DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 1, 2013
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru

Control problems in nonlinear systems

A separation principle of time-varying nonlinear dynamical systems

H.Damak[1] **I.Ellouze**[2] **M.A.Hammami**[3]

University of Sfax Tunisia

E-mail:[3] **MohamedAli.Hammami@fss.rnu.tn**

Abstract.

This paper deals with the separation principle for a class of nonlinear dynamical systems whose dynamics are in general bounded in time. The resultant observer-based state feedback control guarantee the convergence of solutions toward a small neighborhood of the origin of the state oscillation given that the system is both uniformly controllable and observable. An example in dimentional two is given to illustrate the applicability of our result.

2000 Mathematics Subject Classification: 34H15, 37B25.

Keywords: Nonlinear systems, stabilization, Lyapunov functions.

1 Introduction

The problem of state trajectory control for nonlinear systems by output feedback has received many attention. For systems with non-periodically time-varying parameters, an output feedback control design is proposed in [2] for

linear time-varying systems based on the gradient algorithm. In [9], a new design is proposed for the state feedback control of multivariable linear time-varying systems. The new design is based on inversion state transformation and a forward differential Riccati equation.
The condition that we impose on the globally stabilizing state feedback control law are that it does not vanish asymptotically for large values. Then, we will give a separation principle based on analysis results for cascaded systems, as done for instance in ([1] , [4], [7], [8]). The author in [11] study the exponential stabilization for a class of linear systems with mixed time delays in both state and control. However, in contrast to [10] we stress that our results will be formulated for time-varying systems and hence are applicable to tracking problems. Moreover as mentioned above, in [10] the author impose the more restrictive assumption in term of input to state stability (ISS). Our cascades criteria lead to milder conditions.
The main contribution of this paper is the problem of stabilization via a state estimate controller. We give a separation principle for nonlinear systems by a linear output feedback under a generalized conditions. Furthermore, we give an example in dimentional two to show the applicability of the main result.

2 General definitions

We consider the system

$$\begin{cases} \dot{x}(t) = F(t, x(t), u(t)) \\ \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

where $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the system output and $C(t) \in \mathbb{R}^{p\times n}$ is a matrix whose elements are continuous or piecewise continuous functions of time. The function $F : [0, +\infty[\times\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x.

Let $u(t) = u(x(t))$ a continuous stabilizing feedback law for (1).

Let $\tilde{F}(t, x(t)) = F(t, x(t), u(t))$.

Let $x(t) = \phi(t, x)$ be the solution of

$$\dot{x}(t) = \tilde{F}(t, x(t)), \quad x(t_0) = x_0 \tag{2}$$

We now introduce the notions of uniform boundedness and uniform ultimate boundedness of a trajectory of (2) (see [6]).

Definition 1 *The system (2) is uniformly bounded if for all $R_1 > 0$, there exists a $R_2 = R_2(R_1) > 0$, such that for all $t_0 \geq 0$*

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R_2, \quad \forall t \geq t_0.$$

Definition 2 *The system (2) is uniformly ultimately bounded if there exists $R > 0$, such that for all $R_1 > 0$, there exists a $T = T(R_1)$, such that for all $t_0 \geq 0$*

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R, \quad \forall t \geq t_0 + T.$$

Let $r \geq 0$ and $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$. First, we give the definition of uniform stability and uniform attractivity of (2) towards a ball B_r.

Definition 3 (Uniform stability of B_r) *(i) B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all $t_0 \geq 0$*

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (2) are globally uniformly bounded.

Definition 4 (Uniform attractivity of B_r) *B_r is globally uniformly attractive if for all $\varepsilon > r$ and $c > 0$, there exists $T(\varepsilon, c) > 0$, such that for all $t_0 \geq 0$*

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, c), \quad \|x_0\| < c.$$

Definition 5 *The system (2) is globally uniformly practically asymptotically stable if there exists $r \geq 0$, such that B_r is globally uniformly stable and globally uniformly attractive.*

Definition 6 *B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k \geq 0$, such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$*

$$\|x(t)\| \leq k\|x_0\|e^{-\gamma(t - t_0)} + r.$$

The system (2) is globally practically uniformly exponentially stable if there exists $r > 0$, such that B_r is globally uniformly exponentially stable.

