begin{array}{l} P_1(t_1^*, 1) - t_2^*P_2(t_1^*, 1), \text{ if } \delta_{ab} \neq 0, \\ a_1 + b_2, \text{ if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 \neq 0, \\ 1, \quad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 = 0, \\ a_1, \quad \text{if } b_1, b_2 = 0, \end{array}\right. \qquad \widetilde{a}_2 = \left[\begin{array}{l} t_1^*P_2(t_1^*, 1) - P_1(t_1^*, 1), \text{ if } \delta_{ab} \neq 0, \\ 1, \quad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 \neq 0, \\ 0, \quad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 = 0, \\ a_2, \quad \text{if } b_1, b_2 = 0, \end{array}\right.$$

$$\widetilde{c}_1 = \left[\begin{array}{l} P_1(t_2^*, 1) - t_2^*P_2(t_2^*, 1), \text{ if } \delta_{ab} \neq 0, \\ a_1b_2^2 - 2b_1a_2b_2 + a_2^2c_1 - b_2^3 + a_2b_2c_2, \\ \qquad\qquad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 \neq 0, \\ (a_1c_1 - b_1^2 + b_1c_2)c_2^{-2}, \\ \qquad\qquad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 = 0, \\ c_1, \quad \text{if } b_1, b_2 = 0, \end{array}\right. \qquad \widetilde{c}_2 = \left[\begin{array}{l} t_1^*P_2(t_2^*, 1) - P_1(t_2^*, 1), \text{ if } \delta_{ab} \neq 0, \\ 1, \quad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 \neq 0, \\ 1, \quad \text{if } \delta_{ab} = 0,\ \delta_{bc} \neq 0,\ a_2 = 0, \\ c_2, \quad \text{if } b_1, b_2 = 0, \end{array}\right. \tag{23}$$

where $t_1^* = (-\delta_{ac} + R^{1/2})(2\delta_{ab})^{-1}$, $t_2^* = (-\delta_{ac} - R^{1/2})(2\delta_{ab})^{-1}$, for $R < 0$ we assume

$$\widetilde{a}_1 = \left[\begin{array}{l} a_2 r_*^2 + 2b_2 r_* + c_2, \quad \text{if } a_2 \neq 0, \\ \qquad a_1, \quad \text{if } a_2 = 0, \end{array}\right.$$
$$\widetilde{b}_1 = \left[\begin{array}{l} -a_2 r_* - b_2, \quad \text{if } a_2 \neq 0, \\ b_1, \quad \text{if } a_2 = 0, \end{array}\right. \quad \widetilde{b}_2 = \left[\begin{array}{l} b_1 + a_1 r_* - b_2 r_* - a_2 r_*^2, \quad \text{if } a_2 \neq 0, \\ b_2, \quad \text{if } a_2 = 0, \end{array}\right. \tag{24}$$
$$\widetilde{c}_1 = \left[\begin{array}{l} a_2, \quad \text{if } a_2 \neq 0, \\ c_1, \quad \text{if } a_2 = 0, \end{array}\right. \qquad \widetilde{c}_2 = \left[\begin{array}{l} a_2 r_* - a_1, \quad \text{if } a_2 \neq 0, \\ c_2, \quad \text{if } a_2 = 0, \end{array}\right.$$

where r_* is a real root of the cubic polynomial $a_2 r_1^3 + (2b_2 - a_1) r_1^2 + (c_2 - 2b_1) r_1 - c_1$.

Corollary 1 *A system (2) in which $R = \delta_{ac}^2 - 4\delta_{ab}\delta_{bc} \neq 0$, by the linear invertible change of variables (9) is reduced to CF_i^0 $(i = \overline{1,10})$, if the coefficients a_i, b_i, c_i $(i = 1,2)$ satisfy conditions:*

$CF_1^0:$ $R > 0$, $\widetilde{a}_1\widetilde{c}_2 \neq 0$, $\widetilde{a}_2, \widetilde{c}_1 = 0$;

$CF_2^0:$ $R > 0$, $\widetilde{a}_1, \widetilde{c}_2 = 0$;

$CF_3^0:$ *1)* $R > 0$, $0 \neq \widetilde{a}_1\widetilde{c}_1\widetilde{c}_2^{-2} \leq 1/4$, $\widetilde{a}_2 = 0$, *then* $u = 1 + (1 - 4\widetilde{a}_1\widetilde{c}_1\widetilde{c}_2^{-2})^{1/2} \geq 1$; *2)* $R > 0$, $0 \neq \widetilde{a}_2\widetilde{c}_2\widetilde{a}_1^{-2} \leq 1/4$, $\widetilde{c}_1 = 0$, *then* $u = 1 + (1 - 4\widetilde{a}_2\widetilde{c}_2\widetilde{a}_1^{-2})^{1/2} \geq 1$;

$CF_4^0:$ *1)* $R > 0$, $\widetilde{a}_1\widetilde{c}_1\widetilde{c}_2^{-2} > 1/4$, $\widetilde{a}_2 = 0$, *then* $u = \widetilde{a}_1\widetilde{c}_1\widetilde{c}_2^{-2} > 1/4$; *2)* $R > 0$, $\widetilde{a}_2\widetilde{c}_2\widetilde{a}_1^{-2} > 1/4$, $\widetilde{c}_1 = 0$, *then* $u = \widetilde{a}_2\widetilde{c}_2\widetilde{a}_1^{-2} > 1/4$;

$CF_5^0:$ *1)* $R > 0$, $\widetilde{a}_1\widetilde{c}_1^{1/3} = \widetilde{a}_2^{1/3}\widetilde{c}_2 \neq 0$, *then* $u = (\widetilde{a}_1^2 + \widetilde{a}_2\widetilde{c}_2)(2\widetilde{a}_1^2 - 2\widetilde{a}_2\widetilde{c}_2)^{-1} \neq 0; \pm 1/2$, $\sigma = \operatorname{sign} u$; *2)* $R < 0$, $(\widetilde{a}_1 - \widetilde{b}_2)\widetilde{c}_2 - 2\widetilde{b}_1\widetilde{b}_2 = 0$, *then* $u = \widetilde{a}_1(2\widetilde{b}_2)^{-1} \neq 0$; $\sigma = -\operatorname{sign} u$;

$CF_6^0:$ *1)* $R > 0$, $\widetilde{a}_1 = 0$, $\widetilde{c}_2 \neq 0$, *then* $u = \widetilde{c}_2(\widetilde{a}_2\widetilde{c}_1^2)^{-1/3} \neq 0$; *2)* $R > 0$, $\widetilde{a}_1 \neq 0$, $\widetilde{c}_2 = 0$, *then* $u = \widetilde{a}_1(\widetilde{a}_2^2\widetilde{c}_1)^{-1/3} \neq 0$;

$CF_7^0:$ *1)* $R > 0$, $\widetilde{a}_1\widetilde{a}_2\widetilde{c}_1\widetilde{c}_2 \neq 0$, $\widetilde{a}_1^2 + \widetilde{a}_2\widetilde{c}_2 = 0$, *then* $u = 2\widetilde{a}_1(\widetilde{a}_1^3 + \widetilde{a}_2^2\widetilde{c}_1)^{-1/3} \neq 0$; *2)* $R > 0$, $\widetilde{a}_1\widetilde{a}_2\widetilde{c}_1\widetilde{c}_2 \neq 0$, $\widetilde{c}_2^2 + \widetilde{a}_1\widetilde{c}_1 = 0$, *then* $u = 2\widetilde{c}_2(\widetilde{c}_2^3 + \widetilde{c}_1^2\widetilde{a}_2)^{-1/3} \neq 0$;

$CF_8^0:$ *1)* $R > 0$, $\widetilde{a}_1\widetilde{a}_2\widetilde{c}_1\widetilde{c}_2 \neq 0$, $\widetilde{a}_1\widetilde{c}_1^{1/3} \neq \widetilde{a}_2^{1/3}\widetilde{c}_2$, $\widetilde{a}_1^2 + \widetilde{a}_2\widetilde{c}_2 \neq 0$, $\widetilde{c}_2^2 + \widetilde{a}_1\widetilde{c}_1 \neq 0$, $\delta_{\widetilde{a}\widetilde{c}} \neq 0$, *the condition* (17_1) *is fulfilled for polynomials* $Q_1(t), Q_2(t)$ *from (15), being* $u = u_* = \widetilde{a}_1(\widetilde{a}_2^2\widetilde{c}_1)^{-1/3}$, $v = v_* = \widetilde{c}_2(\widetilde{a}_2\widetilde{c}_1^2)^{-1/3}$, *then in* CF_8^0 $u = (u_* t_1'^2 - t_1'^2 t_1'' - v_* t_1'' + 1)(2(t_1'^2 t_1'' + v_* t_1' - u_* t_1' t_1'' - 1))^{-1} \neq 0$, $v = (u_* t_1''^2 - t_1' t_1''^2 - v_* t_1' + 1)(2(t_1' t_1''^2 - u_* t_1' t_1'' + v_* t_1'' - 1))^{-1} \neq 0$; *2)* $R < 0$, $(\widetilde{a}_1 - \widetilde{b}_2)\widetilde{c}_2 - 2\widetilde{b}_1\widetilde{b}_2 \neq 0$, $d_* = (2\widetilde{b}_1 - \widetilde{c}_2)^2 + 4\widetilde{c}_1(2\widetilde{b}_2 - \widetilde{a}_1) \geq 0$, *then* $u = \widetilde{a}_1(2\widetilde{b}_2)^{-1} \neq 0$; $v = (4\widetilde{c}_1(\widetilde{a}_1 - \widetilde{b}_2) + 2\widetilde{b}_1(\widetilde{c}_2 - 2\widetilde{b}_1 + d_*^{1/2}\operatorname{sign}(\widetilde{c}_2 - 2\widetilde{b}_1)))^{-1}(4\widetilde{c}_1\widetilde{b}_2 + \widetilde{c}_2^2 - 2\widetilde{b}_1\widetilde{c}_2 + \widetilde{c}_2 d_*^{1/2}\operatorname{sign}(\widetilde{c}_2 - 2\widetilde{b}_1)) \neq 0$;

$CF_9^0:$ *1)* $R > 0$, $\widetilde{a}_1\widetilde{a}_2\widetilde{c}_1\widetilde{c}_2 \neq 0$, $\widetilde{a}_1\widetilde{c}_1^{1/3} \neq \widetilde{a}_2^{1/3}\widetilde{c}_2$, $\widetilde{a}_1^2 + \widetilde{a}_2\widetilde{c}_2 \neq 0$, $\widetilde{c}_2^2 + \widetilde{a}_1\widetilde{c}_1 \neq 0$, $\delta_{\widetilde{a}\widetilde{c}} \neq 0$, *the condition* (17_2) *is fulfilled for polynomials* $Q_1(t)$, $Q_2(t)$ *from (15), with* $u = u_* = \widetilde{a}_1(\widetilde{a}_2^2\widetilde{c}_1)^{-1/3}$, $v = v_* = \widetilde{c}_2(\widetilde{a}_2\widetilde{c}_1^2)^{-1/3}$, *then in* CF_9^0 $u = (t_2 - u_*)(1 - t_2 v_*)(2t_2(u_* v_* - 1))^{-1} \neq 0$, $v = (u_* t_2 + v_*)|2t_2(u_* v_* - 1)|^{-1/2} \neq 0$, $\sigma = -\operatorname{sign}(t_2(u_* v_* - 1))$; *2)* $R < 0$, $(\widetilde{a}_1 - \widetilde{b}_2)\widetilde{c}_2 - 2\widetilde{b}_1\widetilde{b}_2 \neq 0$, $d_* = (2\widetilde{b}_1 - \widetilde{c}_2)^2 + 4\widetilde{c}_1(2\widetilde{b}_2 - \widetilde{a}_1) < 0$, $\widetilde{a}_1 \neq \widetilde{b}_2$, *then* $u = \widetilde{a}_1(2\widetilde{b}_2)^{-1} \neq 0; 1/2$, $\sigma = -\operatorname{sign} u$, $v = (2\widetilde{b}_1\widetilde{b}_2 - (\widetilde{a}_1 - \widetilde{b}_2)\widetilde{c}_2)|2\widetilde{b}_2((\widetilde{a}_1 - \widetilde{b}_2)(\widetilde{a}_1\widetilde{c}_1 + \widetilde{b}_1\widetilde{c}_2 - \widetilde{c}_1\widetilde{b}_2) - \widetilde{a}_1\widetilde{b}_1^2)|^{-1/2} \neq 0$;

$CF_{10}^0:$ $R < 0$, $(\widetilde{a}_1 - \widetilde{b}_2)\widetilde{c}_2 - 2\widetilde{b}_1\widetilde{b}_2 \neq 0$, $d_* = (2\widetilde{b}_1 - \widetilde{c}_2)^2 + 4\widetilde{c}_1(2\widetilde{b}_2 - \widetilde{a}_1) < 0$, $\widetilde{a}_1 = \widetilde{b}_2$, *then* $u = 2^{3/2}|\widetilde{a}_1\widetilde{b}_1|(4\delta_{\widetilde{a}\widetilde{b}}\delta_{\widetilde{b}\widetilde{c}} - \delta_{\widetilde{a}\widetilde{c}}^2)^{-1/2}$, *at that* $0 < u < \sqrt{2}$.

