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Nonlinear Integral Equations

On Erdélyi-Kober cubic fractional integral equation of Urysohn-Volterra type

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Abstract

Cubic integral equations is the general form of the quadratic integral equations which have several applications in the theory of radiative transfer, in the traffic theory, in the theory of particle transport and in the kinetic theory of gases. In this paper, we present a result on the existence of solutions of the perturbed Erdélyi-Kober fractional cubic integral equation of Uryshon-Volterra type in the Banach space of real functions defined, continuous and bounded on an unbounded interval. We use the Darbo fixed point theorem and a measure of noncompactness in order to prove our main result.

Keywords: Cubic integral equation, measure of noncompactness, Erdélyi-Kober fractional integral, Darbo fixed point theorem, Urysohn-Volterra integral equation

1 Introduction

In this paper, we establish the existence and the asymptotic behaviour of solutions to the perturbed cubic Uryshon-Volterra integral equation involving

Erdélyi-Kober fractional integral, namely

$$x(t) = f(t, x(t)) + \frac{\beta x^2(t)}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1} u(t, s, x(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds, \ t \in \mathbb{R}_+ = [0, +\infty), \qquad (1)$$

where $0 < \alpha < 1$, $\beta > 0$. Let $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$ be the norm for $x \in BC(\mathbb{R}_+)$, where $BC(\mathbb{R}_+)$ is the Banach space of all real functions defined, continuous and bounded on $\mathbb{R}_+$. There, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy special assumptions that will be stated in detail in Section 3.

Cubic integral equations is the general form of the quadratic integral equations which have several applications in the theory of radiative transfer, in the traffic theory, in the theory of particle transport and in the kinetic theory of gases, specially the quadratic integral equation of Chandrasekhar type is applicable in many problems in mechanics, physics and other fields [3, 4, 8, 23]. On the other hand, Erdélyi-Kober fractional integrals are very often used to describe the medium with non-integer mass dimension and also, one can find more applications of fractional integrals of Erdélyi-Kober type in porous media, viscoelasticity and electrochemistry ([10]-[18], [21], [22]).

For a continuous function f, the Erdélyi-Kober fractional integral is defined as [1, 19]

$$I_\beta^\gamma f(\tau) = \frac{\beta}{\Gamma(\gamma)} \int_0^\tau \frac{s^{\beta-1} f(s)}{(\tau^\beta - s^\beta)^{1-\gamma}} ds, \ \beta > 0, \ 0 < \gamma < 1.$$

Eq.(1) considered as Erdélyi-Kober fractional quadratic integral equation with perturbation, see [20].

The aim of this paper is to prove the existence of solutions to Eq.(1) in the space of real functions which are defined, continuous and bounded on an unbounded interval. We use a suitable combination of the technique of measures of noncompactness and the Darbo fixed point principle to obtain our results.

2 Preliminaries

Firstly, we present the concept of a measure of noncompactness [5].
Let the symbol $(E, \|.\|)$ stands for a real Banach space with a zero element 0 and $B(x, r)$ stands for the closed ball of radius r and center x. Also, we denote by B_r the closed ball $B(0, r)$. Next, let $\emptyset \neq X \subset E$ and denote by $\overline{X}$ and

Conv$X$ the closure and convex closure of the set X, respectively. Let $X+Y$ and λX, $\lambda \in \mathbb{R}$, denote the usual algebraic operations on the sets X and Y. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ the subfamily of $\mathfrak{M}_E$ consisting of all relatively compact subsets of E.

Definition 1 *A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness in E if it verifies the following conditions:*

1) $\ker\mu \neq \emptyset$ *and* $\ker\mu \subset \mathfrak{N}_E$, *where* $\ker\mu$ *stands for the family* $\{X \in \mathfrak{M}_E : \mu(X) = 0\}$.
2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3) $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ *for* $0 \leq \lambda \leq 1$.
5) *If* $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ *for* $n = 1,\ 2,\ 3,\ \ldots$ *and* $\lim\limits_{n\to\infty} \mu(X_n) = 0$, *then* $\cap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from 5) belongs to $\ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for every n, then we have that $\mu(X_\infty) = 0$. Secondly, we present the construction of the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be used in the next section (see [6, 7]).