We will illustrate the notion of practical exponential observer of a trajectory of (1)

Definition 7 *A practical exponential observer for (1) is a dynamical system which has the following form:*

$$\dot{\hat{x}}(t) = F(t, \hat{x}(t), u(t)) - L(t)(C(t)\hat{x}(t) - y(t)) \tag{3}$$

where $L(t)$ is the gain matrix and the error equation with $e(t) = \hat{x}(t) - x(t)$, is given by

$$\dot{e}(t) = F(t, \hat{x}(t), u(t)) - F(t, x(t), u(t)) - L(t)C(t)e(t) \tag{4}$$

a Luenberger observer which is expected to produce an estimation of the state in the sense of global practical exponential stability. It means that, the system (4) is globally practically uniformly exponentially stable and the following estimation holds:

$$\|e(t)\| \leq \lambda_1 \|e(t_0)\| \ e^{-\lambda_2 (t - t_0)} + r, \quad \forall \ t \geq t_0$$

with λ_1, λ_2 and r are positives constants.

3 Basic results

We consider now the following dynamical system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)) \\ \\ y(t) = C(t)x(t) \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ is the system output, $u(t) \in \mathbb{R}^m$ is the control input and $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$, $C(t) \in \mathbb{R}^{p \times n}$ are matrices whose elements are bounded continuous or piecewise continuous functions of time. The function $f(t, x)$ is continuous, locally Lipschitz in x. The corresponding nominal system is described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ \\ y(t) = C(t)x(t) \end{cases} \tag{6}$$

Definition 8 *The pair $(A(t), B(t))$ is uniformly controllable if there exist Δ and another constant α depending on Δ, such that the controllability grammian $I(t-\Delta, t)$ satisfies*

$$I(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau)B(\tau)B^T(\tau)\psi^T(t-\Delta, \tau)d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$ and is defined by

$$\frac{\partial \psi(t, t_0)}{\partial t} = A(t)\psi(t, t_0), \quad \psi(t, t) = I,$$

$$\psi(t, t_0)\psi(t_0, s) = \psi(t, s)$$

and

$$\psi(t_0, t) = \psi^{-1}(t, t_0).$$

We find from [9] the state feedback gain $K(t)$, such that the control input

$$u(t) = K(t)x(t) \tag{7}$$

with

$$K(t) = R_1^{-1}(t)\overline{B}^T(t)P(t)$$

where $P(t)$ is the solution of the forward differential Riccati equation

$$\dot{P}(t) = -\overline{A}^T(t)P(t) - P(t)\overline{A}(t) + R_1(t) - P(t)\overline{B}(t)R_2^{-1}(t)\overline{B}^T(t)P(t), P(0) = P_0 > 0 \tag{8}$$

in which

$$\overline{A}(t) = -T(x)A(t)T^{-1}(x), \quad \overline{B}(t) = T(x)B(t),$$

with

$$T(x) = I - 2\frac{x(t)x^T(t)}{x^T(t)x(t)},$$

$R_1(t) > 0$, $R_2(t) > 0$ and $R_1(t), R_2(t), R_1^{-1}(t), R_2^{-1}(t)$ are all uniformly bounded.

3.1 Stabilization

We prove in this subsection the stabilization of system (5) by a state feedback control candidate. It is assumed that the system (6) is uniformly controllable (see [5]).

Proposition 1 *(see [9]) Consider the system (6) and the state feedback control (7) and (8), if the system (6) is uniformly controllable, the closed-loop system is globally exponentially stable.*

Notice that, the system (6) in closed-loop with the linear feedback

$$u(t) = K(t)x(t)$$

is globally exponentially stable, then from [6] we have for all positive definite symmetric matrix $Q_1(t)$,

$$Q_1(t) \geq c_1 I > 0, \quad \forall t \geq 0$$

there exists a positive definite symmetric matrix $P_1(t)$,

$$0 < c_2 I < P_1(t) < c_3 I, \quad \forall t \geq 0$$

which satisfies

$$A_K^T(t)P_1(t)+P_1(t)A_K(t)+\dot{P}_1(t) = -Q_1(t), \text{ where } A_K(t) = A(t)+B(t)K(t) \quad (9)$$

Now, we prove the global practical uniform stabilizability of (5). We shall suppose the following assumption.

