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Control problems in nonlinear systems

ASTATISM IN NONLINEAR CONTROL SYSTEMS WITH APPLICATION TO ROBOTICS

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Abstract.

The notion of astatism broadly used in classical linear control theory is extended to nonlinear systems. Some basic assertions concerning with the properties of astatic systems are presented. A special attention is paid to robust control problems of Lagrangian systems and robotic manipulators as a particular case. It is shown that PID control ensure robust stabilization of a desired position and tracking with a bounded error if the desired velocities are small enough.

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1 Introduction

The astatism is an extensively used notion in classical linear theory. Astatic systems have zero steady-state error if disturbances are constant or tend to constant values. Moreover, for linear systems the astatism ensure a bounded reaction under any unbounded disturbances having bounded derivatives. In linear case the astatism conditions are very simple. It is necessary and sufficient to have stable transfer function with a numerator having zero root. It is of practical interest to show what is a nonlinear astatic system, to state some conditions ensuring the astatism and to clear up if the properties of linear astatic systems are saved in nonlinear case. The paper gives some answers for these questions. Moreover, it is demonstrated that the general results can be used for the explanation of robustness of PID feedback in non-linear Lagrangian systems and in robotics especially.

2 Astatism and boundedness of reactions

Let a system be presented in the form

$$\dot{x} = f(x, w(t)), \quad x(0) = x_0; \qquad y = g(x, w(t)) \tag{1}$$

where x is the state vector, y is the output vector, $w(t)$ is the vector of disturbances (inputs), and $f(x, w)$ satisfies conditions ensuring existence and uniqueness of solutions to (1) under any admissible $w(t) \in W, \quad t \in [0, \infty)$.

Definition. *The system (1) is* **internally astatic** *if it has an isolated equilibrium point $x = 0$ and this equilibrium is asymptotically stable under any $w(t) = w = const, w \in W$.*

The system (1) is **input–output astatic** *if it has an asymptotically stable solution $x = x_\infty(w)$ under any $w(t) = w = const, w \in W$ and*

$$g(x_\infty(w), w) = 0.$$

Let f and g be linear in x and w, or more precisely, let

$$f(x, w) = Ax + Bw, \qquad g(x, w) = Cx + Dw$$

where A, B, C, D are constant matrices. Then the definition presented above coincides with the definition of classical linear theory (see, f.e. [1]) and the astatism conditions are equivalent the following ones:

– A is stable (Hurwitzian),

– $A^{-1}B = 0$ (for internal astatism) or $D - CA^{-1}B = 0$ (for input–output astatism).

However, if f is linear in x only, i.e.

$$f(x, w) = A(w)x + B(w), \tag{2}$$

then the astatism conditions have to include the stability of $A(w)$ under all $w \in W$, i.e. the parametric robustness condition.

It is clear now that the generalized notion of astatism introduced above is much more sophisticated than the traditional notion of linear theory.

It is evident that if the linearized system having structure of right-hand side in the form (2) (with $A(w) = \frac{\partial f}{\partial x}(0, w), B(w) = \frac{\partial f}{\partial w}(0, w)$) is internally astatic then the original non-linear system (1) is internally astatic too.

The basic problem under consideration is the following one. Let the system (1) be internally astatic. Is it true that the output $y(t)$ is bounded under any $w(t) \in W$ which are unbounded but have a bounded derivative?

