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A Sturm-Liouville shaped characteristic differential equation as a guide to establish a quasi-polynomial expression to the Boubaker polynomials

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Abstract

The Boubaker polynomials were established and discussed in several previous studies, as a polynomial sequence related to a solution to the heat transfer equation in a special model of a vertical spray device. In the last years, many attempts to find a characteristic differential equation or a quasi-polynomial expression to these polynomials were yielded.

In this paper, we propose a Sturm-Liouville shaped characteristic differential equation to the Boubaker Polynomials as a supply to further efforts for proposing different analytic expression. The actual attempt is compared to the lastly proposed incomplete differential equation of the m-Boubaker polynomials.

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1 Introduction

The Boubaker polynomials were established in a physics study that yielded a thermal model of the pyrolysis spray (fig. 1.a and 1.b) disposal[1]. It was stated in this study that an attempt to solve the heat transfer equation between the bulk and the substrate, led to the definition of recursive formula. This formula itself led to a serial of polynomial functions[1] that had particular proprieties . These functions, that allowed the proposition of a concrete solution to the heat equation, were presented to the local mathematics community as a new polynomial class[2] . An original study about them has been accepted for publication in an applied mathematics journal[3]. Recently, the m-Boubaker polynomials (modified Boubaker polynomials) have been defined as an ameliorated form that was more appropriated for defining a characteristic differential equation[4] .The Boubaker polynomials are already registered under their final form[5].

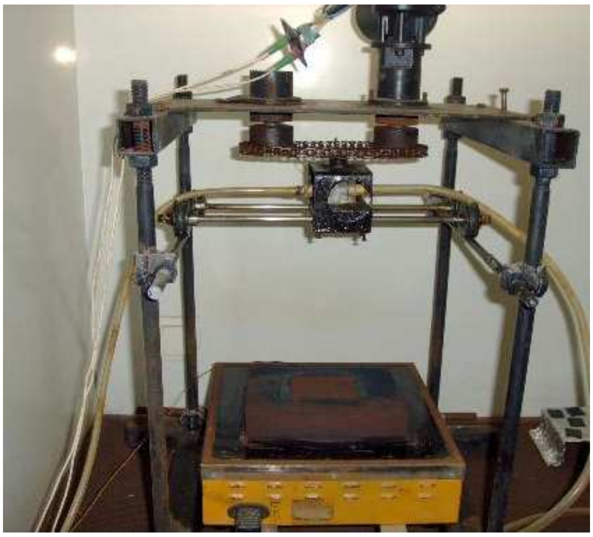

Figure 1.a. : Pyrolysis spray setup

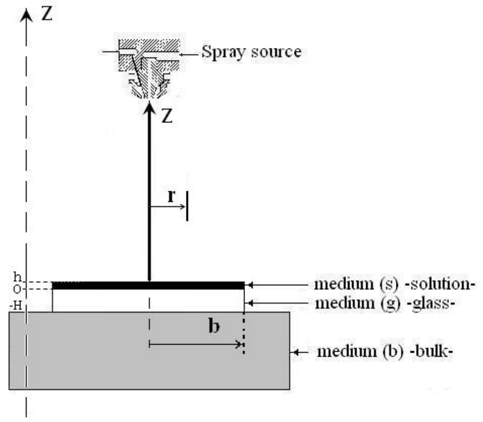

Figure 1.b. : Schematic diagram of pyrolysis spray setup

In the ordinary spray deposition process at ambient temperature, the precursor solution arrives at the substrate in the form of very small droplets. Droplet constituents react and form a chemical compound onto the substrate. The chemical reactants are selected such that substances other than desired compound are volatile at the temperature of deposition and before final solidification.

Uniform deposition is better ensured by moving adequately nozzle or substrate support (fig. 1.b) . In the modeled disposal, substrate support is fixed due to heating source size. Nozzle relative position versus targeted sample is constant in actual studied models. Solution anion to cation ratio, spray rate, substrate temperature, ambient atmosphere and carrier gas properties are invariable and thus considered first as exogenous parameters.