Here for $R > 0$ coefficients of the matrix (18) $\widetilde{a}_1, \widetilde{c}_1, \widetilde{a}_2, \widetilde{c}_2$ are defined in (23), and for $R < 0$ coefficients of the matrix (20) $\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1, \widetilde{b}_2, \widetilde{c}_2$ are defined in (24).

5 Canonical forms for system (2) in case $\mathbf{l = 1}$

5.1 Linear equivalence of systems for $\mathbf{l = 1}$

A system (2) $\dot{x} = P(x)$ for $l = 1$ may be written in the form

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = P_0(x) \begin{pmatrix} p_1x_1 + q_1x_2 \\ p_2x_1 + q_2x_2 \end{pmatrix} = \big\langle(\alpha, \beta),\, x\big\rangle Hx \not\equiv 0 \qquad (\delta_{pq} \neq 0), \tag{25}$$

i. e. in (25) a common factor $P_0 = \alpha x_1 + \beta x_2 \not\equiv 0$, matrix $H = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$.

Hence eigenvalues of H are not equal to zero and have the form

$$\lambda_{1,2} = (p_1 + q_2 \pm \sqrt{D})/2, \tag{26}$$

where $D = (p_1 + q_2)^2 - 4\delta_{pq} = (p_1 - q_2)^2 + 4p_2q_1$.

Proposition 2 *For the purpose of normalizing one of nonzero coefficients of P_0 in system (25) may be taken 1. We assume that if $\alpha \neq 0$, then $\alpha = 1$, and if $\alpha = 0$, then $\beta = 1$.*

Let the change (9) $x = Ly$ $(\det L = \delta \neq 0)$ bring a system (2) of the form (25) to a system (10) $\dot{y} = \widetilde{P}(y)$. Let us take

$$(\widetilde{\alpha},\ \widetilde{\beta}) = (\alpha,\ \beta)L, \quad \widetilde{H} = \begin{pmatrix} \widetilde{p}_1 & \widetilde{q}_1 \\ \widetilde{p}_2 & \widetilde{q}_2 \end{pmatrix} = L^{-1}HL \qquad (\delta_{\widetilde{p}\widetilde{q}} = \det \widetilde{H} = \delta_{pq}), \tag{27}$$

i. e. $\widetilde{\alpha} = \alpha r_1 + \beta r_2$, $\widetilde{\beta} = \alpha s_1 + \beta s_2$, $\widetilde{H} = \delta^{-1} \begin{pmatrix} r_1\delta_{ps} + r_2\delta_{qs} & s_1\delta_{ps} + s_2\delta_{qs} \\ -r_1\delta_{pr} - r_2\delta_{qr} & -s_1\delta_{pr} - s_2\delta_{qr} \end{pmatrix}$

In addition, as matrix product is associative, we have:

$$\big\langle(\alpha, \beta),\, Ly\big\rangle = \big\langle(\alpha, \beta)L,\, y\big\rangle. \tag{28}$$

Theorem 2 *The system (10) obtained form a system (2) of the form (25) by change (9) has the form*

$$\begin{pmatrix} \widetilde{P}_1 \\ \widetilde{P}_2 \end{pmatrix} = \widetilde{P}_0(y) \begin{pmatrix} \widetilde{p}_1y_1 + \widetilde{q}_1y_2 \\ \widetilde{p}_2y_1 + \widetilde{q}_2y_2 \end{pmatrix} = \big\langle(\widetilde{\alpha}, \widetilde{\beta}),\, y\big\rangle\, \widetilde{H}y \quad (\widetilde{P}_0 \not\equiv 0), \tag{29}$$

where coefficients of the polynomial $\widetilde{P}_0 = \widetilde{\alpha}y_1 + \widetilde{\beta}y_2$ and matrix $\widetilde{H}$ are introduced in (27).

So, the case $l = 1$ is invariant with respect to change (9).

Proof The formula (29) follows from equalities:

$$\widetilde{P}(y) \overset{(11)}{=} L^{-1}P(Ly) \overset{(25)}{=} L^{-1}\big\langle(\alpha, \beta),\, Ly\big\rangle\, HLy \overset{(28)}{=} \big\langle(\alpha, \beta)L, y\big\rangle\, L^{-1}HLy \overset{(27)}{=} \big\langle(\widetilde{\alpha}, \widetilde{\beta}),\, y\big\rangle\, \widetilde{H}y.$$

Note that the condition $\widetilde{\alpha}^2 + \widetilde{\beta}^2 = 0$ is equivalent to $\alpha^2 + \beta^2 = 0$, because $\delta_{rs} \neq 0$. $\square$

5.2 Construction of canonical forms when $l = 1$

Without loss of generality we suppose that in (25) $\alpha \neq 0$, as if $\alpha = 0$ then by renumbering (14) we have a system (29) of the form $\begin{pmatrix} \widetilde{P}_1 \\ \widetilde{P}_2 \end{pmatrix} = \beta y_1 \begin{pmatrix} q_2 y_1 + p_2 y_2 \\ q_1 y_1 + p_1 y_2 \end{pmatrix}$.

Now, following to Proposition 2 we take $\alpha = 1$, i. e. the common factor in (25) is always $P_0 = x_1 + \beta x_2$.

To simplify (25) we take at first such a change (9) that reduces matrix H to a Jordan form $\widetilde{H}$ in (29).

It is evidently that the form of the change depends on the sign of the discriminant $D = (p_1 + q_2)^2 - 4\delta_{pq}$ from the formula (26) for eigenvalues of matrix H $\lambda_{1,2} \neq 0$.

Then in (29) with Jordan matrix $\widetilde{H}$ we will perform an arbitrary change (9) choosing its coefficients such that the obtained system be the simplest in the sense of definition 7 – a canonical form (CF^2).

We will mark all the elements of obtained system by the symbol $\breve{}$. Similarly to (27) coefficients of $\breve{P}_0$ have the form

$$\breve{\alpha} = \widetilde{\alpha} r_1 + \widetilde{\beta} r_2, \quad \breve{\beta} = \widetilde{\alpha} s_1 + \widetilde{\beta} s_2. \tag{30}$$

The list of canonical forms of (2) in case of $l = 1$:

$$\mathrm{CF}_1^1 = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{CF}_2^1 = \begin{pmatrix} 0 & \sigma & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathrm{CF}_3^1 = \begin{pmatrix} u & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{CF}_4^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathrm{CF}_5^1 = \begin{pmatrix} u & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where in CF_1^1 $u \neq 0$, in CF_2^1 $\sigma = \pm 1$, in CF_3^1 $0 < |u| < 1$ or $u = 1$, in CF_5^1 $0 < u < 2$.

Theorem 3 *For $l = 1$ system (2) of the form (25) by a linear invertible change of variables (9) is brought to one of 5 linearly nonequivalent CF^1.*

Proof

1) $D > 0$, i. e. in (26) $\lambda_1, \lambda_2 \neq 0$, are real and different. More exactly,

$$\lambda_1 = (p_1 + q_2 + \sigma_* \sqrt{D})/2, \quad \lambda_2 = (p_1 + q_2 - \sigma_* \sqrt{D})/2, \quad \lambda_* = p_1 - q_2 + \sigma_* \sqrt{D},$$

where $\sigma_* = \{\, \mathrm{sign}\,(p_1 - q_2)$ for $p_1 \neq q_2$; 1 for $p_1 = q_2 \,\}$, then $\lambda_* \neq 0$.

The change (9) with $L = \begin{pmatrix} \lambda_* & 2q_1 \\ 2p_2 & -\lambda_* \end{pmatrix}$ reduces (25) to a system (29) of the form

$$\begin{pmatrix} \widetilde{\alpha}\lambda_1 & \widetilde{\beta}\lambda_1 & 0 \\ 0 & \widetilde{\alpha}\lambda_2 & \widetilde{\beta}\lambda_2 \end{pmatrix} \text{ with } \widetilde{\alpha} = 2\beta p_2 + \lambda_*, \ \widetilde{\beta} = 2q_1 - \beta\lambda_*, \ \ \widetilde{H} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{31}$$

Further an arbitrary change (9) brings (31) to the system

$$\delta^{-1} \begin{pmatrix} \breve{\alpha}(\lambda_1 r_1 s_2 - \lambda_2 r_2 s_1) & \breve{\alpha}(\lambda_1 - \lambda_2) s_1 s_2 + \breve{\beta}(\lambda_1 r_1 s_2 - \lambda_2 r_2 s_1) & \breve{\beta}(\lambda_1 - \lambda_2) s_1 s_2 \\ \breve{\alpha}(\lambda_2 - \lambda_1) r_1 r_2 & \breve{\alpha}(\lambda_2 r_1 s_2 - \lambda_1 r_2 s_1) + \breve{\beta}(\lambda_2 - \lambda_1) r_1 r_2 & \breve{\beta}(\lambda_2 r_1 s_2 - \lambda_1 r_2 s_1) \end{pmatrix} \tag{32}$$

with $(\breve\alpha,\,\breve\beta)$ from (30) and $\breve H=\delta^{-1}\begin{pmatrix}\lambda_1 r_1 s_2-\lambda_2 r_2 s_1 & (\lambda_1-\lambda_2)s_1 s_2\\ -(\lambda_1-\lambda_2)r_1 r_2 & \lambda_2 r_1 s_2-\lambda_1 r_2 s_1\end{pmatrix}.$

$\mathbf{1_1}$) $\widetilde\alpha=0$ $(\widetilde\beta\neq 0)$. Then $\breve\beta=0$ for $s_2=0$ and system (32) has the form $\begin{pmatrix}\lambda_2\widetilde\beta r_2 & 0 & 0\\ (\lambda_1-\lambda_2)\widetilde\beta r_1 r_2 s_1^{-1} & \lambda_1\widetilde\beta r_2 & 0\end{pmatrix}$. For $r_1=0,$ $s_1=1,$ $r_2=(\lambda_1)^{-1}\widetilde\beta$ this system is a CF_1^1 with $u=\lambda_1^{-1}\lambda_2\neq 0,1.$

$\mathbf{1_2}$) $\widetilde\beta=0$ $(\widetilde\alpha\neq 0)$. Then $\breve\beta=0$ for $s_1=0$ and the system (32) has the form $\begin{pmatrix}\widetilde\alpha\lambda_1 r_1 & 0 & 0\\ \widetilde\alpha(\lambda_2-\lambda_1)r_1 r_2 s_2^{-1} & \widetilde\alpha\lambda_2 r_1 & 0\end{pmatrix}$. For $r_1=(\widetilde\alpha\lambda_2)^{-1},$ $r_2=0,$ $s_2=1$ this system is a CF_1^1 with $u=\lambda_1\lambda_2^{-1}\neq 0,1.$

$\mathbf{1_3}$) $\widetilde\alpha,\,\widetilde\beta\neq 0$. Then $\breve\beta=0$ for $s_2=-\widetilde\alpha\widetilde\beta^{-1}s_1$ and (32) has the form

$$\begin{pmatrix}\lambda_1\widetilde\alpha r_1+\lambda_2\widetilde\beta r_2 & (\lambda_1-\lambda_2)\widetilde\alpha s_1 & 0\\ (\lambda_1-\lambda_2)\widetilde\beta r_1 r_2 s_1^{-1} & \lambda_2\widetilde\alpha r_1+\lambda_1\widetilde\beta r_2 & 0\end{pmatrix}. \tag{33}$$

$\mathbf{1_3^1}$) $\lambda_1=-\lambda_2,$ then in system (33) $\breve a_1=-\breve b_2/2=\lambda_1(\widetilde\alpha r_1-\widetilde\beta r_2),$ hence for $r_1,s_1=(2\lambda_1\widetilde\alpha)^{-1},$ $r_2=(2\lambda_1\widetilde\beta)^{-1}$ it is a CF_2^1 with $\sigma=1.$

$\mathbf{1_3^2}$) $\lambda_1\neq-\lambda_2,$ then system (33) for $r_1=(\lambda_2\widetilde\alpha)^{-1},$ $s_1=((\lambda_1-\lambda_2)\widetilde\alpha)^{-1},$ $r_2=0$ is a CF_3^1 with $u=\lambda_1\lambda_2^{-1}\neq 0,\pm1.$ For $r_1=0,$ $s_1=((\lambda_1-\lambda_2)\widetilde\alpha)^{-1},$ $r_2=(\lambda_1\widetilde\beta)^{-1}$ it is a CF_3^1 with $u=\lambda_1^{-1}\lambda_2\neq 0,\pm1.$

Hence, choosing a required change one can always obtain $0<|u|<1.$

$\mathbf{2}$) $D=0,$ i.e. in (26) $\lambda=\lambda_{1,2}=(p_1+q_2)/2\neq 0.$ $\mathbf{2_1}$) $q_1\neq 0.$ The change $\begin{pmatrix}0 & 2q_1\\ 2 & q_2-p_1\end{pmatrix}$ brings (25) to a system (29)of the form

$$\begin{pmatrix}\lambda\widetilde\alpha & \lambda\widetilde\beta & 0\\ \widetilde\alpha & \lambda\widetilde\alpha+\widetilde\beta & \lambda\widetilde\beta\end{pmatrix}\quad\text{with}\quad \widetilde\alpha=2\beta,\ \ \widetilde\beta=\beta q_2-\beta p_1+2q_1,\ \ \widetilde H=\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}. \tag{34}$$

$\mathbf{2_1^1}$) $\widetilde\beta=0$ $(\widetilde\alpha\neq 0)$. Then the normalizing (13) with $r_1=(\widetilde\alpha\lambda)^{-1},$ $s_2=\widetilde\alpha^{-1}\lambda^{-2}$ brings (34) to a CF_4^1.

