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Adaptive and robust control

Passification with respect to given input and output for linear SISO systems[1]

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Abstract

New version of passification with respect to given input and output is proposed. It can be considered as an extension of passification introduced in (Fradkov, IEEE CDC 2008). Necessary and sufficient conditions for the proposed version of passification are obtained for linear SISO systems. Solution is based on KYP lemma and Meerov's results concerning high gain stabilization.

Keywords: Passification, absolute stability, KYP lemma, linear systems

1 Introduction

Absolute stability theory plays an important role in the history of control. According to the pioneer paper [11] a nonlinear system with a right hand side consisting of a linear part and a nonlinearity is absolutely stable if it is globally asymptotically stable for all nonlinearities from certain class. A gallery of

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seminal works on absolute stability [10], [13], [14], [16], [9], [15], [1], [8] among others belong to a golden heritage of control. Absolute stability criteria impose restrictions on the system linear part which may be either valid or not. What to do if those restrictions are not valid? In such a case it was proposed by [2] to design a feedback rendering system absolutely stable. The designs proposed in [2] are based on circle and Popov criteria.

However an approach of [2] requires knowledge of the linear part parameters. If the uncertainty is strong, i.e. the system parameters may vary in a broad range then the circle and Popov criteria may be violated for some values of system parameters. In the case of a strong uncertainty a reasonable solution may be based on adaptive control. An adaptive control loop may tune the feedback controller in order to achieve the desired properties of the closed loop system. The system may be called adaptively absolutely stable if there exists an adaptive feedback rendering the system globally asymptotically stable for all uncertainties of its linear part and all nonlinearities from certain class. The adaptive absolute stabilization problem was first formulated and solved for a special case of the circle criterion in [3]. It was assumed that the nonlinearities are bounded by so-called infinite sector, which means that their graphs are located in the first and third quadrants in the plane.

Recall that the circle criterion for infinite sector is equivalent to SPR or passivity of the linear part. It is shown in [3] that condition of passifiability (hyperminimum-phaseness) is sufficient for existence of adaptive algorithm rendering the system absolutely stable. Moreover it is shown that hyperminimum-phaseness is equivalent to existence of quadratic Lyapunov function depending on the system state and adaptation parameters.

The main limitation of [3] was consideration of the systems with matched nonlinearities: control input and nonlinearities were located in the same equations. This limitation arises from the corresponding passification conditions. In order to deal with nonmatched nonlinearities a new version of passification problem extending the result of [7] is introduced in this paper. Necessary and sufficient conditions for a new passification problem are given for SISO systems. They are obtained using version of KYP lemma [17] and Meerov's results concerning high gain stabilization [12]. In the further research these passification results will be applied for adaptive absolute stabilization problem with nonmatched nonlinearities.

2 Problem statement

In this section a new version of passification problem will be introduced. Let us start with conventional passification definition. Consider linear system[2]:

$$\dot{x} = Ax + Bu, \qquad y = C^*x, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, A, B, C are matrices of appropriate size.

Let system (1) be closed with a feedback

$$u = -Ky + v, \tag{2}$$

where v is a new input.

System (1) is called G-passifiable from input u to output y if there exists a feedback (2) such that closed system (1), (2) is passive from input v to output y. This is equivalent to the validity of the following matrix relations for certain matrix K with matrix $H = H^* > 0$:

$$HA(K) + A^*(K)H < 0, \quad HB = CG, \tag{3}$$

$$A(K) = A + BK^*C^*. \tag{4}$$

Let $\delta(s) = \det(sI - A)$. Let the rank of the matrix B be equal to m. It can be shown [5, 6] that system (1) is G-passifiable if and only if it is hyper-minimum-phase (HMP) i.e. polynomial $\varphi(s) = \delta(s) \det G^*W(s)$ is Hurwitz and $G^*C^*B = (G^*C^*B)^* > 0$.