In the last decades, several models were proposed for the pyrolysis spray setup. Chaouachi et al. [1] proposed a thermal transfer model based on resolving the heat transfer equation, that takes into account the boundary conditions, especially at the level of the substrate-precursor interface.

2 The Boubaker polynomials

The Boubaker polynomials merged from an attempt to yield a solution to heat equation. In fact, in a calculation step during resolution process[1], an intermediate calculus sequence raised an interesting recursive formula leading to a class of polynomial functions that performs difference with common classes.

The heat equation[1] inside glass layer medium (*g*) and deposited layer (*s*) was expressed by (1.a):

$$\begin{cases} \dfrac{\partial^2 T_g(z,t)}{\partial z^2} = \dfrac{1}{D_g}\dfrac{\partial T_g(z,t)}{\partial t} - \dfrac{1}{k_g}.(P_b - P_s) \\ \dfrac{\partial^2 T_s(z,t)}{\partial z^2} = \dfrac{1}{D_s}\dfrac{\partial T_s(z,t)}{\partial t} - \dfrac{1}{k_s}.P_s \end{cases} \qquad (1.a)$$

where:

T_g : absolute temperature inside glass medium (in K)
T_s : absolute temperature inside deposited layer (in K)
D_g : glass medium thermal diffusivity (in $m^2.s^{-1}$)
D_s : deposited layer thermal diffusivity (in $m^2.s^{-1}$)
P_b : power transmitted from bulk to glass (in Wm^{-3})
P_s : power transmitted from glass to layer (in Wm^{-3})
k_g : glass medium thermal conductivity (in $W.m^{-1}.K^{-1}$)
k_s : deposited layer thermal conductivity (in $W.m^{-1}.K^{-1}$)

According to bulk size and thermal supply, lower heat conduction toward glass layer (z=-H) could be considered as issued from an infinite source under constant temperature T_b .Boundary conditions concerned mainly temperature distribution continuity at median plane (z=-H) and glass-layer contact plane (z=0).

After proposing a general expression (1.b) for temperature distribution[1] inside glass sample:

$$T_n(z,t) = \frac{1}{N} e^{-\frac{A}{\frac{H}{z}+1}} \sum_{m=0}^{\infty} \xi_m . J_m(t) \quad \text{for}: -H < z < 0 \tag{1.b}$$

where J_m is m-order first kind Bessel function, N is prefixed integer parameter, A and ξ_m are constants to be found; the application of Boundary conditions, and truncation of the infinite sum down to the integer order N lead to the system (1.c) :

$$\begin{cases} Q_1(z)\xi_0 = \xi_1 \\ Q_1(z)\xi_1 = -2\xi_0 + \xi_2 \\ Q_1(z)\xi_m = \xi_{m-1} + \xi_{m+1} \quad \text{for}: 1 < m < N \\ \ldots \\ Q_1(z)\xi_{N-1} = \xi_{N-2} + \xi_N \\ \xi_{N+1}(z) = 0 \end{cases} \tag{1.c}$$

For N=5 for example, the solution (eq. 1.d) is yielded , with ξ_0=1:

$$\vec{\xi} = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} = \begin{pmatrix} 1 \\ Q_1(z) \\ Q_1^2(z) + 2 \\ Q_1^3(z) + Q_1(z) \\ Q_1^4(z) + 2 \\ Q_1^5(z) - Q_1^3(z) + Q_1(z) \end{pmatrix} \tag{1.d}$$

Finally, coefficients ξ_m are calculated for z=0, and for the given parameters values.
For superior values of N, and when z=0; a serial of polynomial function B_m(X) was defined out from the relations (1.c) and (1.d) and resumed in the relation (eq. 1.e):

$$\begin{cases} B_0(X) = 1 \\ B_1(X) = X \\ B_2(X) = X^2 + 2 \\ B_m(X) = X.B_{m-1}(X) - B_{m-2}(X) \quad \text{for} : \mathrm{m} > 2 \end{cases} \tag{1.e}$$

Using the recursive relations (eq. 1.e) and to the study results[1], the authors established, between 2003 and 2007, an explicit monomial form (eq. 2 and 3) and a recursive definition of the Boubaker polynomials coefficients (eq. 4). Demonstrations of equations (3-4) are available in appendixes of previous studies[2-4].