$\mathbf{2_1^2}$) $\widetilde\beta\neq 0.$ Then an arbitrary change (9) brings (34) to a system

$$\delta^{-1}\begin{pmatrix}\breve\alpha(\lambda\delta-r_1 s_1) & \breve\beta(\lambda\delta-r_1 s_1)-\breve\alpha s_1^2 & -\breve\beta s_1^2\\ \breve\alpha r_1^2 & \breve\alpha(\lambda\delta+r_1 s_1)+\breve\beta r_1^2 & \breve\beta(\lambda\delta+r_1 s_1)\end{pmatrix}, \tag{35}$$

where $(\breve\alpha,\breve\beta)$ are from (30), and $\breve H=\delta^{-1}\begin{pmatrix}\lambda\delta-r_1 s_1 & -s_1^2\\ r_1^2 & \lambda\delta+r_1 s_1\end{pmatrix}.$

In the system (35) assume that $\breve\beta=0,$ for which purpose we take $s_2=-\widetilde\alpha\widetilde\beta^{-1}s_1,$ then (35) has the form $\begin{pmatrix}(\widetilde\alpha\lambda+\widetilde\beta)r_1+\widetilde\beta\lambda r_2 & \widetilde\beta s_1 & 0\\ -\widetilde\beta r_1^2 s_1^{-1} & (\widetilde\alpha\lambda-\widetilde\beta)r_1+\widetilde\beta\lambda r_2 & 0\end{pmatrix}$. For $r_1=0,$ $s_1=\widetilde\beta^{-1},$ $r_2=(\widetilde\beta\lambda)^{-1}$ this is a CF_3^1 with $u=1.$

2_2) $q_1 = 0$. Then in (26) $\lambda = p_1 = q_2 \neq 0$.

2_2^1) $p_2 = 0$, i.e. in (25) $H = \begin{pmatrix} p_1 & 0 \\ 0 & p_1 \end{pmatrix}$. An arbitrary change (9) reduces (25) to the system $\begin{pmatrix} \widetilde{\alpha} p_1 & \widetilde{\beta} p_1 & 0 \\ 0 & \widetilde{\alpha} p_1 & \widetilde{\beta} p_1 \end{pmatrix}$ with $\widetilde{\alpha} = r_1 + \beta r_2$, $\widetilde{\beta} = s_1 + \beta s_2$, $\widetilde{H} = H$. For $r_1 = p_1^{-1}$, $s_1 = -\beta$, $r_2 = 0$, $s_2 = 1$ this is a CF_1^1 with $u = 1$.

2_2^2) $p_2 \neq 0$, i.e. in (25) $H = \begin{pmatrix} p_1 & 0 \\ p_2 & p_1 \end{pmatrix}$. The normalizing (13) with $r_1 = 1$, $s_2 = p_2$ reduces (25) to the system (34) from 2_1), being $\widetilde{\alpha} = 1$, $\widetilde{\beta} = \beta p_2$ and $\lambda = p_1$.

3) $D < 0$, i.e. eigenvalues λ_1, λ_2 of H are complex conjugate and $p_2 q_1 < 0$.

The change $\begin{pmatrix} \sqrt{-D} & p_1 - q_2 \\ 0 & 2p_2 \end{pmatrix}$ reduces (25) to a system (29) of the form

$$\begin{pmatrix} \widetilde{\alpha} p_* & \widetilde{\alpha} q_* + \widetilde{\beta} p_* & \widetilde{\beta} q_* \\ -\widetilde{\alpha} q_* & \widetilde{\alpha} p_* - \widetilde{\beta} q_* & \widetilde{\beta} p_* \end{pmatrix} \text{ with } \widetilde{\alpha} = \sqrt{-D} \neq 0,\ \widetilde{\beta} = p_1 - q_2 + 2\beta p_2,\ \widetilde{H} = \begin{pmatrix} p_* & q_* \\ -q_* & p_* \end{pmatrix}, \tag{36}$$

where $p_* = (p_1 + q_2)/2\,(= \operatorname{Re} \lambda_1)$, $q_* = -\sqrt{-D}/2\,(= -\operatorname{Im} \lambda_1) < 0$.

After that a change (9) brings (36) to the system

$$\delta^{-1} \begin{pmatrix} \breve{\alpha}(p_* \delta + q_* \delta_0) & \breve{\alpha} q_* s_0 + \breve{\beta}(p_* \delta + q_* \delta_0) & \breve{\beta} q_* s_0 \\ -\breve{\alpha} q_* r_0 & \breve{\alpha}(p_* \delta - q_* \delta_0) - \breve{\beta} q_* r_0 & \breve{\beta}(p_* \delta - q_* \delta_0) \end{pmatrix}, \tag{37}$$

where $(\breve{\alpha}, \breve{\beta})$ is from (30), matrix $\breve{H} = \delta^{-1} \begin{pmatrix} p_* \delta + q_* \delta_0 & q_* s_0 \\ -q_* r_0 & p_* \delta - q_* \delta_0 \end{pmatrix}$ with $\delta_0 = r_1 s_1 + r_2 s_2$, $r_0 = r_1^2 + r_2^2$, $s_0 = s_1^2 + s_2^2$.

In 1zhord3w) $\breve{\beta} = 0$ for $s_1 = -\widetilde{\alpha}^{-1} \widetilde{\beta} s_2$ and (37) has the form

$$\begin{pmatrix} (\widetilde{\alpha} p_* - \widetilde{\beta} q_*) r_1 + (\widetilde{\alpha} q_* + \widetilde{\beta} p_*) r_2 & \widetilde{\alpha}^{-1}(\widetilde{\alpha}^2 + \widetilde{\beta}^2) q_* s_2 & 0 \\ -\widetilde{\alpha}(r_1^2 + r_2^2) q_* s_2^{-1} & (\widetilde{\alpha} p_* + \widetilde{\beta} q_*) r_1 - (\widetilde{\alpha} q_* - \widetilde{\beta} p_*) r_2 & 0 \end{pmatrix}.$$

3_1) $p_* \neq 0$. Then for $r_1 = \dfrac{(\widetilde{\alpha} q_* - \widetilde{\beta} p_*) \operatorname{sign} p_*}{q_*(\widetilde{\alpha}^2 + \widetilde{\beta}^2)(p_*^2 + q_*^2)^{1/2}}$, $r_2 = \dfrac{(\widetilde{\alpha} p_* + \widetilde{\beta} q_*) \operatorname{sign} p_*}{q_*(\widetilde{\alpha}^2 + \widetilde{\beta}^2)(p_*^2 + q_*^2)^{1/2}}$, $s_1 = \dfrac{\widetilde{\beta}}{q_*(\widetilde{\alpha}^2 + \widetilde{\beta}^2)}$, $s_2 = -\dfrac{\widetilde{\alpha}}{q_*(\widetilde{\alpha}^2 + \widetilde{\beta}^2)}$, this is a CF_5^1 with $u = 2|p_*|(p_*^2 + q_*^2)^{-1/2}$ $(0 < u < 2)$, $\sigma = -1$.

3_2) $p_* = 0$. Then by the same change we obtain a CF_2^1 with $\sigma = -1$. $\square$

Remark 7 CF_3^1 for $u = -1$ is not a canonical form according to principle 2. By the change (9) with $r_1, r_2, s_2 = 1$, $s_1 = 0$ it may be reduced to a CF_2^1 with $\sigma = 1$. As is proved in theorem, for $|u| > 1$ the system is also reduced to a CF_3^1, but with $0 < |u| < 1$.

Remark 8 CF_5^1 for $|u| \geq 2$ is not a canonical form according to principle 3. For $u = u_*$ and $|u_*| = 2$ by change (9) with $r_1 = u_*^{-1}$, $s_1 = 0$, $r_2 = 1 - u_*^{-1}$, $s_2 = -1$ it is reduced to a CF_3^1 with $u = u_*^{-1}$. And for $|u_*| > 2$ by change (9) with $r_1 = 2(u_* \pm (u_*^2-4)^{1/2})^{-1}$, $s_1 = 0$, $r_2 = 1$, $s_2 = -1$ it is reduced to a CF_3^1 with $u = (u_* \mp (u_*^2-4))(u_* \pm (u_*^2-4)^{1/2})^{-1}$, being $0 < |u| < 1$. The form CF_5^1 with $u = u_*$ for $-2 < u_* < 0$ by change (13) with $r_1 = -1$, $s_2 = 1$ is also reduced to a CF_5^1 with $u = -u_*$.

Remark 9 Forms $\mathrm{F}_1^1 = \begin{pmatrix} 0 & u_* & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathrm{F}_2^1 = \begin{pmatrix} u_* & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ with the structure of CF_5^1, which are not listed, are not canonical forms according to principle 3.

The form F_1^1 for $u_* \geq -1/4$ by change (9) with $r_1 = (1-(4u_*+1)^{1/2})(1+2u_*-(4u_*+1)^{1/2})^{-1}$, $s_1 = 0$, $r_2 = 2((4u_*+1)^{1/2}-1-2u_*)^{-1}$, $s_2 = u_*^{-1}$ is reduced to a CF_3^1 with $u = 2u_*((4u_*+1)^{1/2}-1-2u_*)^{-1} \neq 0$; for $u_* < -1/4$ by change (9) with $r_1 = -(-u_*)^{-1/2}$, $s_1 = 0$, $r_2 = (-u_*)^{-3/2}$, $s_2 = -u_*^{-1}$ is reduced to a system CF_5^1 with $u = -(-u_*)^{-1/2}$, being $-2 < u < 0$.

The form F_2^1 with $u = u_*$ by change (9) with $r_1 = ((u_*^2+4)^{1/2}-u_*)/2$, $s_1 = 0$, $r_2 = -1$, $s_2 = 1$ is brought to a system CF_3^1 with $u = u_*((u_*^2+4)^{1/2}-u_*)/2 - 1 < 0$.

Remark 10 The form $\mathrm{F}_3^1 = \begin{pmatrix} u & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ is a CF_4^1 for $u = 1$, and for $u \neq 1$ it is not a canonical form according to principle 2. By change (9) with $r_1, s_2 = 1$, $s_1 = 0$, $r_2 = (u-1)^{-1}$ F_3^1 is reduced to a CF_1^1.

Remark 11 Every CF_i^1 $(i = \overline{1,5})$ by renumbering (14) is brought to $\mathrm{CF}_i^{1\mathrm{p}}$ in accordance with definition 8.

In theorem (3) all the linear invertible changes (9) are given in explicit form. Therefore the conditions assuring the reduction of system (2) to a corresponding CF_i^1, may be written using coefficients of (25).

Corollary 2 *The system (25) in which $p_1q_2 - p_2q_1 \neq 0$ by a linear invertible change (9) is brought to a CF_i^1 $(i = \overline{1,5})$, if the following parameters: the coefficient β of P_0 $(\alpha = 1)$ and elements p_1, q_1, p_2, q_2 of matrix H satisfy conditions:*

CF_1^1 : 1) $D > 0$, $2\beta p_2 + p_1 - q_2 + \sigma_*\sqrt{D} = 0$, then $u = \lambda_1^{-1}\lambda_2 \neq 0, 1$; 2) $D > 0$, $2q_1 - \beta(p_1 - q_2 + \sigma_*\sqrt{D}) = 0$, then $u = \lambda_1\lambda_2^{-1} \neq 0, 1$; 3) $D = 0, q_1 = 0, p_2 = 0$, then $u = 1$;

CF_2^1 : 1) $D > 0$, $p_1 + q_2 = 0$, $2\beta p_2 + 2p_1 + \sigma_*\sqrt{D}, 2q_1 - \beta(2p_1 + \sigma_*\sqrt{D}) \neq 0$, then $\sigma = 1$; 2) $D < 0$, $p_1 + q_2 = 0$, then $\sigma = -1$;

CF_3^1 : 1) $D > 0$, $p_1 + q_2 \neq 0$, $2\beta p_2 + p_1 - q_2 + \sigma_*\sqrt{D} \neq 0$, $2q_1 - \beta(p_1 - q_2 + \sigma_*\sqrt{D}) \neq 0$, then $u = \lambda_1^{-1}\lambda_2$ for $|\lambda_1| > |\lambda_2|$, $u = \lambda_1\lambda_2^{-1}$ for $|\lambda_1| < |\lambda_2|$, i. e. $0 < |u| < 1$; 2) $D = 0$, $q_1 \neq 0$, $2q_1 - \beta p_1 + \beta q_2 \neq 0$, then $u = 1$; 3) $D = 0$, $q_1 = 0$, $p_2 \neq 0$, $\beta \neq 0$, then $u = 1$;

CF_4^1 : 1) $D = 0$, $q_1 \neq 0$, $2q_1 - \beta p_1 + \beta q_2 = 0$; 2) $D = 0$, $q_1 = 0$, $p_2 \neq 0$, $\beta = 0$;

CF_5^1 : $D < 0$, $p_1 + q_2 \neq 0$, then $u = |p_1 + q_2|(p_1q_2 - p_2q_1)^{-1/2}$, $0 < u < 2$.