Let $\emptyset \neq X \subset BC(\mathbb{R}_+)$ be bounded set and fix numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$, we define the modulus of continuity of the function x on the interval $[0,T]$ by

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in [0,T],\ |t-s| \leq \varepsilon\}.$$

Further, we put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon),$$

and

$$\omega_0^\infty(X) = \lim_{T\to\infty} \omega_0^T(X,\varepsilon).$$

For a fixed number $t \in \mathbb{R}_+$, let us define

$$X(t) = \{x(t) : x \in X\}$$

and

$$\mathrm{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Next, we define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by

$$\mu(X) = \omega_0^\infty(X) + c(X), \tag{2}$$

where $c(X) = \limsup_{t\to\infty} \mathrm{diam} X(t)$. The function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$, see [6].

Finally, we present a fixed point theorem due to Darbo [9]. Before giving this theorem, we need the following definition.

Definition 2 *Let M be a nonempty subset of a Banach space E and let $\mathcal{P} : M \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with a constant $k \geq 0$) with respect to a measure of noncompactness μ, if for any bounded subset X of M, we have*

$$\mu(\mathcal{P}X) \leq k\, \mu(X).$$

If $\mathcal{P}$ verifies the Darbo condition with $k < 1$, then $\mathcal{P}$ is said to be a contraction operator with respect to μ.

Theorem 1 *Let $Q \neq \emptyset$ be a bounded, closed and convex subset of the space E and let*

$$\mathcal{P} : Q \to Q$$

be a contraction with respect to the measure of noncompactness μ. Then $\mathcal{P}$ has at least one fixed point in the set Q.

3 Main result

In this section, we will study Eq.(1) under the following assumptions:

(a_1) The functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t, 0)$ is bounded on $\mathbb{R}_+$. Put $f^* = \sup\{|f(t, 0)| : t \in \mathbb{R}_+\}$.

(a_2) There exist continuous functions $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq m(\tau)|x - y|,$$

for any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Moreover, m is bounded. Put $m^* = \sup\{|m(t)| : t \in \mathbb{R}_+\}$.

(a_3) The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist functions $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $q(t) = q : \mathbb{R}_+ \to \mathbb{R}_+$, being Ψ continuous and nondecreasing, with $\Psi(0) = 0$ and q continuous satisfying

$$|u(t,s,x_1) - u(t,s,x_2)| \leq q(t)\Psi(|x_1 - x_2|)$$

for any $t, s \in \mathbb{R}_+$ with $s \leq t$ and for all $x_i \in \mathbb{R}$ $(i = 1, 2)$.

(a_4) The functions $\phi,\ \psi : \mathbb{R}_+ \to R_+$ defined by $\phi(t) = q(t)t^{\alpha\beta}$ and $\psi(t) = u^*(t)t^{\alpha\beta}$ are bounded on $\mathbb{R}_+$, where $u^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $u^*(t) = \max\{|u(t,s,0)| : 0 \leq s \leq t\}$, whose existence is guaranteed by virtue of the continuity of u (assumption (a_3)). Moreover, the function ϕ vanishes at infinity, i.e., $\lim\limits_{t\to\infty} \phi(t) = 0$.

(a_5) There exists a positive solution r_0 of the inequality

$$(m^*r + f^*)\Gamma(\alpha+1) + r^2[\phi^*\Psi(r) + \psi^*] \leq r\Gamma(\alpha+1) \tag{3}$$

satisfying

$$m^*\Gamma(\alpha+1) + 2r_0[\phi^*\Psi(r_0) + \psi^*] < \Gamma(\alpha+1), \tag{4}$$

where $\phi^* = \sup\{\phi(t) : t \in \mathbb{R}_+\}$ and $\psi^* = \sup\{\psi(t) : t \in \mathbb{R}_+\}$.