$$B_n(X) = 1.X^n - (n-4).X^{n-2} + \frac{(n-8)(n-3)}{2}.X^{n-4} - \frac{(n-12)(n-5)(n-4)}{3\times 2}.X^{n-6} + \ldots \tag{2}$$

$$B_n(X) = 1.X^n - (n-4).X^{n-2} + \sum_{p=2}^{\xi(n)} \left[\frac{(n-4p)}{p!} \prod_{j=p+1}^{2p-1} (n-j) \right].(-1)^p.X^{n-2p} \; ; \quad \text{with: } \xi(n) = \frac{2n + ((-1)^n - 1)}{4} \tag{3}$$

$$\begin{cases} B_n(X) = \sum_{j=0}^{\xi(n)} \left[b_{n,j} X^{n-2j} \right]; \xi(n) = \frac{2n + ((-1)^n - 1)}{4} \\ b_{n,0} = 1; \qquad b_{n,1} = -(n-4); \\ b_{n,j+1} = \frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \frac{(n-4j-4)}{(n-4j)} \times b_{n,j} \\ b_{n,\frac{2n+((-1)^n-1)}{4}} = \begin{cases} (-1)^{\frac{n}{2}} \times 2 & \text{if n even} \\ (-1)^{\frac{n+1}{2}} (n-2) & \text{if n odd} \end{cases} \end{cases} \tag{4}$$

According to the relations (1-4), the set of the first Boubaker polynomials was yielded (5) :

$$\begin{aligned}
&B_0(\mathrm{X}) = 1; && B_1(\mathrm{X}) = X; \quad B_2(\mathrm{X}) = X^2 + 2;\\
&B_3(\mathrm{X}) = X^3 + X; && B_4(\mathrm{X}) = X^4 - 2;\\
&B_5(\mathrm{X}) = X^5 - X^3 - 3X; && B_6(\mathrm{X}) = X^6 - 2X^4 - 3X^2 + 2;\\
&B_7(\mathrm{X}) = X^7 - 3X^5 - 2X^3 + 5X; && B_8(\mathrm{X}) = X^8 - 4X^6 + 8X^2 - 2;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{5}$$

The graphics of first Boubaker polynomials are presented in figures 2.a and 2.b :

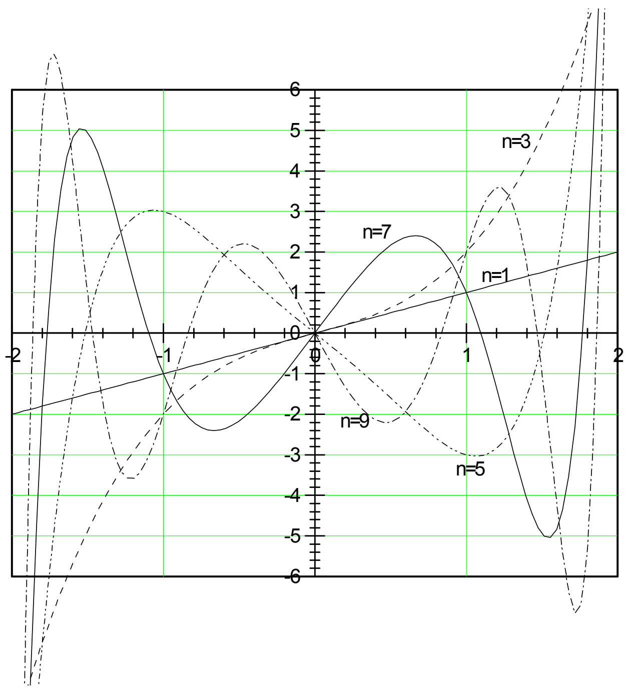

Figure 2.a: The Boubaker polynomials (odd orders)

