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*Functional differential equations*

# On the weakly continuous solutions of a coupled system of functional integral equations

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## Abstract

The existence of weak solutions of integral equations and some coupled systems of integral equations have been studied, on some suitable assumptions, by some authors. In this work we concern with a coupled system of nonlinear Volterra-Urysohn functional integral equations in the reflexive Banach space $E$. The existence of weakly continuous solutions, in the Banach space $C[I, E]$, will be proved under some certain assumptions concerning the functional equations that under the integral sign.
As an application, the coupled system of nonlinear Volterra-Hammerstien functional integral equations is also studied.

*Keywords:* Weak solutions, Coupled system, Functional integral equations, Fixed point theorem.

## 1 Introduction and Preliminaries

Let $I=[0,T]$, and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval $I$. Let $E$ be a reflexive Banach space with norm $\|.\|$ and dual $E^*$.
Denote $C[I,E]$ the Banach space of strongly continuous functions $x:I\to E$ with sup-norm.
Indeed the existence of weak solutions of the integral and differential equations and the coupled systems of integral equations have been extensively investigated by a number of authors and there are many interesting results concerning this problems (see[1]-[3], [5]-[7], [9]-[11] and [13]).
The existence of at least one weak solutions for the coupled systems of integral equations of Uryshon type in a reflexive Banach spaces was proved in (see[6]), where $x$ takes values in reflexive Banach spaces, $f_i$ are weakly sequentially continuous and weakly measurable on $I$ and

$$\|f_i(t,s,x(s))\|\le k_i(t,s),\quad i=1,2.$$

In this paper, we study the existence of weak continuous solutions $x,y\in C[I,E]$ of the the coupled system of nonlinear Volterra-Urysohn integral equations

$$x(t)=h_1(t)+\int_0^t f_1(t,s,y(m_1(s)))ds,\quad t\in I \tag{1}$$

$$y(t)=h_2(t)+\int_0^t f_2(t,s,x(m_2(s)))ds,\quad t\in I \tag{2}$$

under another suitable assumptions of the functions $f_i,\quad i=1,2$ where $m_i:[0,T]\to[0,T],\quad m_i(t)\le t_i,\quad i=1,2$ are continuous.
As an application, we set $f_i(t,s,y(m_i(s)))$ by $k_i(t,s)g_i(s,y(m_i(s))),\quad i=1,2,$ and we study the existence of weak continuous solutions $x,y\in C[I,E]$ of the coupled system of nonlinear Volterra-Hammerstien integral equations

$$x(t)=h_1(t)+\int_0^t k_1(t,s)g_1(s,y(m_1(s)))ds,\quad t\in I \tag{3}$$

$$y(t)=h_2(t)+\int_0^t k_2(t,s)g_2(s,x(m_2(s)))ds,\quad t\in I. \tag{4}$$

Now we present some auxiliary results that will be need in this work.
Let $E$ be a Banach space and let $x:I\to E$, then

(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in X^*$, $\phi(x(.))$ is continuous (measurable) at $t_0$ .
(2) A function $h : E \to E$ is said to be sequentially continuous if $h$ maps weakly convergent sequence in $E$ to weakly convergent sequence in $E$ .

If $x$ is weakly continuous on $I$ , then $x$ is strongly measurable and hence weakly measurable (see[4] and[8]). Note that in reflexive Banach spaces weakly measurable functions are pettis integrable (see[8] and [12] for the definition) if and only if $\phi(x(.))$ is Lebesgue integrable on $I$ for every $\phi \in E^*$.

Now we state a fixed point theorem and some propositions which will be used in the sequal (see[10]).

**Theorem 1** *"O'Regan fixed point theorem"*
*Let $E$ be a Banach space and let $Q$ be a nonempty, bounded, closed and convex subset of the space $(C[0,T], E)$ and let $A : Q \to Q$ be a weakly sequentially continuous and assume that $AQ(t)$ is relatively weakly compact in $E$ for each $t \in [0,T]$. Then $A$ has a fixed point in the set $Q$.*

**Proposition 1** *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

**Proposition 2** *Let $E$ be a normed space with $y \neq 0$. Then there exists a $\phi \in E^*$ with $\|\phi\| = 1$ and $\|y\| = \phi(y)$.*