Before we state and prove our main result, let us denote by $\mathcal{T}$ the operator associated with the right-hand side of Equation (1), i.e., Eq.(1) becomes

$$x = \mathcal{T}x,$$

where

$$(\mathcal{T}x)(t) = (Fx)(t) + x^2(t)\cdot(\mathcal{U}x)(t), \tag{5}$$

$$(\mathcal{U}x)(t) = \frac{\beta}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta - s^\beta)^{1-\alpha}}ds \tag{6}$$

and F is the superposition operator generated by the function $f = f(t,x)$ and defined by

$$(Fx)(t) = f(t, x(t)),$$

where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}_+$, see [2].

Theorem 2 *Under assumptions $(a_1) - (a_5)$, Eq.(1) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$.*

Proof: For better readability, we will divide the proof into several steps.

Step 1: For $x \in BC(\mathbb{R}_+)$ then $\mathcal{T}x$ is continuous on $\mathbb{R}_+$.

To do this, it is sufficient to prove that, if $x \in BC(\mathbb{R}_+)$ then $\mathcal{U}x$ is continuous on $\mathbb{R}_+$. In fact, we take $x \in BC(\mathbb{R}_+)$ and fix $\varepsilon > 0$ and $T > 0$. Suppose that $\tau_1, \tau_2 \in \mathbb{R}_+$ with $|\tau_2 - \tau_1| \leq \varepsilon$ and, without loss of generality, we can assume that $\tau_1 < \tau_2$, then we have

$$
\begin{aligned}
&|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
&= \frac{\beta}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \int_0^{t_1} \frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right| \\
&\leq \frac{\beta}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \int_0^{t_1} \frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right| \\
&\ +\frac{\beta}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \int_0^{t_1} \frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right| \\
&\ +\frac{\beta}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \int_0^{t_1} \frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right| \\
&\leq \frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{s^{\beta-1}|u(t_2,s,x(s))|}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds \\
&\ +\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1} \frac{s^{\beta-1}|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds \\
&\ +\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1} s^{\beta-1}|u(t_1,s,x(s))|\ \left[\left(t_1^\beta - s^\beta\right)^{\alpha-1} - \left(t_2^\beta - s^\beta\right)^{\alpha-1}\right] ds \\
&\leq \frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{s^{\beta-1}[|u(t_2,s,x(s)) - u(t_2,s,0)| + |u(t_2,s,0)|]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds \\
&\ +\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1} \frac{s^{\beta-1}\omega^T_{\|x\|}(u,\varepsilon)}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1} s^{\beta-1}[|u(t_1,s,x(s))-u(t_1,s,0)|+|u(t_1,s,0)|]\\
&\times\left[\left(t_1^\beta-s^\beta\right)^{\alpha-1}-\left(t_2^\beta-s^\beta\right)^{\alpha-1}\right]ds\\
&\le\frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{s^{\beta-1}[q(t_2)\Psi(|x(s)|)+u^*(t_2)]}{\left(t_2^\beta-s^\beta\right)^{1-\alpha}}ds+\frac{\omega^T_{\|x\|}(u,\varepsilon)}{\Gamma(\alpha+1)}\left[t_2^{\alpha\beta}-\left(t_2^\beta-t_1^\beta\right)^\alpha\right]\\
&+\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1} s^{\beta-1}[q(t_1)\Psi(|x(s)|)+u^*(t_1)]\left[\left(t_1^\beta-s^\beta\right)^{\alpha-1}-\left(t_2^\beta-s^\beta\right)^{\alpha-1}\right]ds\\
&\le\frac{q(t_2)\Psi(\|x\|)+u^*(t_2)}{\Gamma(\alpha+1)}\left(t_2^\beta-t_1^\beta\right)^\alpha+\frac{\omega^T_{\|x\|}(u,\varepsilon)}{\Gamma(\alpha+1)}t_2^{\alpha\beta}\\
&+\frac{q(t_1)\Psi(\|x\|)+u^*(t_1)}{\Gamma(\alpha+1)}\left[t_1^{\alpha\beta}-t_2^{\alpha\beta}+\left(t_2^\beta-t_1^\beta\right)^\alpha\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_d^T(u,\varepsilon)=\sup\{|u(t_2,s,x)-u(t_1,s,x)|: s,t_1,t_2\in[0,T],\ t_1\ge s,\ t_2\ge s,\\
|t_2-t_1|\le\varepsilon,\ x\in[-d,d]]\}.
\end{aligned}
$$