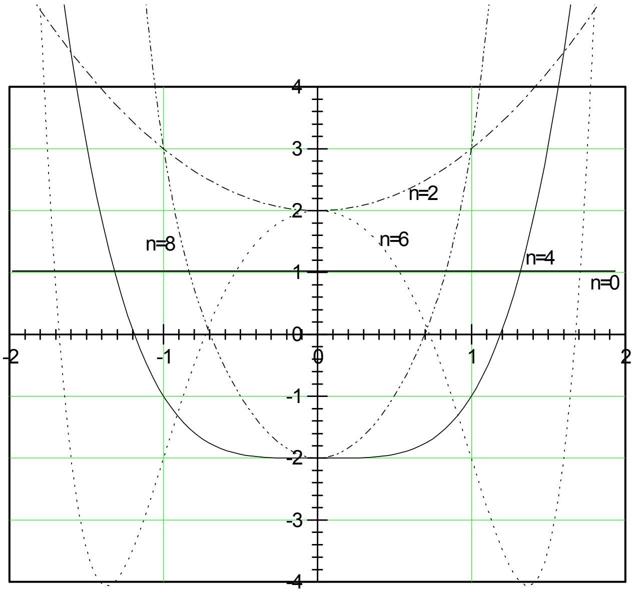

Figure 2.b: The Boubaker polynomials (even orders)

3 The Modified Boubaker polynomials

In 2007, the authors proposed, through a specialized study[5], a new version of these polynomials. Oppositely to the early defined polynomials, the modified Boubaker polynomials, defined by (eq. 6):

$$\tilde{B}_n(X)=2^n.X^n-2^{n-2}(n-4).X^{n-2}+\sum_{p=2}^{\xi(n)}\left[\frac{(n-4p)}{p!}\prod_{j=p+1}^{2p-1}(n-j)\right].2^{n-2p}(-1)^p.X^{n-2p}\;;\xi(n)=\frac{2n+((-1)^n-1)}{4} \tag{6}$$

are solutions to a second order characteristic, but non proper equation (7):

$$16.(1-X^2)\tilde{B}_n''(X)-4.X\tilde{B}_n'(X)+n^2\tilde{B}_n(X)=32.(n-1)T_{n-2}(X);\quad for\ \mathrm{n}>2 \tag{7}$$

where $T_n(X)$, for n>2, are the Chebyshev [6,7] first order polynomials.

This definition allowed an establishment of a quasi-polynomial expression (eq. 8) of the m-Boubaker polynomials[5]:

$$\tilde{B}_n(X)=\left\langle X+\sqrt{X^2-1}\right\rangle^n\left[8X^2-3-8X\sqrt{X^2-1}\right]+\left\langle X-\sqrt{X^2-1}\right\rangle^n\left[8X^2-3+8X\sqrt{X^2-1}\right] \tag{8}$$

or by setting the simplified forms (eq. 9) :

$$\xi=\left\langle X+\sqrt{X^2-1}\right\rangle \quad \text{and} \quad \xi'=\xi^{-1}=\left\langle X-\sqrt{X^2-1}\right\rangle \tag{9}$$

with the properties expressed by (eq.10.a):

$$\xi+\xi'=2X \quad \text{and} \quad \xi.\xi'=1 \tag{10.a}$$

we obtain the simplified analytic relation, (eq. 10.b) :

$$\tilde{B}_n(X)=(\xi)^n[8X\xi'-3]+(\xi')^n[8X\xi-3] \tag{10.b}$$

The graphics of first m-Boubaker polynomials are presented in figures 3.a and 3.b:

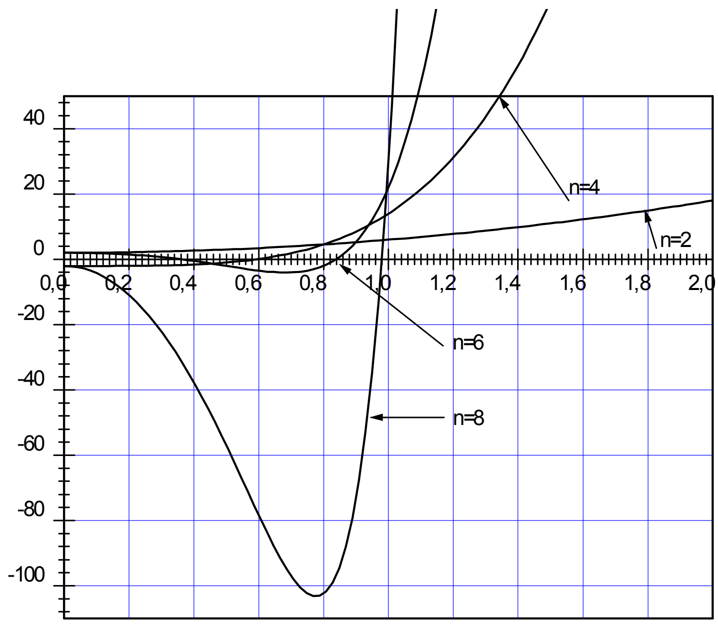

Figure 3.a: The m-Boubaker polynomials (even orders)

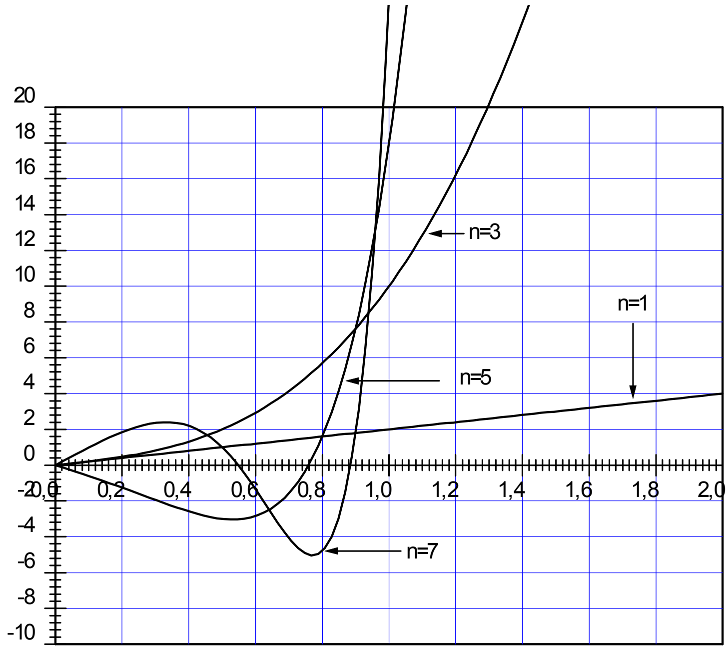

Figure 3.b: The m-Boubaker polynomials (odd orders)

4 The Sturm-Liouville differential equation of the Boubaker polynomials

4.1 The Sturm-Liouville theory

The Sturm-Liouville equation is named after Jacques Charles François Sturm[8], and Joseph Liouville[9], it is a real second-order linear differential equation of the form (eq. 11.a) :

$$-\frac{d}{dx}\left\langle p(x)\frac{dy}{dx}\right\rangle + q(x).y = \lambda\omega(x).y \tag{11.a}$$

where p(x), q(x), and ω(x) are specified at the outset, and in the simplest of cases are continuous on the finite closed interval [a,b]. The function ω (x) is called the "weight" or "density" functions. The value of λ is not specified in the equation; finding the values of λ for which there exist a non-trivial solution of (eq.11.a) satisfying the boundary conditions is part of the problem called the Sturm-Liouville problem .

Such values of λ when they exist are called the eigenvalues of the boundary value problem defined by (eq.11.a) and the prescribed set of boundary conditions. The corresponding solutions are the eigenfunctions of this problem. The resulting theory of the existence and asymptotic behavior of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space is known as Sturm-Liouville theory. This theory is important in applied mathematics, where Sturm-Liouville problems occur very commonly, particularly when dealing with linear partial differential equations which are separable.
Many well-known equations are Sturm-Liouville shaped or can be transformed to be so. We can mention Bessel's equation (eq. 11.b)

$$x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0 \tag{11.b}$$

thet can be transformed to expression (eq. 11.c)

$$(xy')+(\lambda^2x^2-\frac{\nu^2}{x})y=0 \tag{11.c}$$

The Legendre equation (eq. 11.d)

$$(1-x^2)y''-2xy'+\nu(\nu+1)y=0 \tag{11.d}$$

can also be transformed to a Sturm-Liouville shaped equation (eq. 11.e)

$$\left[(1-x^2)y'\right]'+\nu(\nu+1)y=0 \tag{11.e}$$

4.2 The case of Boubaker polynomials

Since their establishment, the Boubaker polynomials were deeply studied and discussed. For a long time, the authors attempted to express these polynomials as solutions to a second order Sturm-Liouville shaped characteristic differential equation. For this purpose, they had to modify the earlier defined polynomials [5], just for giving a Chebyshev shaped expression. While working on the eventual formulations of the Boubaker polynomials, we were led to an expression based on complex functions analysis and formulated the following way:

Let's define a complex function that verifies the condition expressed by (eq. 12):

$$f^*(X,t) = \sum_{m=0}^{+\infty} B_m^*(X).J_m(t),$$
$$\text{with} \quad B_m(X) = \mathrm{Re}\left\langle B_m^*(X)\right\rangle \quad \text{and} \quad J_m : \text{Bessel first kind function} \tag{12}$$

under the constraints resumed in the system (eq.13):

$$\begin{cases} \dfrac{\partial f^*(X,t)}{\partial t} = \displaystyle\sum_{m=0}^{+\infty} B_m^*(X).\frac{dJ_m(t)}{dt} = -j\frac{q(X)}{2} f^*(X,t) = -j\frac{X}{2} f^*(X,t) \\ X\displaystyle\sum_{m=1}^{+\infty} B_m^*(X).j^{-m}.J_m(t) = \sum_{m=1}^{+\infty} B_m^*(X)\left[j^{-m+1}J_{m-1}(t) + j^{-m-1}J_{m+1}(t)\right] \end{cases} \tag{13}$$

Integration of (eq.12) under the conditions expressed by (eq. 13) gives a general solution (eq.14):

$$f^*(X,t) = \Phi^*(X).e^{\left(\frac{X}{2}t\right)} = \sum_{m=0}^{+\infty} B_m^*(X).j^{-m}J_m(t),$$
$$\text{with} \quad \Phi^*(X) \quad \text{a complex } X-\text{dependent function} \tag{14}$$

Then by setting :

$$I_m^*(t)=\frac{I_m(t)}{j^m} \quad \text{and} \quad K_m^*(X)=\frac{B_m^*(X)}{j^m\Phi^*(X)} \tag{15}$$

and as the Bessel functions are orthogonal, we obtain the relation (eq.16):

$$e^{-j\left(\frac{X}{2}t\right)}=\sum_{m=0}^{+\infty}K_m^*(X)I_m^*(t) \quad \text{with} \quad \frac{K_m^*(X)}{2\text{m}}=j^m\int_0^{+\infty}e^{-j\left(\frac{X}{2}t\right)}t^{-1}\,I_m^*(t)dt \tag{16}$$

Using Bessel equation, Fourier inverse transform of (eq. 16) and the successive derivates of K^* and J_m we obtain simultaneously the equations (17.a, 17.b, 17.c and 17.d) :

$$\begin{cases} -\dfrac{m^2}{t}I_m^*(t)=-m^2\displaystyle\int_0^{+\infty}\frac{K_m^*(X)}{2m}e^{-j\left(\frac{X}{2}t\right)}\frac{dX}{4\pi} \\ \dfrac{d}{dX}\left(\dfrac{K_m^*(X)}{2m}\right)=-\dfrac{1}{2}\displaystyle\int_0^{+\infty}e^{-j\left(\frac{X}{2}t\right)}.I_m^*(t)dt \end{cases} \tag{17.a.}$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{I_m^*(t)}{2m}\right)=-\dfrac{1}{2m}\displaystyle\int_0^{+\infty}\frac{d\left(K_m^*(X)\right)}{dX}e^{j\left(\frac{X}{2}t\right)}.\frac{dX}{4\pi} \\ \dfrac{1}{2m}\dfrac{d^2\left(K_m^*(X)\right)}{dX^2}=-\dfrac{1}{4}\displaystyle\int_0^{+\infty}e^{-j\left(\frac{X}{2}t\right)}.t.I_m^*(t)dt \end{cases} \tag{17.b.}$$