Consider now linear system with two vector inputs and two vector outputs:

$$\begin{aligned} \dot{x} &= Ax + Bu + B_1v, \\ y &= C^*x, \quad y_1 = C_1^*x, \end{aligned} \tag{5}$$

To check conventional G-passifiability one needs to find a feedback $u = -Ky$ for system (5) with $B = B_1$ and $C = C_1$ such that (3) holds. Now introduce a new passifiability problem for the case of different B, B_1 and C, C_1. We will seek feedback $u = -Ky$ for system (5) such that besides (3) an additional condition $HB_1 = C_1G_1$ holds for certain matrix G_1. It can be considered as the passifiability for (5) from input $[u, u_1]$ to output $[y, y_1]$ using feedback $u = -Ky$. We will call it as *passification with respect to given input and output.*

[2]The following notations will be used:
$\mathbb{C}^n$ and $\mathbb{R}^n$ are complex and real n-dimensional Euclidean spaces correspondingly;
$col\{a_1, \ldots, a_n\}$ is a column vector with components $a_1, \ldots, a_n$;
$diag\{a_1, \ldots, a_n\}$ is a diagonal matrix with elements $a_1, \ldots, a_n$;
Asterisk is a transposition for real matrices and Hermitian conjugate for complex matrices;
$T > 0$ for Hermitian matrix T means its positive definiteness;
$\mathrm{Re}\, T = (T + T^*)/2$

3 Main result

Further we will be dealing with passifiability with respect to given input and output of SISO linear systems

$$\dot{x} = Ax + bu + b_1u_1, \quad y = c^*x, \quad y_1 = c_1^*x, \tag{6}$$

where $x \in \mathbb{R}^n$ is a state vector, $u, u_1 \in \mathbb{R}^1$ are two scalar inputs and $y, y_1 \in \mathbb{R}^1$ - are two scalar outputs.

Let (6) be closed with the following feedback

$$u = -ky. \tag{7}$$

We need to find conditions for existence of feedback (7) such that system (6), (7) is passive from vector input $[u, u_1]$ to vector output $[y, y_1]$. Passifiability is equivalent to the validity of the following matrix relations for certain k and $H = H^* > 0$:

$$\begin{gathered} HA(k) + A(k)^*H < 0, \quad A(k) = A - kbc^*, \\ HB = C, \end{gathered} \tag{8}$$

where $B = [b, b_1]$, $C = [c, c_1]$.

Introduce notation: $\delta(s) = \det(sI - A)$, $\delta(s,k) = \det(sI - A(k))$ are characteristic polynomials of open and closed loop system (6), transfer functions $W^{bc}(s) = c^*(sI - A)^{-1}b$, $W^{b_1c_1}(s) = c_1^*(sI - A)^{-1}b_1$, $W^{b_1c}(s) = c^*(sI - A)^{-1}b_1$, $W^{bc_1}(s) = c_1^*(sI-A)^{-1}b$. Analogously one can define transfer functions of closed system $W^{bc}(s,k) = c^*(sI - A(k))^{-1}b$, $W^{b_1c_1}(s,k)$, $W^{b_1c}(s,k)$, $W^{bc_1}(s,k)$. The numerator of transfer function $W^{bc}(s)$ is $\varphi^{bc}(s) = \delta(s)W^{bc}(s)$. Analogously one can define numerators of other transfer functions.

Using matrix determinant lemma one can show that the following identity holds:

$$\delta(s,k) = \delta(s) + k\varphi^{bc}(s). \tag{9}$$

Using identities

$$\begin{aligned} (sI-A(k))^{-1} &= (sI - A + kbc^*)^{-1} = (sI - A)^{-1}\left[I - \frac{kbc^*(sI - A)^{-1}}{1 + kc^*(sI - A)^{-1}b}\right] \\ &= [(sI - A)^{-1} + k(sI - A)^{-1}W^{bc}(s) \\ &\quad - k(sI - A)^{-1}bc^*(sI - A)^{-1}][1 + kW^{bc}(s)]^{-1} \end{aligned}$$

one can express partial transfer functions:

$$\begin{aligned} W^{bc}(s,k) &= W^{bc}(s)[1+kW^{bc}(s)]^{-1} = \varphi^{bc}(s)[\delta(s)+k\varphi^{bc}(s)]^{-1} \\ W^{b_1c}(s,k) &= W^{b_1c}(s)[1+kW^{bc}(s)]^{-1} = \varphi^{b_1c}(s)[\delta(s)+k\varphi^{bc}(s)]^{-1} \\ W^{bc_1}(s,k) &= W^{bc_1}(s)[1+kW^{bc}(s)]^{-1} = \varphi^{bc_1}(s)[\delta(s)+k\varphi^{bc}(s)]^{-1} \\ W^{b_1c_1}(s,k) &= [W^{b_1c_1}(s)+kW^{bc}W^{b_1c_1}-kW^{b_1c}W^{bc_1}][1+kW^{bc}(s)]^{-1} \\ &= \varphi^{b_1c_1}(s)[\delta(s)]^{-1} - k\varphi^{b_1c}\varphi^{bc_1}[\delta(s)(\delta(s)+k\varphi^{bc}(s))]^{-1} \end{aligned} \tag{10}$$

Let $W(s,k) = C^*(sI - A(k))^{-1}B$. Then according to KYP lemma [17] in its semisingular form the existence of matrix $H = H^* > 0$, satisfying (8) is equivalent to the following conditions:

i) $\det(sI - A(k))$ is a Hurwitz polynomial;

ii) $\operatorname{Re} W(i\omega,k) > 0$ for all $\omega \in \mathbb{R}^1$;

iii)$\lim_{\omega\to\infty} \omega^2 \operatorname{Re} W(i\omega,k) > 0$.

The results of this paper are based on the following passifiability conditions.

Theorem 1 *Consider system (6). Conditions (8) of passifiability with respect to given input and output hold for all sufficiently large k if and only if the following conditions hold:*

a) polynomial $\varphi^{bc}(s)$ *is Hurwitz with positive coefficients and has degree* $n-1$

b) polynomials φ^{b_1c} *and* φ^{bc_1} *have degrees not exceeding* $n-2$

c) polynomial $\varphi^{b_1c_1}$ *has either degree* $n-1$ *or* $n-2$

d) $\varphi^{bc}(i\omega)\varphi^{b_1c_1}(i\omega) - \varphi^{b_1c}(i\omega)\varphi^{bc_1}(i\omega) \neq 0 \quad \forall\omega$

e) $Re\,\delta(i\omega)\bar{\varphi}^{b_1c_1}(i\omega) > 0$ *for all* ω.

Proof of this theorem is given below.

3.1 Sufficiency

Assume that (a)-(e) hold and check fulfillment of conditions i), ii), iii) for all big enough k:

i) From representation (9) and from results from [12] it follows that $\delta(s,k)$ is Hurwitz for all big enough k if and only if $\varphi^{bc}(s)$ is a Hurwitz polynomial with positive coefficients and either it has degree $n-1$ or it has degree $n-2$ and $\delta_{n-1} > 0$, where δ_{n-1} is the coefficient of $\delta(s)$ of degree $n-1$. Therefore from condition (a) it follows that $\delta(s,k)$ is Hurwitz for all big enough k.

ii) First, let us show that $\det \mathrm{Re}W(i\omega,k) \neq 0$. Indeed,

$$\begin{aligned}\det \mathrm{Re}W(i\omega,k) &= W_{bc}W_{b_1c_1}(1+kW_{bc})^{-1} - kW_{bc}W_{b_1c}W_{bc_1}(1+kW_{bc})^{-2} - \\ &\qquad W_{b_1c}W_{bc_1}(1+kW_{bc})^{-2} \\ &= W_{bc}W_{b_1c_1}(1+kW_{bc})^{-1} - (1+kW_{bc})W_{b_1c}W_{bc_1}(1+kW_{bc})^{-2} \\ &= (W_{bc}W_{b_1c_1} - W_{b_1c}W_{bc_1})(1+kW_{bc})^{-1} \\ &= \frac{\varphi^{bc}(i\omega)\varphi^{b_1c_1}(i\omega) - \varphi^{b_1c}(i\omega)\varphi^{bc_1}(i\omega)}{\delta(i\omega)(\delta(i\omega)+k\varphi^{bc}(i\omega))} \neq 0\end{aligned} \tag{11}$$

by assumptions (d). It has no poles by assumptions (a) and (d) which impliy that $\delta(i\omega) \neq 0$ and $\delta(i\omega) + k\varphi^{bc}(i\omega))$ is Hurwitz for big enough k.