From the above estimates, we infer

$$
\omega^T(\mathcal{U}x,\varepsilon)\le\frac{2\varepsilon^{\alpha\beta}[\hat q(T)\Psi(\|x\|)+\hat u(T)]+T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)}{\Gamma(\alpha+1)},\tag{7}
$$

where

$$
\hat q(T)=\max\{q(t):t\in[0,T]\}
$$

and

$$
\hat u(T)=\max\{u^*(t):t\in[0,T]\}.
$$

Since the function $u(t,s,x)$ is uniformly continuous on the compact set $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$, we deduce that $\omega^T_{\|x\|}(u,\varepsilon)\to0$ as $\varepsilon\to0$.
Therefore, $\lim\limits_{\varepsilon\to0}\omega^T_{\|x\|}(U,\varepsilon)=0$ and this proves that $\mathcal{U}$ is continuous on the interval $[0,T]$ for any $T>0$. This gives us the continuity of $\mathcal{U}x$ on $\mathbb{R}_+$.

Step 2: For $x\in\mathbb{R}_+$, $\mathcal{T}x$ is a bounded function on $\mathbb{R}_+$.

In fact, taking into account our assumptions, for $x\in BC(\mathbb{R}_+)$ and $t\in\mathbb{R}_+$, we have

$$
|(\mathcal{T}x)(t)| \;=\; \left|f(t,x(t))+\frac{\beta x^2(t)}{\Gamma(\alpha)}\int_0^t\frac{s^{\beta-1}u(t,s,x(s))}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds\right|
$$

$$\begin{aligned}
&\leq |f(t,x(t))-f(t,0)|+|f(t,0)| \\
&\quad +\frac{\beta|x^2(t)|}{\Gamma(\alpha)}\int_0^\tau \frac{s^{\beta-1}|u(t,s,x(s))-u(\tau,s,0)|+|u(t,s,0)|}{(t^\beta-s^\beta)^{1-\alpha}}ds \\
&\leq m(t)\|x\|+|f(t,0)| \\
&\quad +\frac{\beta\|x\|^2}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}[q(t)\Psi(|x(s)|)+u^*(t)]}{(t^\beta-s^\beta)^{1-\alpha}}ds \\
&\leq m(t)\|x\|+|f(t,0)|+\frac{\|x\|^2}{\Gamma(\alpha+1)}[q(t)\Psi(\|x\|)+u^*(t)]t^{\alpha\beta} \\
&\leq m^*\|x\|+f^*+\frac{\|x\|^2}{\Gamma(\alpha+1)}[\phi(t)\Psi(\|x\|)+\psi(t)].
\end{aligned}$$

Taking into account (a_2) and (a_4), the last chain of inequalities gives us that $\mathcal{T}x$ is bounded on $\mathbb{R}_+$.

Step 3: $\mathcal{T}$ applies the ball B_{r_0} into itself.

Taking into account that Ψ is a nondecreasing function, from the estimate obtained in Step 2, it follows that

$$\|\mathcal{T}x\|\leq m^*\|x\|+f^*+\frac{\|x\|^2}{\Gamma(\gamma+1)}[\phi^*\Psi(\|x\|)+\psi^*].$$

Taking into account (a_5), we infer that $\mathcal{T}$ applies the ball B_{r_0} into itself.

Step 4: $\mathcal{T}$ is continuous on the ball B_{r_0}.

Since $(\mathcal{T}x)(t)=(Fx)(t)+x^2(t)\cdot(\mathcal{U}x)(t)$ for $t\in\mathbb{R}_+$, it is sufficient to prove that F is continuous on B_{r_0} and

$$(\mathcal{U}x)(t)=\frac{\beta}{\Gamma(\alpha)}\int_0^\tau \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta-s^\beta)^{1-\alpha}}ds$$

is continuous on B_{r_0}.
Let $(x_n)\subset B_{r_0}$ be a sequence such that $x_n\to x$ with $x\in B_{r_0}$.