$$\begin{cases} t.I_m^*(t)=-\dfrac{4}{2m}\displaystyle\int_0^{+\infty}\frac{d^2\left(K_m^*(X)\right)}{dX^2}e^{j\left(\frac{X}{2}t\right)}.\frac{dX}{4\pi} \\ \dfrac{d^2\left(I_m^*(t)\right)}{dt^2}=-\displaystyle\int_0^{+\infty}\frac{d\left(K_m^*(X)\right)}{dX}\frac{jX^2}{4m}e^{j\left(\frac{X}{2}t\right)}.\frac{dX}{4\pi} \\ \dfrac{d\left(K_m^*(X)\right)}{dX}\dfrac{jX^2}{4m}=-\displaystyle\int_0^{+\infty}e^{-j\left(\frac{X}{2}t\right)}.\frac{d^2\left(I_m^*(t)\right)}{dt^2}dt \end{cases} \tag{17.c.}$$

$$\begin{cases} \dfrac{1}{2m}\dfrac{d\left(K_m^*(X).X^2\right)}{dX}=\displaystyle\int_0^{+\infty}e^{-j\left(\frac{X}{2}t\right)}.t.\frac{d^2\left(I_m^*(t)\right)}{dt^2}dt \\ t\dfrac{d^2\left(I_m^*(t)\right)}{dt^2}=-\displaystyle\int_0^{+\infty}\left(2X\frac{1}{2m}\frac{d\left(K_m^*(X)\right)}{dX}+\frac{X^2}{2m}\frac{d^2\left(K_m^*(X)\right)}{dX^2}\right)e^{j\left(\frac{X}{2}t\right)}.\frac{dX}{4\pi} \end{cases} \tag{17.d.}$$

Addition of equations (17.a, 17.b, 17.c and 17.d) gives equation (17.e):

$$2X\frac{1}{2m}\frac{d\left(K_m^*(X)\right)}{dX}+\frac{X^2}{2m}\frac{d^2\left(K_m^*(X)\right)}{dX^2}-\frac{d\left(K_m^*(X)\right)}{dX}.X-4\frac{K_m^*(X)}{2X}-m^2\frac{K_m^*(X)}{2m} \tag{17.e.}$$

Equation (17.e.) can be summed up in the differential equation (eq. 17.f.)

$$(4-X^2)\frac{d^2\left(K_m^*(X)\right)}{dX^2}-X\frac{d\left(K_m^*(X)\right)}{dX}+m^2K_m^*(X)=0 \tag{17.f}$$

This expression (eq.17.f) is the early unfound Sturm-Liouville shaped characteristic differential equation which is proposed in order to investigate the Boubaker polynomials. The previous given ones were either monomial or recursive. It is well known that those forms are not appropriated for setting characteristic quasi-polynomial expressions.

This expression is a valuable guide for further investigations. For example, the calculation of the real roots of the Boubaker polynomials, which is the object of an in-course study, is more suitable with expression (eq. 17.f) than the expression (eq. 1.e.).

Furthermore, in an earlier stage of investigation, the Boubaker polynomials $B_m(X)$ explicit monomial form evoked, while prospected, some singularities for m=4, 8, 12, etc. In fact for the general case: m=4q the 2q rank monomial term is removed from the explicit form so that the whole expression contains only 2q effective terms. Correspondent 4q-order Boubaker polynomials (eq. 18) were easily calculated using both (eq. 17.f) and the well-known Newton binomial development of $(a+b)^n$ term.

$$\begin{aligned}
&B_0(X)=1;\\
&B_4(X)=X^4-2;\\
&B_8(X)=X^8-4X^6+8X^2-2;\\
&B_{12}(X)=X^{12}-8X^{10}+18X^8-35X^4+24X^2-2;\\
&B_{16}(X)=X^{16}-12X^{14}+52X^{12}-88X^{10}+168X^6-168X^4-48X^2-2;\\
&B_{20}(X)=X^{20}-16X^{18}+102X^{16}-320X^{14}+455X^{12}-858X^8+1056X^6-495X^4+80X^2-2;\\
&\ldots\ldots\ldots\ldots
\end{aligned} \tag{18}$$

Thanks to these expressions (eq. 18 and eq. 17.e), it has been conjectured and demonstrated that the number of real positive roots, for high values of q is 2q-1, and are contained exclusively in the domain]0;2[.