$(\mathcal{A}_1)$ Assume that

$$\|f(t,x)\| \leq \gamma(t)\|x\|^{\frac{1}{2}}, \quad \forall t \geq 0, \ \forall \ x \in \mathbb{R}^n, \quad (10)$$

where $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous non-negative function with

$$\int_0^{+\infty} \gamma^2(s)\, ds \leq M_\gamma < +\infty$$

Theorem 1 *Under assumption $(\mathcal{A}_1)$, the system (6) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable.*

Proof. Let consider the Lyapunov function $V(t, x(t)) = x^T(t)P_1(t)x(t)$. The derivative of V along the trajectories of system (5) is given by

$$\dot{V}(t, x(t)) \leq -\frac{c_1}{c_3}V(t, x(t)) + \frac{2c_3\gamma(t)}{c_2^{\frac{3}{4}}}V(t, x)^{\frac{3}{4}}.$$

Using the following change $v(t) = V(t, x(t))^{\frac{1}{4}}$. Then, $v(t)$ satisfies the following estimation

$$v(t) \leq v(t_0)e^{-\frac{c_1}{4c_3}(t-t_0)} + \frac{c_3}{2c_2^{\frac{3}{4}}}\left(\int_{t_0}^{t} \gamma(s)e^{\frac{c_1}{4c_3}(s-t_0)}ds\right)e^{-\frac{c_1}{4c_3}(t-t_0)}.$$

A simple computation shows that,

$$\left(\int_{t_0}^{t} \gamma(s) e^{\frac{c_1}{4c_3}(s-t_0)} ds\right) e^{-\frac{c_1}{4c_3}(t-t_0)} \leq \sqrt{\frac{2c_3 M_\gamma}{c_1}}.$$

Thus, we obtain

$$v(t) \leq v(t_0) e^{-\frac{c_1}{4c_3}(t-t_0)} + \frac{c_3}{2c_2^{\frac{3}{4}}} \sqrt{\frac{2c_3 M_\gamma}{c_1}}.$$

It follows that,

$$\|x(t)\| \leq 2\sqrt{\frac{c_3}{c_2}} \|x_0\| e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3^3 M_\gamma}{c_1 c_2^2}.$$

This implies the global uniform exponential stability of B_κ, with

$$\kappa = \frac{c_3^3 M_\gamma}{c_1 c_2^2}.$$

Hence, the system (5) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable. □

3.2 Conception of the observer

For the concept of observer, we aim at simplifying the design of this system by exploiting the linear form of the nominal system. The system (6) is assumed to be uniformly observable (see[5]).

Definition 9 *The pair $(A(t), C(t))$ is uniformly observable if there exist Δ and another constant α depending on Δ, such that the observability grammian $J(t-\Delta, t)$ satisfies*

$$J(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau) C(\tau) C^T(\tau) \psi^T(t-\Delta, \tau) d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$.

To design an observer, we shall consider the system

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + f(t, \hat{x}(t)) - L(t)(C(t)\hat{x}(t) - y(t)) \qquad (11)$$

where $\hat{x}(t)$ is the state estimate of $x(t)$, and $L(t) \in \mathbb{R}^{n\times p}$ is the observer feedback gain to be determined so that $\hat{x}(t)$ tends to $x(t)$ exponentially. One such design is the well known Kalman filter design ([3]), in which the observer feedback gain $L(t)$ is chosen as

$$L(t) = Q(t)C^T(t)V_2^{-1}(t) \tag{12}$$

where $Q(t)$ satisfies a forward differential Riccati equation

$$\dot{Q}(t) = A(t)Q(t)+Q(t)A^T(t)+V_1(t)-Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t),\ Q(0) = Q_0 > 0 \tag{13}$$

in which $V_1(t) > 0$, $V_2(t) > 0$ and $V_1(t)$, $V_2(t)$, $V_1^{-1}(t)$, $V_2^{-1}(t)$ are all uniformly bounded. The error equation is given by

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A(t) - L(t)C(t))e(t) + f(t, \hat{x}(t)) - f(t, x(t)) \tag{14}$$

Proposition 2 *(see [9]) Consider the system (6) and the observer (12) and (13). If $(A(t), C(t))$ is uniformly observable, the closed-loop system is globally exponentially stable.*