Relation (11) implies invertibility of $\mathrm{Re}W(i\omega,k)$ and one can get:

$$\begin{aligned}\mathrm{Re}W(i\omega,k) &= W(i\omega,k) + W^*(i\omega,k) = W(I + W^{-1}W^*) \\ &= W(W^{-*} + W^{-1})W^* = W\mathrm{Re}W^{-1}W^*.\end{aligned}$$

Thus condition $\mathrm{Re}W(i\omega,k) > 0$ is equivalent to $\mathrm{Re}W^{-1}(i\omega,k) > 0$. Denote $\Delta(i\omega) = \varphi^{bc}(i\omega)\varphi^{b_1c_1}(i\omega) - \varphi^{b_1c}(i\omega)\varphi^{bc_1}(i\omega)$. Calculate $W^{-1}(i\omega,k)$. Dependency on $i\omega$ is omitted:

$$\begin{aligned}W^{-1}(i\omega,k) &= \frac{1+kW_{bc}}{W_{bc}W_{b_1c_1} - W_{b_1c}W_{bc_1}}\begin{bmatrix} W^{b_1c_1} - k\frac{W^{b_1c}W^{bc_1}}{1+kW^{bc}}, & -W^{b_1c}(1+kW^{bc})^{-1} \\ -W^{bc_1}(1+kW^{bc})^{-1}, & W^{bc}(1+kW^{bc})^{-1}\end{bmatrix} \\ &= \frac{1}{W_{bc}W_{b_1c_1} - W_{b_1c}W_{bc_1}}\begin{bmatrix} W^{b_1c_1}(1+kW^{bc}) - kW^{b_1c}W^{bc_1}, & -W^{b_1c} \\ -W^{bc_1}, & W^{bc}\end{bmatrix} \\ &= \frac{\delta^2}{\Delta}\begin{bmatrix} \frac{\varphi^{b_1c_1}(\delta+k\varphi^{bc}) - k\varphi^{b_1c}\varphi^{bc_1}}{\delta^2}, & -\frac{\varphi^{b_1c}}{\delta} \\ -\frac{\varphi^{bc_1}}{\delta}, & \frac{\varphi^{bc}}{\delta}\end{bmatrix} \\ &= \frac{1}{\Delta}\begin{bmatrix} \varphi^{b_1c_1}(\delta+k\varphi^{bc}) - k\varphi^{b_1c}\varphi^{bc_1}, & -\varphi^{b_1c}\delta \\ -\varphi^{bc_1}\delta, & \varphi^{bc}\delta\end{bmatrix} \\ &= \begin{bmatrix} k + \varphi^{b_1c_1}\delta\Delta^{-1}, & -\varphi^{b_1c}\delta\Delta^{-1} \\ -\varphi^{bc_1}\delta\Delta^{-1}, & \varphi^{bc}\delta\Delta^{-1}\end{bmatrix}\end{aligned}$$

Apparently

$$
\operatorname{Re} W^{-1}(i\omega,k) = \begin{bmatrix} k + \operatorname{Re}\{\varphi^{b_1c_1}\delta\Delta^{-1}\}, & -(\varphi^{b_1c}\delta\Delta^{-1} + \bar{\varphi}^{bc_1}\bar{\delta}\bar{\Delta}^{-1}) \\ -(\varphi^{bc_1}\delta\Delta^{-1} + \bar{\varphi}^{b_1c}\bar{\delta}\bar{\Delta}^{-1}), & \operatorname{Re}\{\varphi^{bc}\delta\Delta^{-1}\} \end{bmatrix}
$$
$$
= \frac{1}{|\Delta|^2} \begin{bmatrix} k|\Delta|^2 + \operatorname{Re}\{\varphi^{b_1c_1}\delta\bar{\Delta}\}, & -(\varphi^{b_1c}\delta\bar{\Delta} + \bar{\varphi}^{bc_1}\bar{\delta}\Delta) \\ -(\varphi^{bc_1}\delta\bar{\Delta} + \bar{\varphi}^{b_1c}\bar{\delta}\Delta), & \operatorname{Re}\{\varphi^{bc}\delta\bar{\Delta}\} \end{bmatrix} \quad (12)
$$