Here $D = (p_1 - q_2)^2 + 4p_2q_1$, $\lambda_1 = (p_1 + q_2 + \sigma_*\sqrt{D})/2 \neq 0$, $\lambda_2 = (p_1 + q_2 - \sigma_*\sqrt{D})/2 \neq 0$, $\sigma_* = \{\,\mathrm{sign}\,(p_1 - q_2)$ for $p_1 \neq q_2$; 1 for $p_1 = q_2\,\}$.

6 Canonical forms of system (2) in case of l = 2

6.1 Linear equivalency of systems for l = 2

Assertion 4 *For system (2) the following conditions are equivalent:*

1) $l = 2$, 2) $\exists\, k : P_2 \equiv kP_1$ $(a_2 = ka_1,\ b_2 = kb_1,\ c_2 = kc_1)$, 3) $\delta_{ab}, \delta_{ac}, \delta_{bc} = 0$.

Proof 1) $\Leftrightarrow$ 2) by definition 5 and proposition 1 .

It is evidently that 2) $\Rightarrow$ 3). Inversely, let 3) hold. Then, for instance, $a_1 \neq 0$. Let $k = a_2/a_1$. Because of $a_1b_2 - a_2b_1 = 0$, $b_2 = kb_1$. In much the same way $c_2 = kc_1$. ☐

Assertion 5 *For system (2) the condition* $P_2(x) \equiv 0$ *is invariant with respect to any change (9) with* $r_2 = 0$.

Proof Perform in the system (2)with $P_2 \equiv 0$ a change (9). According to (12) in obtained system we have (10)

$$\widetilde{A} = \delta^{-1} \begin{pmatrix} s_2P_1(r_1, r_2) & s_2(a_1r_1s_1 + b_1\delta_* + c_1r_2s_2) & s_2P_1(s_1, s_2) \\ -r_2P_1(r_1, r_2) & -r_2(a_1r_1s_1 + b_1\delta_* + c_1r_2s_2) & -r_2P_1(s_1, s_2) \end{pmatrix}. \qquad (38)$$

If $\widetilde{P}_2 \equiv 0$, then $r_2 = 0$, because otherwise common factors of $\widetilde{P}_1$ and $\widetilde{P}_2$ vanish, i. e. $\widetilde{P}_1 \equiv 0$. If $r_2 = 0$, then in (38) $\widetilde{P}_2 \equiv 0$. ☐

Assertion 6 *Any change (9) with* $r_2 = -s_2 \neq 0$ *transforms system (2) with* $P_2(x) \equiv 0$ *into the system* (10) *with* $\widetilde{P}_1 \equiv \widetilde{P}_2$.

The assertion 6 immediately follows from (38).

In accordance with assertion 4 for $l = 2$ there is such k, that in (2) $P_2 = kP_1$. Hence system (2) has one of two forms:

$$I)\ b_1 \geq a_1c_1: \quad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = (\alpha x_1 + \beta x_2) \begin{pmatrix} p_1x_1 + q_1x_2 \\ kp_1x_1 + kq_1x_2 \end{pmatrix} \quad \begin{pmatrix} \alpha^2 + \beta^2 \neq 0 \\ p_1^2 + q_1^2 \neq 0 \end{pmatrix}, \qquad (39)$$

i. e. $H = \begin{pmatrix} p_1 & q_1 \\ kp_1 & kq_1 \end{pmatrix}$ has eigenvalues $\lambda_1 = p_1 + kq_1$, $\lambda_2 = 0$. Thus, (39) is the system (25) from the case $l = 1$, with $p_2 = kp_1$, $q_2 = kq_2$ and $\det H = \delta_{pq} = 0$.

By following to proposition 2, we assume that in system (39), if $\alpha \neq 0$, then $\alpha = 1$ and $P_0 = x_1 + \beta x_2$, and if $\alpha = 0$, then $\beta = 1$ and $P_0 = x_2$.

$$II)\ b_1^2 < a_1c_1: \quad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = (a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2) \begin{pmatrix} 1 \\ k \end{pmatrix}. \qquad (40)$$

Remark 12 The system (39) may be written in the form (40), but the form (39) is more preferable as it allows us to use results obtained for (25).

Proposition 3 *To eliminate an ambiguity appearing when factoring out the linear common factor* P_0 *from* P *in system (39) , we arrange for (if it is possible) to factor out such a linear common factor that in matrix* H *the eigenvalue* $\lambda_1 = p_1 + kq_1 \neq 0$.

6.2 Construction of degenerate canonical forms for l = 2

Simplify system (39), following to the scheme of contraction for (25).

By theorem 2 a change (9) brings both system (25) and system (39) to (29) $\widetilde{P} = \left\langle (\widetilde{\alpha}, \widetilde{\beta}),\, y\right\rangle \widetilde{H} y$, where vector $(\widetilde{\alpha}, \widetilde{\beta})$ and matrix $\widetilde{H}$ are defined in (27), but $\delta_{\widetilde{pq}} = \det \widetilde{H} = 0$.

Take a change (9) such that in system (29) matrix $\widetilde{H}$ become Jordan matrix, which is possible due to formula (27_2).

So, if $\lambda_1 = p_1 + kq_1 \neq 0$, then change (9) with $L_1 = \begin{pmatrix} 1 & q_1 \\ k & -p_1 \end{pmatrix}$, and if $\lambda_1 = 0$, then $q_1 \neq 0$ and change (9) with $L_2 = \begin{pmatrix} 1 & 0 \\ k & q_1^{-1} \end{pmatrix}$ transforms system (39) in systems (29) of two following forms:

$$\begin{array}{llll} \widetilde{\alpha} = \alpha + \beta k, & \widetilde{\beta} = \alpha q_1 - \beta p_1, & \widetilde{H} = \begin{pmatrix} p_1 + kq_1 & 0 \\ 0 & 0 \end{pmatrix} & \text{or } \widetilde{A} = \begin{pmatrix} \lambda_1 \widetilde{\alpha} & \lambda_1 \widetilde{\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ \widetilde{\alpha} = \alpha + \beta k, & \widetilde{\beta} = \beta q_1^{-1}, & \widetilde{H} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{or } \widetilde{A} = \begin{pmatrix} 0 & \widetilde{\alpha} & \widetilde{\beta} \\ 0 & 0 & 0 \end{pmatrix}. \end{array} \tag{41}$$

By this means the existence of zero eigenvalue λ_2 of H resulted in $\widetilde{P}_2 \equiv 0$ in systems (41).

Further by change (9) we will simplify as much as possible and normalize systems (41_1) and (41_2), such that the condition $P_2 \equiv 0$ holds. Thereby we reduce the systems to the canonical forms for which principle 1 is not satisfied.

Taking into account assertion 5 we note that an arbitrary change (9) with $r_2 = 0$ brings systems (41_1) and (41_2) to systems

$$\lambda_1 \begin{pmatrix} \widetilde{\alpha} r_1 & 2\widetilde{\alpha} s_1 + \widetilde{\beta} s_2 & (\widetilde{\alpha} s_1 + \widetilde{\beta} s_2) s_1 r_1^{-1} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \widetilde{\alpha} s_2 & (\widetilde{\alpha} s_1 + \widetilde{\beta} s_2) s_2 r_1^{-1} \\ 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

For $l = 2$ the concept of degenerate CF naturally comes into existence.

Definition 9 *For $l = 2$ system (2) is said to be degenerate canonical form (DCF^2), is it is a CF^2 with respect to definition 7, where principle 1 is substituted for the condition $P_2 \equiv 0$.*

Remark 13 A generalized normal form of a system, where DCF^2 is the unperturbed part, is a generalization of the Belitski normal form (see [7], [14]) for the case when the unperturbed part is degenerate but is not linear.

The LIST of degenerate canonical forms of system (2) in case of $l = 2$:

$$\mathrm{DCF}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{DCF}_2^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{DCF}_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{DCF}_4^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{DCF}_5^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4 *For $l = 2$ system (2) of the form (39), (40) by a linear invertible change (9) is reduced to one of 5 linearly nonequivalent DCF^2.*

Proof **I**) System (2) has the form (39).

1) $\lambda_1 = p_1 + kq_1 \neq 0$. System (41_1) is obtained from (39), and system (42_1) is obtained from (41_1).

$\mathbf{1_1}$) $\widetilde{\beta} = 0$ $(\widetilde{\alpha} \neq 0)$. Then (42_1) for $r_1 = (\lambda_1\widetilde{\alpha})^{-1}$, $s_1 = 0$, $s_2 = 1$ is a DCF_1^2.

$\mathbf{1_2}$) $\widetilde{\alpha} = 0$ $(\widetilde{\beta} \neq 0)$. Then (42_1) for $r_1 = 1$, $s_1 = 0$, $s_2 = (\lambda_1\widetilde{\beta})^{-1}$ is a DCF_2^2.

$\mathbf{1_3}$) $\widetilde{\alpha}, \widetilde{\beta} \neq 0$. Then (42_1) for $r_1 = (\lambda_1\widetilde{\alpha})^{-1}$, $s_1 = 0$, $s_2 = (\lambda_1\widetilde{\beta})^{-1}$ is a DCF_4^2.

2) $\lambda_1 = p_1 + kq_1 = 0$ $(q_1 \neq 0)$. Systems (41_2) and (42_2) are obtained from (39).

$\mathbf{2_1}$) $\widetilde{\alpha} = 0$ $(\widetilde{\beta} \neq 0)$. Then (42_2) for $r_1 = \widetilde{\beta}$, $s_1 = 0$, $s_2 = 1$ is a DCF_3^2.

$\mathbf{2_2}$) $\widetilde{\beta} = 0$ $(\widetilde{\alpha} \neq 0)$. Then in system (41_2) $\widetilde{P}_1 = \widetilde{\alpha}x_1x_2$. By proposition 3 the case 2_2) is impossible. Such a situation applies to the case 1).

$\mathbf{2_3}$) $\widetilde{\alpha}, \widetilde{\beta} \neq 0$. Then in (41_2) $\widetilde{P}_1 = (\widetilde{\alpha}x_1 + \widetilde{\beta}x_2)x_2$. By proposition 3 the case 2_3) is impossible. Such a situation applies to the case 1).

II) System (2) has the form (40).

In accordance with (12) any change (9) brings (2) with $P_2 = kP_1$ to a system with coefficients

$$\begin{array}{l}
\widetilde{a}_1 = (s_2 - ks_1)P_1(r_1, r_2)\delta^{-1}, \quad \widetilde{a}_2 = (kr_1 - r_2)P_1(r_1, r_2)\delta^{-1},\\
\widetilde{b}_1 = (s_2 - ks_1)(a_1r_1s_1 + b_1(r_1s_2 + r_2s_1) + c_1r_2s_2)\delta^{-1},\\
\widetilde{b}_2 = (kr_1 - r_2)(a_1r_1s_1 + b_1(r_1s_2 + r_2s_1) + c_1r_2s_2)\delta^{-1},\\
\widetilde{c}_1 = (s_2 - ks_1)P_1(s_1, s_2)\delta^{-1}, \quad \widetilde{c}_2 = (kr_1 - r_2)P_1(s_1, s_2)\delta^{-1}.
\end{array}$$

For $r_2 = kr_1$ $(\delta = r_1(s_2 - ks_1) \neq 0)$ the obtained system has the form

$$\begin{pmatrix} (a_1 + 2b_1k + c_1k^2)r_1 & 2(a_1s_1 + kb_1s_1 + b_1s_2 + kc_1s_2) & P_1(s_1, s_2)r_1^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

For $r_1 = (a_1 + 2b_1k + c_1k^2)^{-1}$, $s_1 = -(b_1 + kc_1)(a_1 + 2b_1k + c_1k^2)^{-1}(a_1c_1 - b_1^2)^{-1/2}$, $s_2 = (a_1 + kb_1)(a_1 + 2b_1k + c_1k^2)^{-1}(a_1c_1 - b_1^2)^{-1/2}$ it is a DCF_5^2. $\square$

Remark 14 Forms $\mathrm{DF}_1^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathrm{DF}_2^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ with the structure of DCF_5^2, which are not in the list, are not canonical ones according to principles 2 and 3. A form DF_1^2 by change (9) with $r_1 = 1$, $s_1 = -1$, $r_2 = 0$, $s_2 = 1$ is reduced to a DCF_2^2. A form DF_2^2 by change (9) with $r_1 = 1$, $s_1 = 1/2$, $r_2 = 0$, $s_2 = -1/2$ is reduced to a DCF_4^2.

Corollary 3 *I) A system (39) by a linear invertible change (9) may be brought to a* DCF_i^2 $(i = \overline{1,4})$, *if the following parameters: coefficients* α, β *of the common factor* P_0, *elements* p_1, q_1 *of* H *and the proportionality coefficient* k *satisfy the conditions:*

$DCF_1^2:$ 1) $\alpha = 1,\ q_1 = \beta p_1,\ kq_1 \neq -p_1,$ 2) $\alpha = 0,\ \beta = 1,\ p_1 = 0,\ kq_1 \neq 0;$

$DCF_2^2:$ 1) $\alpha = 1,\ \beta k = -1,\ kq_1 \neq -p_1,$ 2) $\alpha = 0,\ \beta = 1,\ p_1 \neq 0,\ k = 0;$

$DCF_3^2:$ 1) $\alpha = 1,\ \beta k = -1,\ kq_1 = -p_1,$ 2) $\alpha = 0,\ \beta = 1,\ p_1 = 0,\ k = 0;$

$DCF_4^2:$ 1) $\alpha = 1,\ \beta k \neq -1,\ kq_1 \neq -p_1,\ q_1 \neq \beta p_1,$ 2) $\alpha = 0,\ \beta = 1,\ p_1 \neq 0,\ k \neq 0,\ kq_1 \neq -p_1.$

II)System (40) by a linear invertible change (9) is brought to a DCF_5^2.