Notice that, if the system (6) in closed-loop with the observer (12) and (13) is globally uniformly exponentially stable, then for all positive definite symmetric matrix $Q_2(t)$,

$$Q_2(t) \geq b_1 I > 0, \quad \forall t \geq 0$$

there exists a positive definite symmetric matrix $P_2(t)$,

$$0 < b_2 I < P_2(t) < b_3 I, \quad \forall t \geq 0$$

which satisfies

$$A_L^T(t)P_2(t)+P_2(t)A_L(t)+\dot{P}_2(t) = -Q_2(t),\ \text{where } A_L(t) = A(t)-L(t)C(t) \tag{15}$$

$(\mathcal{A}_2)$ Assume that

$$\|f(t, x) - f(t, y)\| \leq \gamma(t)\|x - y\|^{\frac{1}{2}}, \quad \forall t \geq 0,\ \forall\ \ x, y \in \mathbb{R}^n, \quad f(t, 0) = 0 \tag{16}$$

where $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous non-negative function with

$$\int_0^{+\infty} \gamma^2(s)\, ds \leq M_\gamma < +\infty.$$

Theorem 2 *Under assumption $(\mathcal{A}_2)$, the system* (11) *is a practical exponential observer for the system (5).*

Proof. Let consider the Lyapunov function $Y(t,e(t)) = e^T(t)P_2(t)e(t)$. The derivative of Y along the trajectories of system (14) is given by

$$\dot{Y}(t,e(t)) \leq -\frac{b_1}{b_3}Y(t,e(t)) + \frac{2b_3\gamma(t)}{b_2^{\frac{3}{4}}}Y(t,e(t))^{\frac{3}{4}}.$$

Using the following change $y(t) = Y(t,e(t))^{\frac{1}{4}}$. Then, $y(t)$ satisfies the following estimation

$$y(t) \leq y(t_0)e^{-\frac{b_1}{4b_3}(t-t_0)} + \frac{b_3}{2b_2^{\frac{3}{4}}}\left(\int_{t_0}^{t}\gamma(s)e^{\frac{b_1}{4b_3}(s-t_0)}ds\right)e^{-\frac{b_1}{4b_3}(t-t_0)}.$$

A simple computation shows that,

$$\left(\int_{t_0}^{t}\gamma(s)e^{\frac{b_1}{4b_3}(s-t_0)}ds\right)e^{-\frac{b_1}{4b_3}(t-t_0)} \leq \sqrt{\frac{2b_3M_\gamma}{b_1}}.$$

Thus, we obtain

$$y(t) \leq y(t_0)e^{-\frac{b_1}{4b_3}(t-t_0)} + \frac{b_3}{2b_2^{\frac{3}{4}}}\sqrt{\frac{2b_3M_\gamma}{b_1}}.$$

Hence,

$$\|e(t)\| \leq 2\sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{b_3^3M_\gamma}{b_1b_2^2}.$$

This implies the global uniform exponential stability of B_η, with

$$\eta = \frac{b_3^3M_\gamma}{b_1b_2^2}.$$

We deduce that, the system (14) is globally practically exponentially stable. Hence, the system (11) is a practical exponential observer for the system (5). □

3.3 Separation principle

Now, in order to obtain a separation principle for (5). We consider the system (5) controlled by the linear feedback control $u(t) = K(t)\hat{x}(t)$ and estimated with the observer (11).

Theorem 3 *Under assumption ($\mathcal{A}_2$) and the fact*

$$\frac{b_1}{b_3} < \frac{c_1}{c_3}$$

the system

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + f(t,\hat{x}(t)) - L(t)C(t)e(t) \\ \\ \dot{e}(t) = (A(t) - L(t)C(t))e(t) + f(t,\hat{x}(t)) - f(t,x(t)) \end{cases} \tag{17}$$

is globally practically uniformly exponentially stable.

Proof. In order to study the stabilization problem via an observer, we consider the system

$$\dot{\hat{x}}(t) = \psi(t,\hat{x}(t)) + g(t,\hat{x}(t))e(t) \tag{18}$$

$$\dot{e}(t) = h(t,\hat{x}(t),e(t)) \tag{19}$$

where,

$$\psi(t,\hat{x}(t)) = (A(t) + B(t)K(t))\hat{x}(t) + f(t,\hat{x}(t)), \qquad g(t,\hat{x}(t)) = -L(t)C(t)$$

and

$$h(t,\hat{x}(t),e(t)) = (A(t) - L(t)C(t))e(t) + f(t,\hat{x}(t)) - f(t,x(t)).$$

We have, $\dot{\hat{x}}(t) = \psi(t,\hat{x}(t))$ is globally practically uniformly exponentially stable with Lyapunov function associated to this system can be taken as