Let us show that $\operatorname{Re} W^{-1}(i\omega,k)$ is positive definite using Sylvester criterion. We should check that its top left corner element and its determinant are positive. The top left corner element is following:

$$
k|\Delta|^2 + \operatorname{Re}\{\varphi^{b_1c_1}\delta\bar{\Delta}\}. \quad (13)
$$

Term $|\Delta|$ is positive and separated from zero on imaginary axis since it tends to infinity for large ω and reaches maximal and minimal value for bounded ω as a continuous function on compact. Degree of $|\Delta|^2$ is either $4n-4$ or $4n-6$ dependent on degree of $\varphi^{b_1c_1}$. Degree of $\operatorname{Re}\{\varphi^{b_1c_1}\delta\bar{\Delta}\}$ is the same. Therefore (13) is positive for all big enough k.

Determinant of $\operatorname{Re} W^{-1}(i\omega,k)$ is as follows:

$$
\begin{aligned}
&\operatorname{Re}\{\varphi^{bc}\delta\bar{\Delta}\}(k|\Delta|^2 + \operatorname{Re}\{\varphi^{b_1c_1}\delta\bar{\Delta}\}) - |\varphi^{bc_1}\delta\bar{\Delta} + \bar{\varphi}^{b_1c}\bar{\delta}\Delta|^2 \\
&\quad = k|\Delta|^2|\varphi^{bc}|^2\operatorname{Re}\{\delta\bar{\varphi}^{b_1c_1}\} - k|\Delta|^2\operatorname{Re}\{\varphi^{bc}\delta\bar{\varphi}^{b_1c}\bar{\varphi}^{bc_1}\} \\
&\quad\quad + \operatorname{Re}\{\varphi^{bc}\delta\bar{\Delta}\}\operatorname{Re}\{\varphi^{b_1c_1}\delta\bar{\Delta}\} - |\varphi^{bc_1}\delta\bar{\Delta} + \bar{\varphi}^{b_1c}\bar{\delta}\Delta|^2 \quad (14)
\end{aligned}
$$

The first term is positive and separated from zero on imaginary axis. Straightforward calculation can show that under assumptions (b) and (c) the first term has the largest degree. Therefore determinant will be positive for all ω and large enough k.

iii) Calculate the limit of each element of matrix $\omega^2\operatorname{Re} W(i\omega,k)$. Let us start with the top left corner element. For large ω we have:

$$\begin{aligned}
\omega^2 \mathrm{Re}\, W^{bc}(i\omega, k) &= \omega^2 \mathrm{Re}\, \frac{\varphi^{bc}(i\omega)}{\delta(i\omega) + k\varphi^{bc}(i\omega)} \\
&= \omega^2 \mathrm{Re}\, \frac{(i\omega)^{n-1}\varphi^{bc}_{n-1} + (i\omega)^{n-2}\varphi^{bc}_{n-2} + \mathcal{O}(\omega^{n-3})}{(i\omega)^n + (i\omega)^{n-1}\delta_{n-1} + k(i\omega)^{n-1}\varphi^{bc}_{n-1} + \mathcal{O}(\omega^{n-2})} \\
&= \mathrm{Re}\, \frac{\omega^2\varphi^{bc}_{n-1} - i\omega\varphi^{bc}_{n-2} + \mathcal{O}(1)}{i\omega + \delta_{n-1} + k\varphi^{bc}_{n-1} + \mathcal{O}(\omega^{-1})} \\
&= \mathrm{Re}\, \frac{(\omega^2\varphi^{bc}_{n-1} - i\omega\varphi^{bc}_{n-2})(\delta_{n-1} + k\varphi^{bc}_{n-1} - i\omega) + \mathcal{O}(\omega)}{\omega^2 + (\delta_{n-1} + k\varphi^{bc}_{n-1} + \mathcal{O}(\omega^{-1}))^2} \\
&= \frac{\omega^2\varphi^{bc}_{n-1}\delta_{n-1} + k\omega^2(\varphi^{bc}_{n-1})^2 - \omega^2\varphi^{bc}_{n-2} + \mathcal{O}(\omega)}{\omega^2 + (\delta_{n-1} + k\varphi^{bc}_{n-1} + \mathcal{O}(\omega^{-1}))^2}.
\end{aligned} \tag{15}$$