6.3 The construction of main and additional CF for $\mathbf{l = 2}$

Generally speaking, with a view to further normalization of perturbed systems it is unsufficiently to bring systems (41) and together with them systems (39) or (40) to a DCF^2. Principle 1 has to be hold for a total normalization. Hence we will transform every DCF_i^2 in a nondegenerate CF^2 by change(9).

Remark 15 A CF^2 obtained from a DCF_i^2 has as a rule more nonzero elements, which is a reasonable "pay" for greater opportunities on the further normalization of perturbations.

Remark 16 A specifics of the case $l = 2$ is such that due to a proportionality of coefficients in polynomials P_1, P_2 in systems (39) or (40) principle 1 in the definition of canonical form is fully applied, i. e. namely for $l = 2$ the requirement $P_1 \equiv P_2 \quad (k = 1)$ is actual. But principle 3 loses its value completely.

The LIST of canonical forms of system (2) in case of $l = 2$:

$$\mathrm{CF}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{CF}_2^2 = \begin{pmatrix} 1 & 0 & \sigma \\ 1 & 0 & \sigma \end{pmatrix} \quad (\sigma = \pm 1), \quad \mathrm{CF}_3^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix},$$
$$\mathrm{CF}_4^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \qquad \mathrm{ACF}_2^2 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Theorem 5 *Every DCF_i^2 $(i = \overline{1,5})$ by a linear invertible change (9) is reduced to a CF^2.*

Proof By using the assertion 6 we will sequentially perform (9) for $\mathrm{DCF}_1^2, \ldots, \mathrm{DCF}_5^2$ with $r_2 = -s_2 \neq 0$ and select the remaining coefficients such that to obtain a CF^2. Let $\delta_1 = (r_1 + s_1)^{-1}$.

A DCF_1^2 is brought to a system with $a_1 = r_1^2\delta_1,\ b_1 = r_1 s_1 \delta_1,\ c_1 = s_1^2 \delta_1$ that for $r_1 = 1,$ $s_1 = 0,\ r_2 = -1,\ s_2 = 1$ is a CF_1^2.

A DCF_2^2 is brought to a system with $a_1 = -r_1 s_2 \delta_1,\ 2b_1 = s_2(r_1 - s_1)\delta_1,\ c_1 = s_1 s_2 \delta_1$ that for $r_1 = 1,\ s_1 = 1,\ r_2 = 2,\ s_2 = -2$ is a CF_2^2 with $\sigma = -1$, and for $r_1 = 1,\ s_1 = 0,\ r_2 = 1,$ $s_2 = -1$ is a ACF_2^2.

A DCF_3^2 is reduced to a system with $a_1 = s_2^2\delta_1,\ b_1 = -s_2^2\delta_1,\ c_1 = s_2^2\delta_1$ that for $r_1 = 1,$ $s_1 = 1,\ r_2 = -2^{1/2},\ s_2 = 2^{1/2}$ is a CF_3^2.

A DCF_4^2 is reduced to a system with $a_1 = r_1(r_1 - s_2)\delta_1,\ 2b_1 = (2r_1 s_1 + r_1 s_2 - s_1 s_2)\delta_1,$ $c_1 = s_1(s_1 + s_2)\delta_1$ which for $r_1 = 1,\ s_1 = 0,\ r_2 = -1,\ s_2 = 1$ is a CF_4^2.

A DCF_5^2 is reduced to a system with $a_1 = (r_1^2 + s_2^2)\delta_1,\ b_1 = (r_1 s_1 - s_2^2)\delta_1,\ c_1 = (s_1^2 + s_2^2)\delta_1$ which for $r_1 = 1, s_1 = 1,\ r_2 = -1\ \ s_2 = 1$ is a CF_2^2 with $\sigma = 1$. $\square$

Remark 17 As was proved, DCF_i^2 are reduced to CF_i^2 for $i = 1, 3, 4$. Forms DCF_2^2 and DCF_5^2 are reduced to a CF_2^2 with $\sigma = -1$ and $\sigma = 1$ respectively.

Remark 18 The form $\mathrm{F}_1^2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ (not in the list) is not a canonical one in accordance with principle 2. By the change (9) with $r_1 = 0,\ s_1 = 1/2,\ r_2 = 1,\ s_2 = -1/2$ it is reduced to a DCF_4^2.

Remark 19 Forms ACF_2^2 and CF_2^2 in which $\sigma = -1$ are connected by the change (9) with $r_1, s_1 = -2, r_2 = -4, s_2 = 0$, not a renumbering. At that a CF_2^2 is the main one in accordance with principle 5b.

The COMPLETE LIST of the forms having 3,4 and 6 nonzero elements

$$\underline{\mathrm{CF}_1^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_2, \quad \mathrm{CF}_1^1 = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_3, \quad \mathrm{CF}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}_4,$$

$$\mathrm{CF}_1^{1\mathrm{r}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & u \end{pmatrix}_3, \quad \underline{\mathrm{CF}_4^2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_4, \quad \mathrm{CF}_2^1 = \begin{pmatrix} 0 & \sigma & 0 \\ 1 & 0 & 0 \end{pmatrix}_5,$$

$$\mathrm{CF}_1^{2\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_4, \quad \mathrm{CF}_2^{1\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \sigma & 0 \end{pmatrix}_5, \quad \underline{\mathrm{CF}_2^0} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}_6;$$

$$\mathrm{CF}_3^0 = \begin{pmatrix} 1 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}_4, \quad \mathrm{CF}_3^1 = \begin{pmatrix} u & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_5, \quad \mathrm{CF}_5^1 = \begin{pmatrix} u & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}_6,$$

$$\mathrm{CF}_4^0 = \begin{pmatrix} u & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_5, \quad \mathrm{CF}_5^0 = \begin{pmatrix} u & 0 & \sigma \\ 0 & 1 & 0 \end{pmatrix}_6, \quad \mathrm{CF}_6^0 = \begin{pmatrix} u & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}_7,$$

$$\mathrm{CF}_4^{1\mathrm{r}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}_6, \quad \mathrm{F}_1^{1\mathrm{r}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & u & 0 \end{pmatrix}_7, \quad \mathrm{CF}_7^0 = \begin{pmatrix} 0 & u & 1 \\ 1 & 0 & 0 \end{pmatrix}_8;$$

$$\mathrm{CF}_3^{0\mathrm{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 1 \end{pmatrix}_4, \quad \mathrm{CF}_3^{1\mathrm{r}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & u \end{pmatrix}_5, \quad \mathrm{CF}_5^{1\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & u \end{pmatrix}_6,$$

$$\mathrm{CF}_4^{0\mathrm{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & u \end{pmatrix}_5, \quad \mathrm{CF}_5^{0\mathrm{r}} = \begin{pmatrix} 0 & 1 & 0 \\ \sigma & 0 & u \end{pmatrix}_6, \quad \mathrm{CF}_6^{0\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & u \end{pmatrix}_7,$$

$$\mathrm{CF}_4^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}_6, \quad \mathrm{F}_1^1 = \begin{pmatrix} 0 & u & 0 \\ 1 & 1 & 0 \end{pmatrix}_7, \quad \mathrm{CF}_7^{0\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & u & 0 \end{pmatrix}_8;$$

$$\underline{\mathrm{CF}_8^0} = \begin{pmatrix} u & 1 & 0 \\ 0 & 1 & v \end{pmatrix}_6, \quad \mathrm{CF}_9^{0\mathrm{r}} = \begin{pmatrix} v & 1 & 0 \\ \sigma & 0 & u \end{pmatrix}_7, \quad \mathrm{ACF}_2^2 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}_8,$$

$$\mathrm{ACF}_9^0 = \begin{pmatrix} u & 0 & \sigma \\ 0 & 1 & v \end{pmatrix}_7, \quad \underline{\mathrm{CF}_2^2} = \begin{pmatrix} 1 & 0 & \sigma \\ 1 & 0 & \sigma \end{pmatrix}_8, \quad \mathrm{F}_4^0 = \begin{pmatrix} u & 0 & 1 \\ 1 & v & 0 \end{pmatrix}_9,$$

$$\mathrm{ACF}_2^{2\mathrm{r}} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}_8, \quad \mathrm{F}_4^{0\mathrm{r}} = \begin{pmatrix} 0 & v & 1 \\ 1 & 0 & u \end{pmatrix}_9, \quad \underline{\mathrm{F}_5^0} = \begin{pmatrix} 0 & u & 1 \\ 1 & v & 0 \end{pmatrix}_{10};$$

$$\mathrm{F}_3^0 = \begin{pmatrix} u & v & 1 \\ 0 & 0 & 1 \end{pmatrix}_7, \quad \mathrm{CF}_{10}^0 = \begin{pmatrix} 1/2 & u & -1 \\ 0 & 1 & 0 \end{pmatrix}_8, \quad \mathrm{F}_2^0 = \begin{pmatrix} u & v & 1 \\ 1 & 0 & 0 \end{pmatrix}_9,$$

$$\mathrm{F}_3^{0\mathrm{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & v & u \end{pmatrix}_7, \quad \mathrm{CF}_{10}^{0\mathrm{r}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & u & 1/2 \end{pmatrix}_8, \quad \mathrm{F}_2^{0\mathrm{r}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & v & u \end{pmatrix}_9;$$

$$\underline{\mathrm{CF}_3^2} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}_{12},$$

here each matrix is marked by a subscript and symmetric forms are underlined.

Part III

GNF of systems with degenerate CF in unperturbed part

7 Normalization of systems with DCF_1^2

Let in system (1) the unperturbed part $P(x)$ by a linear invertible change be reduced to the $\mathrm{DCF}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We assume at once that system (1) has the form

$$\dot{x}_1 = x_1^2 + X_1(x_1, x_2), \quad \dot{x}_2 = X_2(x_1, x_2). \tag{43}$$

Then system (6) may be written as

$$(s-3)h_1^{(s-1,p+1-s)} = \widehat{Y}_1^{(s,p+1-s)}, \quad (s-1)h_2^{(s-1,p+1-s)} = \widehat{Y}_2^{(s,p+1-s)} \quad (s = \overline{0, p+1};\ p \geq 2). \tag{44}$$

For solvability of system (44) it is necessary and sufficient the following relations were fulfilled

$$\widehat{Y}_1^{(0,p+1)} = 0, \quad \widehat{Y}_2^{(0,p+1)} = 0, \quad \widehat{Y}_1^{(3,p-2)} = 0, \quad \widehat{Y}_2^{(1,p)} = 0,$$

being there are no restrictions for coefficients $h_1^{(2,p-2)}$ and $h_2^{(0,p)}$ in change (9). By using denotation introduced for equations (5) and (7) we rewrite obtained resonance relations via coefficients of system (4):

$$Y_1^{(0,p+1)} = \widetilde{c}, \quad Y_1^{(3,p-2)} = \widetilde{c}, \quad Y_2^{(0,p+1)} = \widetilde{c}, \quad Y_2^{(1,p)} = \widetilde{c}. \tag{45}$$

Theorem 6 *The system (43) is formally equivalent to a system (4) with the unperturbed part $P = (y_1^2, 0)$ iff for any $p \geq 2$ coefficients of homogeneous polynomials $Y_i^{(p+1)}$ satisfy 4 resonance equations (45).*

Corollary 4 *The set consisting of $Y_1^{(0,p+1)}$, $Y_1^{(3,p-2)}$, $Y_2^{(0,p+1)}$, $Y_2^{(1,p)}$ is the unique resonant set.*

Theorem 7 *A system (43) by a formal change (9) may be reduced to a GNF (4), where for any $p \geq 2$ all the coefficients $Y_i^{(p+1)}$ $(i = 1, 2)$ are equal to zero, except possibly four coefficients from a resonance set, i. e. any GNF has the form:*

$$\dot{y}_1 = y_2^2 + \sum_{p=2}^{\infty} (Y_1^{(0,p+1)} y_2^{p+1} + Y_1^{(3,p-2)} y_1^3 y_2^{p-2}), \quad \dot{y}_2 = \sum_{p=2}^{\infty} (Y_2^{(0,p+1)} y_2^{p+1} + Y_2^{(1,p)} y_1 y_2^p).$$

Remark 20 Any GNF of system (43) has a rigid structure of orders of resonant terms, which is a distinctive feature of resonant normal forms and as a rule is not fulfilled for generalized ones.

8 Normalization of systems with DCF_2^2

Let in system (1) the unperturbed part $P(x)$ by a linear invertible change be reduced to the $\mathrm{DCF}_2^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We assume at once that system (1) has the form

$$\dot{x}_1 = x_1x_2 + X_1(x_1,x_2), \quad \dot{x}_2 = X_2(x_1,x_2). \tag{46}$$

Then system (6) is written as the following

$$(s-1)h_1^{(s,p-s)} - h_2^{(s-1,p-s+1)} = \widehat{Y}_1^{(s,p+1-s)}, \; sh_2^{(s,p-s)} = \widehat{Y}_2^{(s,p+1-s)} \;\; (s = \overline{0,p+1};\; p \geq 2). \tag{47}$$

In the subsystem (47_2) for $s = 0, p+1$ we have

$$\widehat{Y}_2^{(0,p+1)} = 0, \qquad \widehat{Y}_2^{(p+1,0)} = 0,$$

being $h_2^{(0,p)}$ is free, and for $s = \overline{1,p}$ we have $h_2^{(s,p-s)} = s^{-1}\widehat{Y}_2^{(s,p-s+1)}$.