$$v(t,\hat{x}(t)) : \mathbb{R}_+ \times D \longrightarrow \mathbb{R}^n,$$

with $D = \{x \in \mathbb{R}^n / \|x\| > 1\}$ and $v(t,\hat{x}(t)) = (\hat{x}^T(t)P_1(t)\hat{x}(t))^{\frac{1}{4}}$, which satisfies

$$\sqrt{c_2}^{\frac{1}{2}}\|\hat{x}(t)\|^{\frac{1}{2}} \leq v(t,\hat{x}(t)) \leq \sqrt{c_3}^{\frac{1}{2}}\|\hat{x}(t)\|^{\frac{1}{2}}$$

$$\frac{\partial v}{\partial t}(t,\hat{x}(t)) + \frac{\partial v}{\partial \hat{x}(t)}\psi(t,\hat{x}(t)) \leq -\frac{c_1}{4c_3}v(t,\hat{x}(t)) + \frac{c_3}{2c_2^{\frac{3}{4}}}\gamma(t)$$

and

$$\left\|\frac{\partial v}{\partial \hat{x}}(t,\hat{x}(t))\right\| \leq \frac{c_3}{2c_2^{\frac{3}{4}}}.$$

The derivative of v along the trajectories of system (18) is given by

$$\dot{v}(t,\hat{x}(t)) \leq -\frac{c_1}{2c_3}v(t,\hat{x}(t)) + \frac{c_3}{2c_2^{\frac{3}{4}}}\gamma(t) + \frac{c_3}{2c_2^{\frac{3}{4}}}\|L(t)C(t)\| \times \left(2\sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{b_3^3 M_\gamma}{b_1 b_2^2}\right).$$

Since $L(t)C(t)$ is uniformly bounded for all $t \geq t_0 \geq 0$, then there exists $R_1 > 0$, such that

$$\|L(t)C(t)\| \leq R_1, \quad \forall t \geq t_0 \geq 0.$$

Then,

$$\dot{v}(t,\hat{x}(t)) \leq -\frac{c_1}{2c_3}v(t,\hat{x}(t)) + \lambda\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{c_3}{2c_2^{\frac{3}{4}}}\gamma(t) + R$$

with

$$\lambda = \frac{c_3}{c_2^{\frac{3}{4}}}R_1\sqrt{\frac{b_3}{b_2}},$$

$$R = \frac{c_3}{2c_2^{\frac{3}{4}}}R_1\frac{b_3^3 M_\gamma}{b_1 b_2^2}.$$

Using the following change

$$y(t) = v(t)e^{\frac{c_1}{2c_3}(t-t_0)}.$$

We obtain,

$$y(t) \leq y(t_0) + \int_{t_0}^{t}\lambda\|e(t_0)\|e^{\left(-\frac{b_1}{2b_3}+\frac{c_1}{2c_3}\right)(s-t_0)}ds + \frac{c_3}{2c_2^{\frac{3}{4}}}\int_{t_0}^{t}\gamma(s)e^{\frac{c_1}{2c_3}(s-t_0)}ds + \int_{t_0}^{t}Re^{\frac{c_1}{2c_3}(s-t_0)}ds.$$

Then,

$$v(t) \leq v(t_0)e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{2\lambda b_3 c_3}{b_3 c_1 - b_1 c_3}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{c_3}{2c_2^{\frac{3}{4}}}\sqrt{\frac{c_3 M_\gamma}{c_1}} + \frac{2Rc_3}{c_1}.$$

Then,

$$\|\hat{x}(t)\| \le 2\sqrt{\frac{c_3}{c_2}}\|\hat{x}_0\|e^{-\frac{b_1}{b_3}(t-t_0)} + \frac{8\lambda^2 b_3^2 c_3^2}{\sqrt{c_2}(b_3c_1 - b_1c_3)^2}\|e(t_0)\|^2 e^{-\frac{b_1}{b_3}(t-t_0)}$$
$$+ \frac{c_3^3 M_\gamma}{2c_2^2 c_1} + \frac{8R^2 c_3^2}{\sqrt{c_2}c_1^2}, \quad \forall t \ge t_0 \ge 0, \quad \forall \|\hat{x}_0\| > 1, \quad \forall \|e(t_0)\| > 1.$$