Therefore

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W^{bc}(i\omega, k) = \varphi^{bc}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{bc}_{n-2} \tag{16}$$

Analogously one can show that

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W^{b_1c}(i\omega, k) = \varphi^{b_1c}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{b_1c}_{n-2} \tag{17}$$

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W^{bc_1}(i\omega, k) = \varphi^{bc_1}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{bc_1}_{n-2} \tag{18}$$

Finally calculate limit of the bottom right-corner element $\mathrm{Re}\, W^{b_1c_1}(i\omega, k)$. It is can be expressed in the following form

$$\mathrm{Re}\, W^{b_1c_1}(i\omega, k) = \mathrm{Re} W^{b_1c_1}(i\omega) - kW^{b_1c}(i\omega)W^{bc_1}(i\omega)[1 + kW(i\omega)]^{-1} \tag{19}$$

Consider the first term for large ω:

$$\begin{aligned}
\omega^2 \mathrm{Re} W^{b_1c_1}(i\omega) &= \omega^2 \mathrm{Re} \frac{(i\omega)^{n-1}\varphi^{b_1c_1}_{n-1} + (i\omega)^{n-2}\varphi^{b_1c_1}_{n-2} + \mathcal{O}(\omega^{n-3})}{(i\omega)^n + (i\omega)^{n-1}\delta_{n-1} + \mathcal{O}(\omega^{n-2})} \\
&= \mathrm{Re} \frac{\omega^2\varphi^{b_1c_1}_{n-1} - i\omega\varphi^{b_1c_1}_{n-2} + \mathcal{O}(1)}{i\omega + \delta_{n-1} + \mathcal{O}(\omega^{-1})} \\
&= \mathrm{Re} \frac{(\omega^2\varphi^{b_1c_1}_{n-1} - i\omega\varphi^{b_1c_1}_{n-2})(\delta_{n-1} - i\omega) + \mathcal{O}(\omega)}{\omega^2 + \delta^2_{n-1}} \\
&= \frac{(\omega^2\varphi^{b_1c_1}_{n-1} - i\omega\varphi^{b_1c_1}_{n-2})(\delta_{n-1} - i\omega) + \mathcal{O}(\omega)}{\omega^2 + \delta^2_{n-1}} \\
&= \frac{\omega^2\varphi^{b_1c_1}_{n-1}\delta_{n-1} - \omega^2\varphi^{b_1c_1}_{n-2} + \mathcal{O}(\omega)}{\omega^2 + \delta^2_{n-1}}
\end{aligned}$$

Thus limit of $\omega^2 \mathrm{Re} W^{b_1 c_1}(i\omega)$ is $\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2}$. Now consider the second term for large ω. Straightforward calculation yields

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W^{b_1 c}(i\omega) W^{bc_1}(i\omega)[1 + kW^{bc}(i\omega)]^{-1} = -\varphi^{b_1 c}_{n-1}\varphi^{bc_1}_{n-1} \tag{20}$$

Since both limits in (19) exist, the limit of their difference also exists. It equals to:

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W^{b_1 c_1}(i\omega, k) = \varphi^{b_1 c_1}_{n-1}\delta_{n-1} + k\varphi^{b_1 c}_{n-1}\varphi^{bc_1}_{n-1} - \varphi^{b_1 c_1}_{n-2} \tag{21}$$

We will apply Sylvester criterion for checking positive definiteness of $\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W(i\omega, k)$. Top left corner element is positive for $\varphi^{bc}_{n-1} \neq 0$ and large enough k:

$$\varphi^{bc}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{bc}_{n-2} > 0. \tag{22}$$