## 2 Existence of weak solutions

This section deals with the existence of weak continuous solutions for the coupled system of nonlinear Volterra-Urysohn integral equations (1)-(2).
Let $f_i : I \times E \to E, \quad i = 1, 2$ be a nonlinear single-valued maps, assume that $f_i, \quad i = 1, 2$ satisfy the following assumptions:
(I) $f_i(t, s, .), \quad i = 1, 2$ are weakly sequentially continuous in $x \in E$ for each $t, s \in I \times I$.
(II) $f_i(t, ., x(.)), \quad i = 1, 2$ are weakly measurable on $I$ for each $x \in E$.
(III) $f_i(., s, x(.)), \quad i = 1, 2$ are continuous on $I$ for each $x \in E$.
(IV) $\|f_i(t, s, x(s))\| \leq a_i(t, s) + b_i \|x(s)\|, \; i = 1, 2, \; a_i : I \times I \to R_+$ is integrable in $s$ and continuous in $t$, $b_i$ are constants,
and $\int_0^t a_i(t, s)ds < M_i, \quad i = 1, 2 \quad, t \in I$.

(V) $h_i(t) : I \to I, \quad i = 1, 2$ are continuous functions.
(VI) $b_i T < 1, \quad i = 1, 2.$

**Definition 1** *Let $X$ be the class of all ordered pairs $(u, v), \quad u, v \in C[I, E]$, with norm $\|(u, v)\| = \|u\| + \|v\|$.*

**Definition 2** *By a weak solution of the coupled system (1)-(2) we mean the ordered pair of functions $(x, y) \in X, \quad x, y \in C[I, E]$ such that*

$$\phi(x(t)) = \phi(h_1(t)) + \int_0^t \phi(f_1(t, s, y(m_1(s))))ds, \quad t \in I$$

$$\phi(y(t)) = \phi(h_2(t)) + \int_0^t \phi(f_2(t, s, x(m_2(s))))ds, \quad t \in I$$

*for all $\phi \in E^*$.*

Now for the existence of a weak continuous solution of (1)-(2) we have the following theorem

**Theorem 2** *Let the assumptions (I)-(VI) be satisfied. Then the coupled system of the nonlinear functional integral equations (1)-(2) has at least one weak continuous solution $(x, y) \in X, \quad x, y \in C[I, E]$.*

**Proof.** Let

$$\begin{aligned} U(t) &= (x(t), y(t)) \\ &= (h_1(t) + \int_0^t f_1(t, s, y(m_1(s)))ds,\ h_2(t) + \int_0^t f_2(t, s, x(m_2(s)))ds), \quad t \in I \end{aligned}$$

Let $A$ be defined by

$$AU(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t))$$

where

$$A_1 y(t) = h_1(t) + \int_0^t f_1(t, s, y(m_1(s)))ds, \quad t \in I$$

$$A_2 x(t) = h_2(t) + \int_0^t f_2(t, s, x(m_2(s)))ds, \quad t \in I$$

Now let the set $Q_r$ be defined by

$$Q_r = \{U = (x, y) : x, y \in C[I, E], \quad \|y(t)\| \le r_1,\ \|x(t)\| \le r_2, \quad r = r_1 + r_2\}$$

Let $U \in Q_r$ be an arbitrary ordered pair, then we have from proposition 2

$$\begin{aligned}
\|A_1 y(t)\| &= \phi(A_1 y(t)) \\
&= \phi(h_1(t)) + \int_0^t \phi(f_1(t,s,y(m_1(s))))ds \\
&= \|h_1\| + \int_0^t \|f_1(t,s,y(m_1(s)))\|ds \\
&\leq \|h_1\| + \int_0^t \{a_1(t,s) + \|y(m_1(s))\|\}ds \\
&\leq \|h_1\| + \int_0^t a_1(t,s)ds + b_1 \int_0^t \|y(m_1(s))\|ds \\
&\leq \|h_1\| + M_1 + b_1 \int_0^t \|y(m_1(s))\|ds \\
&\leq \|h_1\| + M_1 + b_1 r_1 T
\end{aligned}$$

Therefore,
$\|A_1 y(t)\| \leq \|h_1\| + M_1 + b_1 r_1 T = r_1$, where $r_1 = \frac{\|h_1\| + M_1}{1-(b_1 T)}$.