Let

$$k = \max\left(2\sqrt{\frac{c_3}{c_2}},\ \frac{8\lambda^2 b_3^2 c_3^2}{\sqrt{c_2}(b_3c_1 - b_1c_3)^2}\right).$$

Hence,

$$\|\hat{x}(t)\| \le k\|(\hat{x}_0, e(t_0))\|^2 e^{-\frac{b_1}{b_3}(t-t_0)} + \frac{2c_3^3 M_\gamma}{c_2^2 c_1} + \frac{8R^2 c_3^2}{\sqrt{c_2}c_1^2},$$

$\forall t \ge t_0 \ge 0$, $\forall \|\hat{x}_0\| > 1$, $\forall \|e(t_0)\| > 1$. Then, the cascade system (17) is globally practically uniformly exponentially stable. □

We give now an example to illustrate the applicability of our result.

Example. Consider the system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)) \\ \\ y(t) = C(t)x(t) \end{cases} \tag{20}$$

with $x(t) = (x_1(t), x_2(t))^T$,

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}$$
$$C(t) = \begin{pmatrix} 1 & e^{-2t} \end{pmatrix}$$

and

$$f(t, x(t)) = \gamma(t)h(x)$$

with $\gamma(t) = e^{-t}$, $h(x)$ satisfies $\|h(x) - h(y)\| \le k\|x - y\|^{\frac{1}{2}}, \quad k > 0,\ \forall\ x, y \in \mathbb{R}^n$ and $h(0) = 0$. The proposed control (7) is then applied to the system with the following design parameters $P(0) = I$, $R_1(t) = I$, $R_2(t) = I$ in (8). The matrix

$P(t)$ is calculated by solving the Ricatti equation (8). The function $f(t, x(t))$ is continuous and satisfies the assumption $(\mathcal{A}_1)$ because

$$\int_0^{+\infty} e^{-2t} = \frac{1}{2}.$$

We conclude that the system (5) can be globally practically uniformly exponentially stable. The observer feedback gain $L(t)$ be chosen as (12) by solving the Riccati equation (13). We conclude that the system (11) is a practical exponential observer for the system (20). Thus, if $\frac{b_1}{b_3} < \frac{c_1}{c_3}$ the theorem 3 is satisfied. We conclude that, the system (17) is globally uniformly practically exponentially stable. □

4 Conclusion

This paper presents a separation principle of time-varying nonlinear dynamical systems. It is shown that the system can be globally exponentially stabilizable by means of an estimated state feedback control given by an observer design.

References

[1] M.Arcak and P.Kokotovic, Observer-based control of systems with slope-restricted nonlinearities, *IEEE Transactions on Automatic Control* (2001), 1146-1150.

[2] M.Chen, Control of linear time-varying systems by the gradient algorithm, *IEEE* (1997), 4549-4553.

[3] R.E.Kalman and R.S.Bucy, New Results in Linear Filtering and Prediction Theory, *ASME Journal of Basic Engineering* (1961), 95-108.

[4] I.Ellouze and M.Hammami, A separation principle of time-varying dynamical system: A practical approach, *Mathematical Modelling and Analysis* **12** (2007), No. 2, 1-16.

[5] W.J.Rugh, Linear System Theory, Prentice-Hall, Englewood Cliffs (1993).

[6] H.K.Khalil, Nonlinear Systems, Prentice-Hall, New York (2002).

[7] A.Loria and E.Panteley, A separation principle for Euler-Langrange systems, *In New Directions in Nonlinear Observer Design*, H. Nijmeijer and T.I. Fossen, Eds. (London: Springer Verlag), (1999), 229-247.

[8] A.Loria and J.de Leon Morales, On persistently exciting observers and a non-linear separation principle: application to the stabilization of a generator, *International Journal of Control* **76** (2003), No. 6, 607-617.

[9] M.S.Chen and C.Y.Kao, Control of linear time-varying systems using forward Riccati equation, *Journal of Dynamics systems, Measurement and Control* **119** (1997), 536-540.

[10] J.Tsinias, Sontag's input to state stability condition and global stabilization using state detection, *Systems and Control Letters* **20** (1993), 219-226.

[11] L.V.Hien and H.V.Thi, Exponential Stabilization of Linear Systems with Mixed Delays in State and Control, *e-journal Differential Equations and Control Processes* (2009), No. 2, 32-42.