Firstly, we will prove that $Fx_n\to Fx$. In fact, for $t\in\mathbb{R}_+$ and, taking into account (a_2), we have

$$\begin{aligned}
|(Fx_n)(t)-(Fx)(t)| &= |f(t,x_n(t))-f(t,x(t))| \\
&\leq m(t)|x_n(t)-x(t)| \\
&\leq m(t)\|x_n-x\|.
\end{aligned}$$

Since m is a bounded function, we infer

$$\|Fx_n-Fx\|\leq L\|x_n-x\|,$$

where $L = \sup\{m(t) : t \in \mathbb{R}_+\}$, and this proves that F is continuous on B_{r_0}.

Next, we will prove that $\mathcal{U}x_n \to \mathcal{U}x$. In fact, for $t \in \mathbb{R}_+$, and, taking into account our assumptions, it follows

$$\begin{aligned}
|(\mathcal{U}x_n)(t) - (\mathcal{U}x)(t)| &= \Bigg| \frac{\beta}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}u(t,s,x_n(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad - \frac{\beta}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds \Bigg| \\
&\leq \frac{\beta}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}|u(t,s,x_n(s)) - u(t,s,x(s))|}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{\beta}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}q(t)\Psi(|x_n(s) - x(s)|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{q(t)\Psi(\|x_n - x\|)t^{\alpha\beta}}{\Gamma(\alpha+1)} \\
&\leq \frac{\phi(t)\Psi(\|x_n - x\|)}{\Gamma(\alpha+1)} \\
&\leq \frac{\phi^*\Psi(\|x_n - x\|)}{\Gamma(\alpha+1)}.
\end{aligned}$$

From the above estimation we deduce that $\|\mathcal{U}x_n - \mathcal{U}x\| \to 0$ when $n \to \infty$. This proves that $\mathcal{T}$ is continuous on the ball B_{r_0}.

Step 5**:** An estimate of $\mathcal{T}$ with respect to the quantity c.

We take $\emptyset \neq X \subset B_{r_0}$ and $x, y \in X$. Then, for $t \in \mathbb{R}_+$ and, taking into account our assumptions, we get

$$\begin{aligned}
&|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \\
&\leq |f(t,x(t)) - f(t,y(t))| \\
&\quad + \frac{\beta}{\Gamma(\alpha)} \left| x^2(t) \int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds - y^2(t) \int_0^t \frac{s^{\beta-1}u(t,s,y(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds \right| \\
&\leq m(t)|x(t) - y(t)| + \frac{\beta|x^2(t) - y^2(t)|}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}|u(t,s,x(s))|}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta|y^2(t)|}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}|u(t,s,x(s)) - u(t,s,y(s))|}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq m(t)\mathrm{diam}X(t) + \frac{\beta|x(t) - y(t)|\,|x(t) + y(t)|}{\Gamma(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t \frac{s^{\beta-1}[|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|]}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&+\frac{\beta\|y\|^2}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}q(t)\Psi(|x(s)-y(s)|)}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&\leq m(t)\mathrm{diam}X(t)+\frac{2\beta\|x\|\mathrm{diam}X(t)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}[q(t)\Psi(|x(s)|)+u^*(t)]}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&+\frac{\beta\|y\|^2}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}q(t)\Psi(\|x-y\|)}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&\leq m(t)\mathrm{diam}X(t)+\frac{2\beta\|x\|q(t)\Psi(\|x\|)\mathrm{diam}X(t)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&+\frac{2\beta\|x\|u^*(t)\mathrm{diam}X(t)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}}ds\\
&+\frac{\beta\|y\|^2q(t)\Psi(\|x-y\|)}{\Gamma(\gamma)}\int_0^t \frac{s^{\beta-1}}{(\tau^\beta - s^\beta)^{1-\gamma}}ds\\
&\leq m(t)\mathrm{diam}X(t)+\frac{2r_0\phi(t)\Psi(r_0)\mathrm{diam}X(t)}{\Gamma(\alpha+1)}+\frac{2r_0\psi(t)\mathrm{diam}X(t)}{\Gamma(\alpha+1)}\\
&+\frac{r_0^2\phi(t)\Psi(2r_0)}{\Gamma(\alpha+1)}
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\mathrm{diam}(\mathcal{T}X)(\tau) \ \leq\ & m(t)\mathrm{diam}X(t)+\frac{2r_0\phi(t)\Psi(r_0)\mathrm{diam}X(t)}{\Gamma(\alpha+1)}+\frac{2r_0\psi(t)\mathrm{diam}X(t)}{\Gamma(\alpha+1)}\\
&+\frac{r_0^2\phi(t)\Psi(2r_0)}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Finally, by assumption (a_5), it follows