Similarly,
$\|A_2 x(t)\| \leq \|h_2\| + M_2 + b_2 r_2 T = r_2$, where $r_2 = \frac{\|h_2\| + M_2}{1-(b_2 T)}$.
Since

$$\begin{aligned}
\|AU(t)\| &= \|A_1 y(t)\| + \|A_2 x(t)\| \\
&\leq \|h_1\| + \|h_2\| + M_1 + M_2 + b_1 r_1 T + b_2 r_2 T
\end{aligned}$$

Then

$$\|AU\| \leq r$$

Hence, $AU \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e. $A : Q_r \to Q_r$, and the class of functions $\{AQ_r\}$ is uniformly bounded.
Now $Q_r$ is nonempty, closed, convex and uniformly bounded.
As a consequence of proposition 1, then $AQ_r$ is relatively weakly compact.

Now, we prove that $A : X \to X$. For this, let $x, y \in C[I, E]$. Let $t_1, t_2 \in I$, $t_1 < t_2$ (without loss of generality assume that $\|AU(t_2) - AU(t_1)\| \neq 0$)), then

$$\begin{aligned}
A_1y(t_2)-A_1y(t_1) &= (h_1(t_2)-h_1(t_1))+\int_0^{t_2} f_1(t_2,s,y(m_1(s)))ds \\
&- \int_0^{t_1} f_1(t_1,s,y(m_1(s)))ds \\
&= (h_1(t_2)-h_1(t_1))+\int_0^{t_1} f_1(t_2,s,y(m_1(s)))ds \\
&+ \int_{t_1}^{t_2} f_1(t_2,s,y(m_1(s)))ds-\int_0^{t_1} f_1(t_1,s,y(m_1(s)))ds \\
&= (h_1(t_2)-h_1(t_1)) \\
&+ \int_0^{t_1}[f_1(t_2,s,y(m_1(s)))-f_1(t_1,s,y(m_1(s)))]ds \\
&+ \int_{t_1}^{t_2} f_1(t_2,s,y(m_1(s)))ds
\end{aligned}$$

Therefore as a consequence of proposition 2, we obtain

$$\begin{aligned}
\|A_1y(t_2)-A_1y(t_1)\| &= \phi(A_1y(t_2)-A_1y(t_1)) \\
&= \phi(h_1(t_2)-h_1(t_1))+\int_0^{t_1}\phi(f_1(t_2,s,y(m_1(s))) \\
&- f_1(t_1,s,y(m_1(s)))))ds+\int_{t_1}^{t_2}\phi(f_1(t_2,s,y(m_1(s))))ds \\
&= \|h_1(t_2)-h_1(t_1)\| \\
&+ \int_0^{t_1}\|f_1(t_2,s,y(m_1(s)))-f_1(t_1,s,y(m_1(s)))\|ds \\
&+ \int_{t_1}^{t_2}\|f_1(t_2,s,y(m_1(s)))\|ds \\
&\le \|h_1(t_2)-h_1(t_1)\| \\
&+ \int_0^{t_1}\|f_1(t_2,s,y(m_1(s)))-f_1(t_1,s,y(m_1(s)))\|ds \\
&+ \int_{t_1}^{t_2}a_1(t_2,s)ds+b_1\int_{t_1}^{t_2}\|y(m_1(s)))\|ds
\end{aligned}$$

Similarly,

$$\begin{aligned}\|A_2x(t_2)-A_2x(t_1)\| &\le \|h_2(t_2)-h_2(t_1)\|\\ &+ \int_0^{t_1}\|f_2(t_2,s,x(m_2(s)))-f_2(t_1,s,x(m_2(s)))\|ds\\ &+ \int_{t_1}^{t_2}a_2(t_2,s)ds+b_2\int_{t_1}^{t_2}\|x(m_2(s)))\|ds\end{aligned}$$

Therefore,

$$\begin{aligned}\|AU(t_2)-AU(t_1)\| &= \|(A_1y(t_2),A_2x(t_2))-(A_1y(t_1),A_2x(t_1))\|\\ &= \|((A_1y(t_2)-A_1y(t_1)),(A_2x(t_2),A_2x(t_1)))\|\\ &= \|A_1y(t_2)-A_1y(t_1)\|+\|A_2x(t_2)-A_2x(t_1)\|\end{aligned}$$

which proves that $A:X\to X$.