The investigation of the real positive roots of these polynomials could also be achieved using dichotomy method (D.M.). It yielded for example the table 1.

N=4q	First positive real root
4	1,1894
8	0,5078
12	0,3114
16	0,2236
20	0,1742
24	0,1428
28	0,1208

Table 1: 4q-order Boubaker polynomials first positive real roots

It has been conjectured, but not demonstrated, that the 4q-order Boubaker polynomials first positive real roots follow a particular sequence .

The localization of some roots could be achieved graphically (figure 4) :

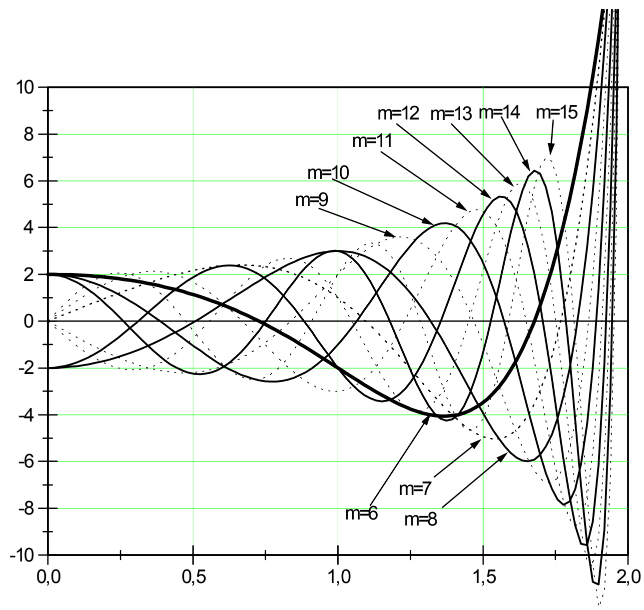

Figure 4: Localization of some Boubaker polynomials positive roots

The graphics of first 4q-order Boubaker polynomials could also be represented (figure 5) :

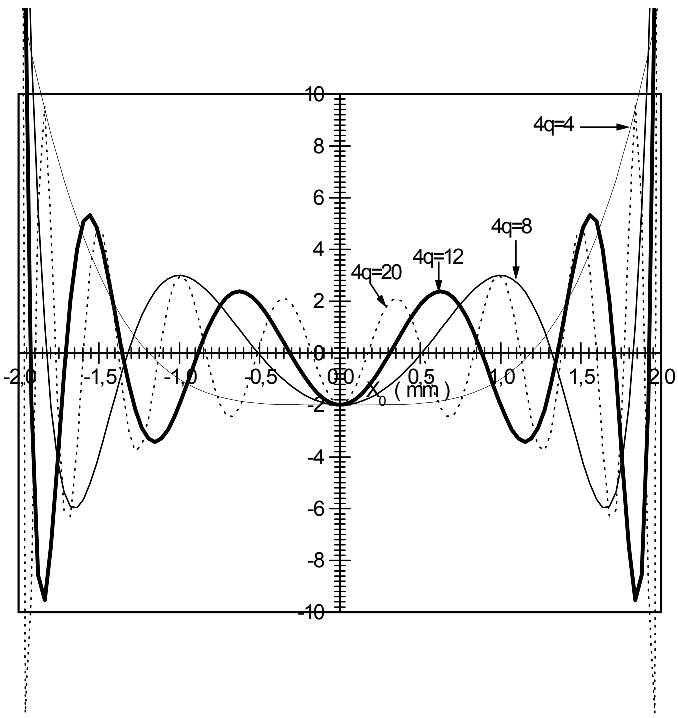

Figure 5: Some 4q-order Boubaker polynomials

5 Perspectives

We proposed a Sturm-Liouville shaped characteristic differential equation to the recently defined Boubaker polynomials[1-5,6-9]. The main advantages of this differential equation polynomials are illustrated by some examples concerning real roots and 4q-order polynomials. We are working now on an eventual quasi-polynomial expression to these polynomials, expression that would be an important support for investigating Boubaker polynomials real positive roots and eventual sub-families particularities.

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