$$
c(\mathcal{T}X)\leq\left(m^*+\frac{2r_0\psi^*}{\Gamma(\alpha+1)}\right)c(X). \tag{8}
$$

Step 6**:** An estimate of $\mathcal{T}$ with respect to the quantity ω_0^∞.

We take $\emptyset\neq X\subset B_{r_0}$, $\varepsilon>0$ and $T>0$. For $x\in X$, we take $t_1,t_2\in[0,T]$ with $|t_2-t_1|\leq\varepsilon$ and we can assume that $t_1<t_2$. Then, we have

$$
\begin{aligned}
&|(\mathcal{T}x)(t_2)-(\mathcal{T}x)(t_1)|\\
&\leq|f(t_2,x(t_2))-f(t_1,x(t_1))|+\left|x^2(t_2)(\mathcal{U}x)(t_2)-x^2(t_1)(\mathcal{U}x)(t_1)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |f(t_2, x(t_2)) - f(t_2, x(t_1))| + |f(t_2, x(t_1)) - f(t_1, x(t_1))| \\
&\quad + \left| x^2(t_2)(\mathcal{U}x)(t_2) - x^2(t_1)(\mathcal{U}x)(t_2) \right| + \left| x^2(t_1)(\mathcal{U}x)(t_2) - x^2(t_1)(\mathcal{U}x)(t_1) \right| \\
&\leq m(t_2)|x(t_2) - x(t_1)| + \omega_f^T(\varepsilon) + \left| x^2(t_2) - x^2(t_1) \right| \ |(\mathcal{U}x)(t_2)| \\
&\quad + \left| x^2(t_1) \right| \ |(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
&\leq m(t_2)\omega^T(x, \varepsilon) + \omega_f^T(\varepsilon) + \|x\|^2 \omega^T(\mathcal{U}x, \varepsilon) \\
&\quad + \frac{\beta |x(t_2) - x(t_1)| \ |x(t_2) + x(t_1)|}{\Gamma(\alpha)} \int_0^{t_2} \frac{s^{\beta-1}|u(t_2, s, x(s))|}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\leq m(t)\omega^T(x, \varepsilon) + \omega_f^T(\varepsilon) + \|x\|^2 \omega^T(\mathcal{U}x, \varepsilon) \\
&\quad + \frac{2\beta \|x\| \omega^T(x, \varepsilon)}{\Gamma(\alpha)} \int_0^{t_2} \frac{s^{\beta-1}\left[|u(t_2, s, x(s)) - u(t_2, s, 0)| + |u(t_2, s, 0)|\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds.
\end{aligned}
$$

Taking into account step 1, we had obtained that

$$
\omega^T(\mathcal{U}X, \varepsilon) \leq \frac{2\varepsilon^{\alpha\beta}\left[\hat{q}(T)\Psi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)}{\Gamma(\alpha + 1)},
$$