It remains to prove that $A$ is weakly sequentially continuous.
Let $\{U_n\}$ be a sequence in $Q_r$ converges weakly to $U \quad \forall t\in I$, then $\{y_n\},\ \{x_n\}$ converges weakly to $y,\ x$, respectively, i.e. $y_n(t)\rightharpoonup y,\ x_n(t)\rightharpoonup x, \quad \forall t\in I$ weakly.
Since $f_1(t,s,y(m_1(s)))$ and $f_2(t,s,x(m_2(s)))$ are weakly sequentially continuous in $y$ and $x$, then $f_1(t,s,y_n(m_1(s)))$ and $f_2(t,s,x_n(m_2(s)))$ converges weakly to $f_1(t,s,y(m_1(s)))$ and $f_2(t,s,x(m_2(s)))$ respectively.
Thus $\phi(f_1(t,s,y_n(m_1(s))))$ and $\phi(f_2(t,s,x_n(m_2(s))))$ converges strongly to $f_1(t,s,y(m_1(s)))$ and $\phi(f_2(t,s,x(m_2(s))))$ respectively.
By applying Lebesgue dominated convergence theorem for Pettis integral, then we get

$$\begin{aligned}\phi(\int_0^t f_1(t,s,y_n(m_1(s)))ds) &= \int_0^t \phi(f_1(t,s,y_n(m_1(s))))ds\\ &\to \int_0^t \|f_1(t,s,y(m_1(s)))\|ds, \quad \forall\ \phi\in E^*, \quad t\in I\end{aligned}$$

and

$$\begin{aligned}\phi(\int_0^t f_2(t,s,x_n(m_2(s)))ds) &= \int_0^t \phi(f_2(t,s,x_n(m_2(s))))ds\\ &\to \int_0^t \|f_2(t,s,x(m_2(s)))\|ds, \quad \forall\ \phi\in E^*, \quad t\in I.\end{aligned}$$

Then $\phi(AU_n((t))) \to \phi(AU((t))), \quad \forall\phi \in E^*, \quad t \in I$. Hence, $A$ is weakly sequentially continuous (i.e. $AU_n(t) \to AU(t), \forall t \in I$ weakly).
Since all conditions of O'Regan theorem are satisfied, then the operator $A$ has at least one fixed point $U \in Q_r$, and hence the coupled system of nonlinear functional integral equations (1)-(2) has at least one weak solution.

## 3 Application

This section, as an application, deals with the existence of weak continuous solution for the coupled system of nonlinear Volterra-Hammerstien integral equations (3)-(4).
Let $g_i : I \times E \to E, \quad i = 1, 2$ be a nonlinear single-valued maps, assume that $g_i, k_i \quad i = 1, 2$ satisfy the following assumptions:
(i) $g_i(t, .), \quad i = 1, 2$ are weakly sequentially continuous in $x \in E$ for each $t \in I$.
(ii) $g_i(., x), \quad i = 1, 2$ are weakly measurable on $I$.
(iii) There exists an integrable functions $a_i(t)$ and a constants $b_i > 0$ such that $\|g_i(t, x(m_i(t)))\| \leq a_i(t) + b_i \, \|x(m_i(t))\|, \; i = 1, 2$.
(iv) $k_i \; : \; I \times I \; \to \; R_+$ are integrable in $s$ and continuous in $t$ and $\int_0^t k_i(t, s)a_i(s)ds < M_i, \quad \int_0^t k_i(t, s)ds < K_i, \quad i = 1, 2 \quad , t \in I$.
(v) $h_i(t) : I \to I, \quad i = 1, 2$ are continuous functions.

**Definition 3** *By a weak solution of the coupled system (3)-(4) we mean the ordered pair of functions* $(x, y) \in X, \quad x, y \in C[I, E]$ *such that*

$$\phi(x(t)) = \phi(h_1(t)) + \int_0^t k_1(t, s)\phi(g_1(s, y(m_1(s))))ds, \quad t \in I$$

$$\phi(y(t)) = \phi(h_2(t)) + \int_0^t k_2(t, s)\phi(g_2(s, x(m_2(s))))ds, \quad t \in I$$

*for all* $\phi \in E^*$.

Now for the existence of a weak continuous solution of (3)-(4) we have the following theorem

**Theorem 3** *Let the assumptions (i)-(v) be satisfied. Then the coupled system of the nonlinear functional integral equations (3)-(4) has at least one weak continuous solution* $(x, y) \in X, \quad x, y \in C[I, E]$.

**Proof.** Let

$$f_i(t, s, y(m_i(s))) = k_i(t, s)g_i(s, y(m_i(s))), \quad i = 1, 2.$$

From the assumptions on $g_i$, $k_i$, we find that the assumptions of Theorem 2.2 are satisfied. Then result follows.

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