Let us calculate its determinant:

$$\begin{aligned}
\det \lim_{\omega\to\infty} &\omega^2 \mathrm{Re}\, W(i\omega, k) \\
&= (\varphi^{bc}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{bc}_{n-2})(\varphi^{b_1 c_1}_{n-1}\delta_{n-1} + k\varphi^{b_1 c}_{n-1}\varphi^{bc_1}_{n-1} - \varphi^{b_1 c_1}_{n-2}) \\
&\quad - (\varphi^{bc_1}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{bc_1}_{n-2})(\varphi^{b_1 c}_{n-1}(\delta_{n-1} + k\varphi^{bc}_{n-1}) - \varphi^{b_1 c}_{n-2}) \\
&= k^2((\varphi^{bc}_{n-1})^2\varphi^{b_1 c}_{n-1}\varphi^{bc_1}_{n-1} - (\varphi^{bc}_{n-1})^2\varphi^{b_1 c}_{n-1}\varphi^{bc_1}_{n-1}) \\
&\quad + k((\varphi^{bc}_{n-1})^2(\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2})\varphi^{bc_1}_{n-1}\varphi^{bc}_{n-1}(\varphi^{b_1 c}_{n-1}\delta_{n-1} - \varphi^{b_1 c}_{n-2})) \\
&\quad + (\varphi^{bc}_{n-1}\delta_{n-1} - \varphi^{bc}_{n-2})(\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2}) \\
&\quad - (\varphi^{bc_1}_{n-1}\delta_{n-1} - \varphi^{bc_1}_{n-2})(\varphi^{b_1 c}_{n-1}\delta_{n-1} - \varphi^{b_1 c}_{n-2})
\end{aligned} \tag{23}$$

Since $\varphi^{b_1 c}_{n-1} = \varphi^{bc_1}_{n-1} = 0$ the previous chain of inequalities can be continued:

$$\begin{aligned}
\det \lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W(i\omega, k) &= k(\varphi^{bc}_{n-1})^2(\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2}) \\
&\quad + (\varphi^{bc}_{n-1}\delta_{n-1} - \varphi^{bc}_{n-2})(\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2}).
\end{aligned} \tag{24}$$

Assumptions (c) and (d) implies that either $\varphi^{b_1 c_1}_{n-1}\delta_{n-1} - \varphi^{b_1 c_1}_{n-2}$ or $\varphi^{b_1 c_1}_{n-1} = 0$ and $\varphi^{b_1 c_1}_{n-2} < 0$. Therefore (24) is positive.

3.2 Necessity

Assume that i), ii), iii) hold for all large enough k and prove conditions (a)-(e).

From point i) of the sufficiency part polynomial $\varphi(s)$ is Hurwitz polynomial with positive coefficients and either it has degree $n-1$ or it has degree $n-2$ and $\delta_{n-1} > 0$.

From point iii) of the sufficiency part limit of top left corner element $\mathrm{Re}\, W(i\omega, k)$ should be positive that is (22) should be hold. Since φ_{n-2}^{bc} is positive inequality (22) is hold only if $\varphi_{n-1}^{bc} \neq 0$ which implies (a).

Since $\mathrm{Re}\, W(i\omega, k)$ satisfies i), ii), iii) it is SPR and its numerator is Hurwitz. Therefore $\det \mathrm{Re} W(i\omega, k) \neq 0$ for all ω. Using expression (11) we see that $(W_{bc}W_{b_1c_1} - W_{b_1c}W_{bc_1}) \neq 0$ which implies (d).

Similarly to sufficiency part invertibility of $\mathrm{Re}\, W(i\omega, k)$ implies positive definiteness of $\mathrm{Re}\, W^{-1}(i\omega, k)$. Evaluating its determinant (14) we get that it should be positive by Sylvester criterion. If φ^{bc_1} or φ^{b_1c} has degree $n-1$ then positiveness of (14) will be violated since in this case negative term $-|\varphi^{bc_1}\delta\bar{\Delta} + \bar{\varphi}^{b_1c}\bar{\delta}\Delta|^2$ has degree $8n-6$, while other terms have degrees $8n-8$. Also it will be violated if degree of $\varphi^{b_1c_1}$ is less than $n-2$. Moreover positiveness of (14) implies (e).

4 Conclusion

In the paper a new version of passification with respect to given input and output is proposed. The necessary and sufficient conditions for a new passification problem for SISO systems are given.

In the future these results will be applied for adaptive absolute stabilization based on circle and Popov criteria for the case when inputs and nonlinearities belong to different equations. Besides that these results might be extended to a general MIMO case and to more general quadratic constraints, different from infinite sectors.

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