In the subsystem (47_1) for $s = 1$ we obtain $0 \cdot h_1^{(1,p-1)} - h_2^{(0,p)} = \widehat{Y}_1^{(1,p+1)}$. This equation is uniquely decidable by using $h_2^{(0,p)}$, and $h_1^{(1,p-1)}$ remains free. For $s = 0, \overline{2,p}$ system (47_1) is decidable by using coefficients $h_1^{(s,p-s)}$. For $s = p+1$ we have the relation

$$p\widehat{Y}_1^{(p+1,0)} + \widehat{Y}_2^{(p,1)} = 0.$$

By using denotations introduced for equations (5) and (7), we rewrite obtained resonant relations via coefficients of system (4):

$$Y_2^{(p+1,0)} = \widetilde{c}, \qquad Y_2^{(0,p+1)} = \widetilde{c}, \qquad pY_1^{(p+1,0)} + Y_2^{(p,1)} = \widetilde{c}. \tag{48}$$

Theorem 8 *A system (46) is formally equivalent to a system (4) with the unperturbed part $P = (y_1y_2, 0)$ iff for any $p \geq 2$ coefficients of homogeneous polynomials $Y_i^{(p+1)}$ satisfy resonance equations (48).*

Corollary 5 *There are two resonance sets containing $Y_2^{(0,p+1)}$, $Y_2^{(p+1,0)}$ and either $Y_1^{(p+1,0)}$ or $Y_2^{(p,1)}$.*

Theorem 9 *A system (46) by a formal change (9) may be brought to a GNF (4), where for any $p \geq 2$ all the coefficients $Y_i^{(p+1)}$ $(i = 1,2)$ are equal to zero, except possibly 3 coefficients of one of two resonance sets, i. e. any GNF has one of two following structures:*

$$\dot{y}_1 = y_2^2 + \sum_{p=2}^{\infty} Y_1^{(p+1,0)} y_1^{p+1}, \quad \dot{y}_2 = \sum_{p=2}^{\infty} (Y_2^{(0,p+1)} y_2^{p+1} + Y_2^{(p+1,0)} y_1^{p+1});$$

$$\dot{y}_1 = y_2^2, \quad \dot{y}_2 = \sum_{p=2}^{\infty} (Y_2^{(0,p+1)} y_2^{p+1} + Y_2^{(p,1)} y_1^p y_2 + Y_2^{(p+1,0)} y_1^{p+1}).$$

9 Normalization of systems with $\mathbf{DCF}_3^2$

Let in a system (1) the unperturbed part $P(x)$ by a linear invertible change be reduced to the $\mathrm{DCF}_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We assume at once that system (1) has the form

$$\dot{x}_1 = x_2^2 + X_1(x_1, x_2), \quad \dot{x}_2 = X_2(x_1, x_2). \tag{49}$$

Then system (6) may be written in the form

$$(s+1)h_1^{(s+1,p-s-1)} - 2h_2^{(s,p-s)} = \widehat{Y}_1^{(s,p+1-s)}, \ (s+1)h_2^{(s+1,p-s-1)} = \widehat{Y}_2^{(s,p+1-s)} \ (s = \overline{0, p+1}). \tag{50}$$

In the subsystem (50_2) for $s = p, p+1$ we have relations

$$\widehat{Y}_2^{(p,1)} = 0, \qquad \widehat{Y}_2^{(p+1,0)} = 0,$$

and for $s = \overline{0, p-1}$ we obtain $h_2^{(s+1,p-s-1)} = (s+1)^{-1}\widehat{Y}_2^{(s,p+1-s)}$, being $h_2^{(0,p)}$ is free.

In the subsystem (50_1) for $s = p, p+1$ there are relations

$$\widehat{Y}_1^{(p+1,0)} = 0, \qquad \widehat{Y}_1^{(p,1)} + 2h_2^{(p,0)} = 0,$$

where $h_2^{(p,0)} = p^{-1}\widehat{Y}_2^{(p-1,2)}$, i.e. the second relation has the form $p\widehat{Y}_1^{(p,1)} + 2\widehat{Y}_2^{(p-1,2)} = 0$.

Components $h_1^{(0,p)}$ and $h_2^{(0,p)}$ are free because they are not contained in system (50).

By using denotations introduced for equations (5) and (7), we rewrite obtained resonance relations via coefficients of system (4):

$$Y_1^{(p+1,0)} = \widetilde{c}, \quad Y_2^{(p,1)} = \widetilde{c}, \quad Y_2^{(p+1,0)} = \widetilde{c}, \quad pY_1^{(p,1)} + 2Y_2^{(p-1,2)} = \widetilde{c}. \tag{51}$$

Theorem 10 *A system (49) is formally equivalent to a system (4) with the unperturbed part $P = (y_2^2, 0)$ iff for any $p \geq 2$ coefficients of homogeneous polynomials $Y_i^{(p+1)}$ satisfy 4 resonant equations (51).*

Corollary 6 *There are two resonance sets containing $Y_1^{(p+1,0)}$, $Y_2^{(p,1)}$, $Y_2^{(p+1,0)}$ and either $Y_1^{(p,1)}$ or $Y_2^{(p-1,2)}$.*

Theorem 11 *A system (49) by a formal change (9) may be reduced to a GNF (4), where for any $p \geq 2$ all the coefficients $Y_i^{(p+1)}$ $(i = 1, 2)$ are equal to zero, except possibly 4 coefficients from one of two resonance sets, i.e. any GNF has one of two structures:*

$$\dot{y}_1 = y_2^2 + \sum_{p=2}^{\infty}(Y_1^{(p+1,0)}y_1^{p+1} + Y_1^{(p,1)}y_1^p y_2), \quad \dot{y}_2 = \sum_{p=2}^{\infty}(Y_2^{(p,1)}y_1^p y_2 + Y_2^{(p+1,0)}y_1^{p+1});$$

$$\dot{y}_1 = y_2^2 + \sum_{p=2}^{\infty} Y_1^{(p+1,0)}y_1^{p+1}, \quad \dot{y}_2 = \sum_{p=2}^{\infty}(Y_2^{(p-1,2)}y_1^{p-1}y_2^2 + Y_2^{(p,1)}y_1^p y_2 + Y_2^{(p+1,0)}y_1^{p+1}).$$

10 Normalization of systems with DCF_4^2

Let in a system (1) the unperturbed part $P(x)$ by a linear invertible change be reduced to the $\mathrm{DCF}_4^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We assume at once that system (1) has the form

$$\dot{x}_1 = x_1^2 + x_1 x_2 + X_1(x_1, x_2), \quad \dot{x}_2 = X_2(x_1, x_2). \tag{52}$$

Then system (6) is written in the form

$$\begin{array}{l} (s-3)h_1^{(s-1,p-s+1)} + (s-1)h_1^{(s,p-s)} - h_2^{(s-1,p-s+1)} = \widehat{Y}_1^{(s,p+1-s)} \quad , \\ (s-1)h_2^{(s-1,p-s+1)} + s h_2^{(s,p-s)} = \widehat{Y}_2^{(s,p+1-s)} \qquad (s = \overline{0, p+1};\ p \geq 2). \end{array} \tag{53}$$

In subsystem (53_2) for $s = 0$ we have the relation

$$\widehat{Y}_2^{(0,p+1)} = 0,$$

being $h_2^{(0,p)}$ is free. For $s = \overline{1,p}$ from (53_2) coefficients $h_2^{(s,p-s)} = s^{-1} \sum_{j=1}^{s} (-1)^{s-j} \widehat{Y}_2^{(j,p+1-j)}$ are uniquely determined.

The last equation in (53_2) has the form: $p h_2^{(p,0)} = \widehat{Y}_2^{(p+1,0)}$.

Substituting in the equation $h_2^{(p,0)}$, we obtain the second relation

$$\sum_{j=1}^{p+1} (-1)^j \widehat{Y}_2^{(j,p+1-j)} = 0.$$

Substituting $h_2^{(s,p-s)}$ from (53_2) in (53_1), we obtain the system

$$a_s h_1^{(s-1,p-s+1)} + b_s h_1^{(s,p-s)} = \breve{Y}_1^{(s,p+1-s)} \qquad (s = \overline{0, p+1}), \tag{54}$$

where $a_s = s - 3$, $b_s = s - 1$, $\breve{Y}_1^{(s,p+1-s)} = \widehat{Y}_1^{(s,p+1-s)} + h_2^{(s-1,p-s+1)}$.

Take last $p - 1$ equations of (54) and form a new subsystem

$$\Theta h_1 = \breve{Y},$$

where $\Theta = \begin{pmatrix} 0 & b_3 & 0 & \dots & 0 & 0 \\ 0 & a_4 & b_4 & \dots & 0 & 0 \\ 0 & 0 & a_5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{p-1} & 0 \\ 0 & 0 & 0 & \dots & a_p & b_p \\ 0 & 0 & 0 & \dots & 0 & a_{p+1} \end{pmatrix}$ – bidiagonal $(p-1)$ - matrix, vectors $h_1 = (h_1^{(2,p-2)}, \dots, h_1^{(p,0)})$, $\breve{Y} = (\breve{Y}_1^{(3,p-2)}, \dots, \breve{Y}_1^{(p+1,0)})$.

By the Gauss method this system may be transformed in the system

$$\Theta_g h_1 = Y_g, \tag{55}$$

where $\Theta_g = \operatorname{diag}\{0, a_4, \dots, a_{p+1}\}$, vector $Y_g = (Y_g^{(3,p-2)}, \dots, Y_g^{(p+1,0)})$ has components $Y_g^{(p+1,0)} = \breve{Y}_1^{(p+1,0)}$, $Y_g^{(s,p+1-s)} = \breve{Y}_1^{(s,p+1-s)} - a_{s+1}^{-1} b_s Y_g^{(s+1,p-s)}$.

It is clear that $Y_g^{(s,p+1-s)} = (s-2)^{-1}\sum_{j=s-1}^{p}(-1)^{j-s+1}(j-1)\check{Y}_1^{(j+1,p-j)}$ $(s=\overline{p+1,3})$.

For $s = \overline{p+1,4}$ system (55) is uniquely decidable with respect to coefficients $h_1^{(p,0)},\ldots,h_1^{(3,p-3)}$, and its first equation $(s=3)$ has the form: $0\cdot h_1^{(2,p-2)} = Y_g^{(3,p-2)}$, and the term $h_1^{(2,p-2)}$ is free.

Substituting into this equations expressions for $Y_g^{(3,p-2)}$, $\check{Y}_1^{(j+1,p-j)}$, $h_2^{(j,p-j)}$, we obtain:

$$\sum_{j=2}^{p}(-1)^j(j-1)\widehat{Y}_1^{(j+1,p-j)} + \sum_{j=2}^{p}(-1)^j\frac{j-1}{j}\sum_{k=1}^{j}(-1)^{j-k}\widehat{Y}_2^{(k,p+1-k)} = 0$$

or the resonance relation

$$\sum_{j=2}^{p}(-1)^j(j-1)\widehat{Y}_1^{(j+1,p-j)} + \sum_{j=1}^{p}(-1)^j\sum_{k=j-1}^{p-1}\frac{k}{k+1}\widehat{Y}_2^{(j,p+1-j)} = 0.$$

First 3 equations of (54) have the form:

$$-h_1^{(0,p)} = \widehat{Y}_1^{(0,p+1)},\quad -2h_1^{(0,p)} = \widehat{Y}_1^{(1,p)} + h_2^{(0,p)},\quad -h_1^{(1,p-1)} + h_1^{(2,p-2)} = \widehat{Y}_1^{(2,p+1)} + h_2^{(1,p-1)}.$$

We uniquely find $h_1^{(0,p)}$ from the first equation and using free $h_2^{(0,p)}$ uniquely decide the second equation. In the third equation either $h_1^{(1,p-1)}$ or $h_1^{(2,p-2)}$ is free. Using introduced denotations for (5) and (7) we rewrite obtained resonance relations via coefficients of (4):

$$\begin{gathered} Y_2^{(0,p+1)} = \widetilde{c},\qquad \sum_{j=1}^{p+1}(-1)^jY_2^{(j,p+1-j)} = \widetilde{c},\\ \sum_{j=2}^{p}(-1)^j(j-1)Y_1^{(j+1,p-j)} + \sum_{j=1}^{p}(-1)^j\sum_{k=j-1}^{p-1}\frac{k}{k+1}Y_2^{(j,p+1-j)} = \widetilde{c}. \end{gathered}\tag{56}$$

Theorem 12 *A system (52) is formally equivalent to (4) with the unperturbed part* $P = (y_1^2+y_1y_2,0)$ *iff for any* $p\ge 2$ *coefficients of homogeneous polynomials* $Y_i^{(p+1)}$ $(i=1,2)$ *satisfy 3 resonance equations (56).*

Corollary 7 *Any resonance set contains 3 coefficients:*
1) the coefficient $Y_2^{(0,p+1)}$, *2) any coefficient of* $Y_2^{(s,p+1-s)}$ $(1\le s\le p+1)$,
3) any coefficient of $Y_1^{(s,p+1-s)}$ $(3\le s\le p+1)$ *or any coefficient of* $Y_2^{(s,p+1-s)}$ $(1\le s\le p+1)$ *which is differ from the coefficient selected in 2), except* $Y_2^{(1,p)}$ *and* $Y_2^{(2,p-1)}$ *which can not be in a resonance set simultaneously , because for them* $\det\Upsilon^p = 0$.