and, therefore, from the chain of inequalities obtained above, we deduce

$$
\begin{aligned}
\omega^T(\mathcal{T}X, \varepsilon) \ &\leq \ m(t_2)\omega^T(x, \varepsilon) + \omega_f^T(\varepsilon) \\
&\quad + \frac{2\|x\|^2 \varepsilon^{\alpha\beta}\left[\hat{q}(T)\Psi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)}{\Gamma(\alpha + 1)} \\
&\quad + \frac{2\beta \|x\| \omega^T(x, \varepsilon)}{\Gamma(\alpha)} \int_0^{t_2} \frac{s^{\beta-1}[q(t_2)\Psi(|x(s)|) + u^*(t_2)]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\leq \ m^*\omega^T(x, \varepsilon) + \omega_f^T(\varepsilon) + \frac{2r_0^2 \varepsilon^{\alpha\beta}\left[\hat{q}(T)\Psi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{r_0}^T(u, \varepsilon)}{\Gamma(\alpha + 1)} \\
&\quad + \frac{2r_0 \omega^T(x, \varepsilon)[q(t_2)\Psi(r_0) + u^*(t_2)]}{\Gamma(\alpha + 1)} t_2^{\alpha\beta} \\
&\leq \ m^*\omega^T(x, \varepsilon) + \omega_f^T(\varepsilon) + \frac{2r_0^2 \varepsilon^{\alpha\beta}\left[\hat{q}(T)\Psi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{r_0}^T(u, \varepsilon)}{\Gamma(\alpha + 1)} \\
&\quad + \frac{2r_0 \omega^T(x, \varepsilon)[\phi(t_2)\Psi(r_0) + \psi(t_2)]}{\Gamma(\alpha + 1)},
\end{aligned}
$$

where

$$
\omega_f^T(\varepsilon) = \sup\{|f(t_2, x) - f(t_1, x)| : t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \varepsilon, \ x \in [-r_0, r_0]\}.
$$

Therefore,

$$\omega^T(\mathcal{T}X,\varepsilon) \le m^*\omega^T(x,\varepsilon)+\omega_f^T(\varepsilon)+\frac{2r_0^2\varepsilon^{\alpha\beta}\left[\hat{q}(T)\Psi(r_0)+\hat{u}(T)\right]+T^{\alpha\beta}\omega_{r_0}^T(u,\varepsilon)}{\Gamma(\alpha+1)} + \frac{2r_0[\phi^*\Psi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon).$$

Since the functions $f=f(\tau,x)$ and $u=u(\tau,s,x)$ are uniformly continuous on the sets $[0,T]\times[-r_0,r_0]$ and $[0,T]\times[0,T]\times[-r_0,r_0]$, respectively, $\omega_f^T(\varepsilon)\to 0$ and $\omega_{r_0}^T(u,\varepsilon)\to 0$ as $\varepsilon\to 0$, and, therefore, from the last estimate, it follows

$$\omega_0^T(\mathcal{T}X)\le\left(m^*+\frac{2r_0[\phi^*\Psi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\right)\omega_0^T(X)$$

and, consequently,

$$\omega_0^\infty(\mathcal{T}X)\le\left(m^*+\frac{2r_0[\phi^*\Psi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\right)\omega_0^\infty(X).$$

Step 7: $\mathcal{T}$ is contraction with respect to the measure of noncompactness μ.

In fact, linking the results obtained in steps 5 and 6, we deduce

$$\begin{aligned}\mu(\mathcal{T}X) &= \omega_0^\infty(\mathcal{T}X)+c(\mathcal{T}X)\\ &\le \left(m^*+\frac{2r_0\phi^*\Psi(r_0)+2r_0\psi^*}{\Gamma(\alpha+1)}\right)\omega_0^\infty(X)+\left(m^*+\frac{2r_0\psi^*}{\Gamma(\alpha+1)}\right)c(X)\\ &\le \left(m^*+\frac{2r_0\phi^*\Psi(r_0)+2r_0\psi^*}{\Gamma(\alpha+1)}\right)(\omega_0^\infty(X)+c(X))\\ &\le \left(m^*+\frac{2r_0\phi^*\Psi(r_0)+2r_0\psi^*}{\Gamma(\alpha+1)}\right)\mu(X).\end{aligned}$$

Taking into account (a_5), since $m^*+\frac{2r_0\phi^*\Psi(r_0)+2r_0\psi^*}{\Gamma(\alpha+1)}<1$ this proves that $\mathcal{T}X$ is a contraction with respect to the measure of noncompactness μ.

Finally, by Darbo's fixed point theorem, Eq.(1) has at least one solution $x\in BC(R_+)$ with $\|x\|\le r_0$.

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