Theorem 13 *A system (52) by a formal change (9) may be transformed in GNF (4), where for any* $p\ge 2$ *all the coefficients* $Y_i^{(p+1)}$ $(i=1,2)$ *are equal to zero, except possibly 3 coefficients from a resonance set.*

Example 1 A system (52) by a formal change (9) may be reduced to GNF (4), where in the first equation there is no perturbation term:

$$\dot{y}_1 = y_1^2 + y_1y_2,\qquad \dot{y}_2 = \sum_{p=2}^{\infty}(Y_2^{(0,p+1)}y_2^{p+1} + Y_2^{(1,p)}y_1y_2^p + Y_2^{(3,p-2)}y_1^3y_2^{p-2}).$$

It should be noted that there is no GNF in which we could annul y_2^s $(s\le 3)$, because any resonant set contains either $Y_1^{(s,p+1-s)}$ or $Y_2^{(s,p+1-s)}$ with $s\ge 3$.

11 Normalization of systems with $\mathbf{DCF}_5^2$

Let in a system (1) the unperturbed part $P(x)$ by a linear invertible change be reduced to the $\mathrm{DCF}_5^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We assume at once that the system (1) has the form

$$\dot{x}_1 = x_1^2 + x_2^2 + X_1(x_1, x_2), \quad \dot{x}_2 = X_2(x_1, x_2). \tag{57}$$

Then system (6) is written in the form

$$\begin{array}{l} (s-3)h_1^{(s-1,p+1-s)} + (s+1)h_1^{(s+1,p-1-s)} - 2h_2^{(s,p-s)} = \widehat{Y}_1^{(s,p+1-s)}, \\ (s-1)h_2^{(s-1,p+1-s)} + (s+1)h_2^{(s+1,p-1-s)} = \widehat{Y}_2^{(s,p+1-s)} \qquad (s = \overline{0, p+1};\ p \geq 2). \end{array} \tag{58}$$

Depending on the parity of the index s system (58) decomposes in two independent subsystems. Hence it is conveniently to consider following expansions:

$$p = 2r + \mu \quad (r \geq 1,\ \mu \in \{0,1\}), \quad s = 2\tau + \mu + \nu \quad (-(\nu+\mu)/2 \leq \tau \leq r,\ \nu \in \{0,1\}).$$

As a consequence of it, for any $r \geq 1$, $\mu \in \{0,1\}$ system (58) has the form:

$$\begin{array}{l} (2\tau+\mu+\nu-3)h_1^{(2\tau+\mu+\nu-1,2(r-\tau)+1-\nu)} + (2\tau+\mu+\nu+1)h_1^{(2\tau+\mu+\nu+1,2(r-\tau)-1-\nu)} - \\ \qquad -2h_2^{(2\tau+\mu+\nu,2(r-\tau)-\nu)} = \widehat{Y}_1^{(2\tau+\mu+\nu,2(r-\tau)+1-\nu)}, \\ (2\tau+\mu+\nu-1)h_2^{(2\tau+\mu+\nu-1,2(r-\tau)+1-\nu)} + (2\tau+\mu+\nu+1)h_2^{(2\tau+\mu+\nu+1,2(r-\tau)-1-\nu)} = \\ \qquad = \widehat{Y}_2^{(2\tau+\mu+\nu,2(r-\tau)+1-\nu)}. \end{array} \tag{58^μ}$$

Taking in (58_1^μ) $\nu = 1$ and in (58_2^μ) $\nu = 0$, and then vice versa, for any $p = 2r+\mu$ from system (58^μ) we obtain two independent systems

$$\begin{array}{l} (2\tau+\mu-2)h_1^{(2\tau+\mu,2(r-\tau))} + (2\tau+\mu+2)h_1^{(2\tau+\mu+2,2(r-\tau)-2)} - 2h_2^{(2\tau+\mu+1,2(r-\tau)-1)} = \\ \qquad = \widehat{Y}_1^{(2\tau+\mu+1,2(r-\tau))} \qquad (-(1+\mu)/2 \leq \tau \leq r), \\ (2\tau+\mu-1)h_2^{(2\tau+\mu-1,2(r-\tau)+1)} + (2\tau+\mu+1)h_2^{(2\tau+\mu+1,2(r-\tau)-1)} = \widehat{Y}_2^{(2\tau+\mu,2(r-\tau)+1)} \\ \qquad (0 \leq \tau \leq r); \end{array} \tag{59}$$

$$\begin{array}{l} (2\tau+\mu-3)h_1^{(2\tau+\mu-1,2(r-\tau)+1)} + (2\tau+\mu+1)h_1^{(2\tau+\mu+1,2(r-\tau)-1)} - 2h_2^{(2\tau+\mu,2(r-\tau))} = \\ \qquad = \widehat{Y}_1^{(2\tau+\mu,2(r-\tau)+1)} \qquad (0 \leq \tau \leq r), \\ (2\tau+\mu)h_2^{(2\tau+\mu,2(r-\tau))} + (2\tau+\mu+2)h_2^{(2\tau+\mu+2,2(r-\tau)-2)} = \widehat{Y}_2^{(2\tau+\mu+1,2(r-\tau))} \\ \qquad (-(1+\mu)/2 \leq \tau \leq r). \end{array} \tag{60}$$

1) Investigation of system (59) for $\mu = 0$ and system (60) for $\mu = 1$.

A system (59) for $\mu = 0$ and a system (60) for $\mu = 1$ have the form

$$\begin{array}{l} (2\tau-2)h_1^{(2\tau,2(r-\tau)+\mu)} + (2\tau+2)h_1^{(2\tau+2,2(r-\tau)-2+\mu)} - 2h_2^{(2\tau+1,2(r-\tau)-1+\mu)} = \widehat{Y}_1^{(2\tau+1,2(r-\tau)+\mu)} \\ \qquad (0 \leq \tau \leq r), \\ (2(\tau+\mu)-1)h_2^{(2(\tau+\mu)-1,2(r-\tau)+1-\mu)} + (2(\tau+\mu)+1)h_2^{(2(\tau+\mu)+1,2(r-\tau)-1-\mu)} = \\ \qquad = \widehat{Y}_2^{(2(\tau+\mu),2(r-\tau)+1-\mu)} \qquad (-\mu \leq \tau \leq r). \end{array} \tag{61}$$

For $\tau = \overline{-\mu, r-1}$ coefficients

$$h_2^{(2(\tau+\mu)+1,2(r-\tau)-1-\mu)} = \frac{1}{2(\tau+\mu)+1}\sum_{j=0}^{\tau+\mu}(-1)^{\tau-j+\mu}\widehat{Y}_2^{(2j,2(r-j)+1+\mu)}$$

are uniquely determined from (61_2).

The last equation in (61_2) has the form: $(2(r+\mu)-1)h_2^{(2(r+\mu)-1,1-\mu)} = \widehat{Y}_2^{(2(r+\mu),1-\mu)}$.

By substituting $h_2^{(2(r+\mu)-1,1-\mu)}$ into this equation we obtain the resonance relation

$$\sum_{j=0}^{r+\mu}(-1)^j\widehat{Y}_2^{(2j,2(r-j)+1+\mu)} = 0. \tag{62}$$

By substituting $h_2^{(2\tau+1,2(r-\tau)-1+\mu)}$ from (61_2) into (61_1) we have the system

$$a_\tau h_1^{(2\tau,2(r-\tau)+\mu)} + b_\tau h_1^{(2\tau+2,2(r-\tau)-2+\mu)} = \breve{Y}_1^{(2\tau+1,2(r-\tau)+\mu)} \qquad (\tau = \overline{0,r}), \tag{63}$$

where $a_\tau = 2\tau-2$, $b_\tau = 2\tau+2$, $\breve{Y}_1^{(2\tau+1,2(r-\tau)+\mu)} = \widehat{Y}_1^{(2\tau+1,2(r-\tau)+\mu)} + 2h_2^{(2\tau+1,2(r-\tau)-1+\mu)}$.

Select last r equations in system (63)and form a subsystem

$$\Theta h_1 = \breve{Y},$$

where $\Theta = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{r-1} & b_{r-1} \\ 0 & 0 & 0 & \dots & 0 & a_r \end{pmatrix}$ – bidiagonal $(r \times r)$ - matrix, vectors $h_1 = (h_1^{(2,2r-2+\mu)}, \dots, h_1^{(2r,\mu)})$, $\breve{Y} = (\breve{Y}_1^{(3,2r-2+\mu)}, \dots, \breve{Y}_1^{(2r+1,\mu)})$.

By the Gauss method this system may be transformed in the system

$$\Theta_g h_1 = Y_g, \tag{64}$$

where $\Theta_g = \operatorname{diag}\{0, a_2, \dots, a_r\}$, vector $Y_g = (Y_g^{(3,2r-2+\mu)}, \dots, Y_g^{(2r+1,\mu)})$ has components $Y_g^{(2r+1,\mu)} = \breve{Y}_1^{(2r+1,\mu)}$, $Y_g^{(2\tau+1,2(r-\tau)+\mu)} = \breve{Y}_1^{(2\tau+1,2(r-\tau)+\mu)} - a_{\tau+1}^{-1}b_\tau Y_g^{(2\tau+3,2(r-\tau)-2+\mu)}$ $(\tau = \overline{r-1,1})$.

It is clear that $Y_g^{(2\tau+1,2(r-\tau)+\mu)} = \tau^{-1}\sum_{j=\tau}^{r}(-1)^{j-\tau}j\breve{Y}_1^{(2j+1,2(r-j)+\mu)}$ $(\tau = \overline{r,1})$.

For $\tau = \overline{r,2}$ system (64) is uniquely decidable with respect to coefficients $h_1^{(4,2r-4+\mu)}, \dots, h_1^{(2r,\mu)}$, and its first equation has the form $0 \cdot h_1^{(2,2r-2+\mu)} = Y_g^{(3,2r-2+\mu)}$, being $h_1^{(2,2r-2+\mu)}$ is free. Substituting in this equation formulas for $Y_g^{(3,p-2+\mu)}$, $\breve{Y}_1^{(2j+1,2(r-j)+\mu)}$, $h_2^{(2j+1,2(r-j)-1+\mu)}$, we obtain:

$$\sum_{j=1}^{r}(-1)^{j-1}j\widehat{Y}_1^{(2j+1,2(r-j)+\mu)} + 2\sum_{j=1}^{r-1+\mu}\frac{j}{2j+1}\sum_{k=0}^{j}(-1)^{k+1}\widehat{Y}_2^{(2k,2(r-k)+1+\mu)} = 0$$

or the resonance relation

$$\sum_{j=1}^{r}(-1)^{j-1}j\widehat{Y}_1^{(2j+1,2(r-j)+\mu)}+2\sum_{j=0}^{r-1+\mu}(-1)^{j+1}\widehat{Y}_2^{(2j,2(r-j)+1+\mu)}\sum_{k=j}^{r-1}\frac{k}{2k+1}=0. \tag{65}$$

The first equation in (63) has the form:

$$-2h_1^{(0,2r+\mu)}+2h_1^{(2,2r-2+\mu)}=\check{Y}_1^{(1,2r+\mu)}.$$

It is evidently decidable and the coefficient $h_1^{(0,2r+\mu)}$ (or $h_1^{(2,2r-2+\mu)}$) is free.

2) Investigation of system (59) for $\mu=1$ and system (60) for $\mu=0$.

The system (60) for $\mu=0$ and the system (59) for $\mu=1$ have the form

$$\begin{aligned}&(2(\tau+\mu)-3)h_1^{(2(\tau+\mu)-1,2(r-\tau)+1-\mu)}+(2(\tau+\mu)+1)h_1^{(2(\tau+\mu)+1,2(r-\tau)-1-\mu)}-\\ &\qquad -2h_2^{(2(\tau+\mu),2(r-\tau)-\mu)}=\widehat{Y}_1^{(2(\tau+\mu),2(r-\tau)+1-\mu)} \qquad (-\mu\le\tau\le r),\\ &2\tau h_2^{(2\tau,2(r-\tau)+\mu)}+(2\tau+2)h_2^{(2\tau+2,2(r-\tau)-2+\mu)}=\widehat{Y}_2^{(2\tau+1,2(r-\tau)+\mu)} \qquad (0\le\tau\le r).\end{aligned} \tag{66}$$

For $\tau=\overline{0,r-1}$ from the subsystem (66_2) coefficients $h_2^{(2\tau+2,2(r-\tau)-2+\mu)}=(2\tau+2)^{-1}\times\sum_{j=0}^{\tau}(-1)^{\tau-j}\widehat{Y}_2^{(2j+1,2(r-j)+\mu)}$ are uniquely found and the coefficient $h_2^{(0,2r+\mu)}$ is free.

The last equation in (66_2) has the form: $2rh_2^{(2r,\mu)}=\widehat{Y}_2^{(2r+1,\mu)}$.

Substituting in it obtained $h_2^{(2r,\mu)}$, we have the resonance relation

$$\sum_{j=0}^{r}(-1)^j\widehat{Y}_2^{(2j+1,2(r-j)+\mu)}=0. \tag{67}$$

Further we substitute $h_2^{(2\tau+2,2(r-\tau)-2+\mu)}$ from (66_2) in (66_1) and obtain the system

$$a_\tau h_1^{(2(\tau+\mu)-1,2(r-\tau)+1-\mu)}+b_\tau h_1^{(2(\tau+\mu)+1,2(r-\tau)-1-\mu)}=\check{Y}_1^{(2(\tau+\mu),2(r-\tau)+1-\mu)} \qquad (\tau=\overline{-\mu,r}), \tag{68}$$

where $a_\tau=2(\tau+\mu)-3$, $b_\tau=2(\tau+\mu)+1$, $\check{Y}_1^{(2(\tau+\mu),2(r-\tau)+1-\mu)}=\widehat{Y}_1^{(2(\tau+\mu),2(r-\tau)+1-\mu)}+2h_2^{(2(\tau+\mu),2(r-\tau)-\mu)}$.

Take last $r+\mu$ equations of system (68)and form the subsystem

$$\Theta h_1=\check{Y},$$

where $\Theta=\begin{pmatrix}a_{1-\mu}&b_{1-\mu}&0&\dots&0&0\\0&a_{2-\mu}&b_{2-\mu}&\dots&0&0\\0&0&a_{3-\mu}&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&a_{r-1}&b_{r-1}\\0&0&0&\dots&0&a_r\end{pmatrix}$ – bidiagonal $(r+\mu)$ - matrix, vectors $h_1=(h_1^{(1,2r-1+\mu)},\dots,h_1^{(2(r+\mu)-1,1-\mu)})$, $\check{Y}=(\check{Y}_1^{(2,2r-1+\mu)},\dots,\check{Y}_1^{(2(r+\mu),1-\mu)})$.

This system is uniquely decidable because the coefficients on principal diagonal satisfy conditions $a_\tau = 2(\tau+\mu) - 3 \neq 0 \quad (\tau = \overline{1-\mu, r})$.

The first equation in (68) $(\tau = -\mu)$ has the form: $h_1^{(1,2r-1+\mu)} = \widehat{Y}_1^{(0,2r+1+\mu)} + 2h_2^{(0,2r+\mu)}$ and is uniquely decidable by using the free coefficient $h_2^{(0,2r+\mu)}$.

By using denotations introduced for equations (5) and (7), we rewrite obtained resonance relations (65), (62), (67) for $\mu = 0$ and $\mu = 1$ via coefficients of system (4):

$$\begin{gathered}\sum_{j=1}^{r}(-1)^{j-1}jY_1^{(2j+1,2(r-j))} + 2\sum_{j=0}^{r-1}(-1)^{j+1}Y_2^{(2j,2(r-j)+1)}\sum_{k=j}^{r-1}\frac{k}{2k+1} = \widetilde{c},\\ \sum_{j=0}^{r}(-1)^{j}Y_2^{(2j,2(r-j)+1)} = \widetilde{c}; \qquad \sum_{j=0}^{r}(-1)^{j}Y_2^{(2j+1,2(r-j))} = \widetilde{c} \qquad (\mu=0);\end{gathered} \tag{69}$$

$$\begin{gathered}\sum_{j=1}^{r}(-1)^{j-1}jY_1^{(2j+1,2(r-j)+1)} + 2\sum_{j=0}^{r}(-1)^{j+1}Y_2^{(2j,2(r-j)+2)}\sum_{k=j}^{r}\frac{k}{2k+1} = \widetilde{c},\\ \sum_{j=0}^{r+1}(-1)^{j}Y_2^{(2j,2(r-j)+2)} = \widetilde{c}; \qquad \sum_{j=0}^{r}(-1)^{j}Y_2^{(2j+1,2(r-j)+1)} = \widetilde{c} \qquad (\mu=1).\end{gathered} \tag{70}$$

Theorem 14 *A system (57) is formally equivalent to a system (4) with the unperturbed part* $P(y_1^2+y_2^2, 0)$ *iff for any* $p = 2r+\mu$ $(r \geq 1,\ \mu \in \{0,1\})$ *coefficients of homogeneous polynomials* $Y_i^{(p+1)}$ $(i=1,2)$ *satisfy 3 resonance relations, namely:*

1) for $p = 2r$ $(r \geq 1, \mu = 0)$ *coefficients* $Y_1^{(2\tau+1,2(r-\tau))}$ $(1 \leq \tau \leq r)$, $Y_2^{(2\tau,2(r-\tau)+1)}$ $(0 \leq \tau \leq r)$ *satisfy equations* (69_1), (69_2), *and coefficients* $Y_2^{(2\tau+1,2(r-\tau))}$ $(0 \leq \tau \leq r)$ *satisfy equation* (69_3);

2) for $p = 2r+1$ $(r \geq 1, \mu = 1)$ *coefficients* $Y_1^{(2\tau+1,2(r-\tau)+1)}$ $(1 \leq \tau \leq r)$, $Y_2^{(2\tau,2(r-\tau)+2)}$ $(0 \leq \tau \leq r+1)$ *satisfy equations* (70_1), (70_2), *coefficients* $Y_2^{(2\tau+1,2(r-\tau)+1)}$ $(0 \leq \tau \leq r)$ *satisfy equation* (70_3).

Corollary 8 *Any resonance set contains 3 coefficients.*

For $p = 2r$ $(r \geq 1, \mu = 0)$ *these coefficients are the following:*
1) any one of $Y_2^{(2\tau,2(r-\tau)+1)}$ $(0 \leq \tau \leq r)$; *2) any one of* $Y_2^{(2\tau+1,2(r-\tau))}$ $(0 \leq \tau \leq r)$; *3) any one of* $Y_1^{(2\tau+1,2(r-\tau))}$ $(1 \leq \tau \leq r)$ *or any one of* $Y_2^{(2\tau,2(r-\tau)+1)}$ $(0 \leq \tau \leq r)$ *which is differ from the coefficient selected in 1), except the pair* $Y_2^{(0,2r+1)}$ *and* $Y_2^{(2,2r-1)}$ *that can not be in the resonance set simultaneously, because for them* $\det \Upsilon^p = 0$.

For $p = 2r+1$ $(r \geq 1, \mu = 1)$ *the resonance set contains the following coefficients:*
1) any one of $Y_2^{(2\tau,2(r-\tau)+2)}$ $(0 \leq \tau \leq r+1)$; *2) any one of* $Y_2^{(2\tau+1,2(r-\tau)+1)}$ $(0 \leq \tau \leq r)$; *3) any one of* $Y_1^{(2\tau+1,2(r-\tau)+1)}$ $(1 \leq \tau \leq r)$ *or any one of* $Y_2^{(2\tau,2(r-\tau)+2)}$ $(0 \leq \tau \leq r+1)$ *which is differ from the coefficient selected in 1), except the pair* $Y_2^{(0,2r+2)}$ *and* $Y_2^{(2,2r)}$ *that can not be in the resonance set simultaneously, because for them* $\det \Upsilon^p = 0$.

Theorem 15 *An arbitrary system (57) by a formal change (9) may be transformed in a GNF (4), where for any* $p \geq 2$ *all the coefficients* $Y_i^{(p+1)}$ $(i = 1,2)$ *equal zero, except possibly 3 coefficients from a resonance set described in corollary 8.*

Example 2 A system (57) by a formal change (9) may be reduced to a GNF (4) that is linear in y_2 :

$$\dot{y}_1 = y_1^2 + y_1 y_2 + \sum_{r=1}^{\infty} (Y_1^{(2r+1,0)} + Y_1^{(2r+1,1)} y_2) y_1^{2r+1},$$
$$\dot{y}_2 = \sum_{r=1}^{\infty} (Y_2^{(2r+1,0)} y_1 + Y_2^{(2r+2,0)} y_1^2 + Y_2^{(2r,1)} y_2 + Y_2^{(2r+1,1)} y_1 y_2) y_1^{2r}.$$

Part IV

Conclusion

As was mentioned in part I, normalization of a real system (1) $\dot{x}_i = P_i(x) + X_i(x)$, where $P_i = a_i x_1^2 + 2b_i x_1 x_2 + c_i x_2^2$ is the unperturbed part, $X_i = \sum_{p=2}^{\infty} X_i^{(p+1)}(x)$ – the perturbation, $X_i^{(r)}$ is a homogeneous polynomial by the order r $(i = 1, 2)$, is naturally divides into 2 phase.

I) On the first phase by using linear invertible changes (9) we simplify the unperturbed part of system (1), i. e. the vector of homogeneous quadratic polynomial $P = (P_1, P_2)$.

In part II the set of systems (2) is divided into 19 linearly nonequivalent classes. For each class canonical form (see definition 7) is the simplest representative, being the form is an analogue to a Jordan matrix for linear systems.

Depending on the order l of the common factor of polynomials P_1 and P_2 (see definition 5) the set of canonical forms is divided into 3 subsets.

It is turned out that if P_1 and P_2 do not have a common factor $(l = 0)$, (and according to assertion 1 this is equivalent to the fact that corresponding them resultant $R \neq 0$), then system (2) may be brought to one of 10 canonical forms: CF_1^0 – CF_{10}^0. If P_1 and P_2 have a common factor of the first order, then system (2) is reduced to one of 5 canonical forms: CF_1^1 – CF_5^1. At last, if P_1 and P_2 are proportional $(l = 2)$, then system (2) is reduced to one of 4 canonical forms: CF_1^2 – CF_4^2.

We consider two questions in detail.

1) The renumbering (14) brings any CF_i^l (of course, if the form is not invariant with respect to this change) to an additional canonical form $\mathrm{CF}_i^{l\mathrm{r}}$ (see definition 8) for which principle 5 from the definition of CF is not hold.

And for $l = 2$ there exists another variant of additional CF: ACF_2^2 in accordance with remark 19 is obtained from a CF_2^2 with $\sigma = -1$ by a linear change differing from the renumbering and has its own $\mathrm{ACF}_2^{2\mathrm{r}}$.

2) In case of $l = 2$ if we divide CF_2^2 into two forms depending on the sign of σ, then every of 5 CF^2 is linearly equivalent to its degenerate CF (see definition 9). The advantage of DCF^2 is that it has the index lesser than corresponding CF^2. That easily allows us to investigate GNF of systems (1) with DCF^2 in unperturbed part. But the absence of P_2 does not allow us to annul some summands in the perturbation, which is possible for nondegenerate CF if there are technical tools.

II) The second phase implies that for a system (1) with the unperturbed part CF^l or DCF^2 to obtain all the generalized normal forms (see definition 4) in explicit form by almost identical change of variables (3).

In part III this problem has been solved for systems with $\mathrm{DCF}_1^2, \ldots, \mathrm{DCF}_5^2$ in unperturbed part.

Since before this article the classification of canonical forms introduced in [2, § 2], where principles 5 and 6 of the definition of canonical forms were not clearly formulated, we consider results for other CF_i^l or their close analogs.

In case of $l = 0$ systems (1) with the following 4 forms: with $\mathrm{CF}_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ – in [2, § 6], with $\mathrm{CF}_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ – in [4, § 11], with $\mathrm{CF}_4^{0*} = \begin{pmatrix} 1 & 0 & u_* \\ 0 & 0 & 1 \end{pmatrix}$ differing from $\mathrm{CF}_4^0 = \begin{pmatrix} u & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ by a normalization – in [2, § 6], with $\mathrm{CF}_5^0 = \begin{pmatrix} u & 0 & \sigma \\ 0 & 1 & 0 \end{pmatrix}$, where $\sigma = -\operatorname{sign} u$ $(R < 0)$, – in [5, § 12] (unfortunately, CF_5^0 with $R > 0$ have not been revealed yet) were investigated previously.

In case of $l = 1$ systems (1) with the following 4 forms: with $\mathrm{CF}_1^1 = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ – in [2, § 5], with $\mathrm{CF}_2^1 = \begin{pmatrix} 0 & \sigma & 0 \\ 1 & 0 & 0 \end{pmatrix}$ – in [3, § 8], with $\mathrm{CF}_3^1 = \begin{pmatrix} u & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, including $u = -1$, – in [3, § 7], with $\mathrm{CF}_4^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ – in [4, § 9] were investigated previously.

Thus, on one of cases two sets of formally equivalent GNF have been obtained. In the first set the unperturbed part is presented by a CF_2^1 with $\sigma = 1$, and in the second one – by a form CF_3^1, but with $u = -1$, and in accordance with remark 7 these CF are linearly equivalent.

In case of $l = 2$ systems (1) with the following 3 forms: with $\mathrm{CF}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ – in [2, § 4], with $\mathrm{ACF}_2^{2*} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ differing from $\mathrm{ACF}_2^2 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ by a normalization – in [4, § 10], with $\mathrm{CF}_4^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ – in [2, § 3].

In this situation in 3 cases two sets of formally equivalent GNF were obtained. By theorem 5 DCF_i^2 is linear equivalent to a CF_i^2 $(i = 1, 3, 4)$, DCF_2^2 is equivalent to a CF_2^2 with $\sigma = -1$ (and ACF_2^2), and DCF_5^2 is equivalent CF_2^2 with $\sigma = 1$.

We also note that system (1) investigated in [4, § 10] is up to the present the unique system in which the unperturbed part has 4 nonzero summands (ACF_2^{2*}).

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