Theorem 1[1]**.** *Let the system (1) be internally astatic on a set of time–constant disturbances $w \in W$ and moreover, there exists a Lyapunov function $V(x, w)$ such that*

$$V(x, w) \to \infty \qquad \text{under} \quad \|x\| \to \infty \qquad \text{for any} \quad w \in W \tag{3}$$

and its time derivative $\dot V$ defined on trajectories of the system (1) under $w =$ const satisfies the conditions

$$\dot V \le -\beta \|x\|^2, \quad \left\| \frac{\partial V}{\partial w} \right\| \le \gamma \|x\|^2 \qquad \text{under} \quad \|x\| \le R, \quad w \in W \tag{4}$$

Let for any $\delta < \beta\gamma^{-1}$

$$\|\dot w(t)\| \le \delta, \qquad w(t) \in W \tag{5}$$

then there exist $R_0 > 0$ such that

$$\|x_0\| \le R_0 \implies \lim_{t \to \infty} \|x(t)\| = 0.$$

[1]A similar assertion is contained in [2, Ch.5]

Theorem 2. *Let $x = 0$ be asymptotically stable equilibrium of the system (1) under any constant $w \in W$ and moreover there exists a Lyapunov function $V(x, w)$ satisfying (3) (i.e. radially unbounded) and*

$$\dot{V} \leq -Q(x), \qquad \text{under} \quad \|x\| \leq R$$

where $Q(x) > 0, \|x\| \neq 0$, $Q(0) = 0$, $\frac{Q(x)}{\|x\|} \to \infty$ under $\|x\| \to \infty$, and

$$\left\|\frac{\partial V}{\partial w}\right\| \leq c\|x\|, \qquad c = const > 0, \qquad \text{under any} \quad w \in W \tag{6}$$

Then, under any $\Delta > 0$, one can show $\delta > 0$ and $R_0 > 0$ such that

$$\overline{\lim_{t\to\infty}}\|x\| \leq \Delta$$

if $w(t)$ satisfying (5) and $\|x_0\| \leq R_0$

The Theorem 2 gives some sufficient conditions under which the astatism yields a boundedness of reactions if the disturbances have a bounded rate of changements. Naturally, the conditions are much harder and "more local" then in the linear case.

Note that the conditions of theorems 1,2 concerning with a "regular" behavior of Lyapunov functions in w are nontrivial.

Let $f(x, w)$ be continuously differentiable in x and w, and uniformly Lipschitzian in x under all $w \in W$. Let the system (1) have an asymptotically stable solution under $w = const \in W$. Then following Massera theorem (see, f.e. [3]), one can state only that there exists a smooth Lyapunov function $V(x, w)$ such that under $\|x\| \leq R$:

$$a(\|x\|, w) \leq V(x, w) \leq b(\|x\|, w), \qquad \dot{V}(x, w) \leq -c(\|x\|, w),$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq d(w), \qquad \left\|\frac{\partial V}{\partial w}\right\| \leq e(w),$$

where a, b, c are smooth positive anywhere (except zero x) functions monotonously increasing with $\|x\|$ (i.e. class $\mathcal{K}$ functions) and d, e are non-negative smooth function. Sometimes one can show a limited upper bounds which are independent on w; and it is possible not only for a bounded W. In those cases any solutions to (1) are **uniformly ultimately bounded**, i.e. $x(t)$ converges in a ball if $\|\dot{w}\|$ and $\|x_0\|$ are small enough. In other terminology the system (1) is **dissipative**. Proof of this assertion is analogous to the proof of Malkin theorem on dissipativity under nonvanishing bounded disturbances (see, f.e. [3]).

3 Astatic servomechanisms

In many cases the description (1) does not allow to express completely the disturbance influence on the system behavior. In particular, it takes place in the tracking problems where a desired trajectory plays role of a disturbance, and zero value of tracking errors is the desired equilibrium point (or it is required to ensure uniform ultimate boundedness of those errors).

Let, f.e., it be desired that the state $x(t)$ of the closed–loop system

$$\dot{x} = F(x,u), \qquad u = k(x,x_d),$$

where $k(x,x_d)$ is a static feedback control law ($F(x_d, k(x_d,x_d)) \equiv 0$), converges to a differentiable process $x_d(t)$, i.e. the output $y(t) = x(t) - x_d(t)$ tends to zero. Denoting

$$y = x - x_d \qquad \text{and} \qquad f(x,x_d) = F(x,k(x,x_d))$$

one obtain

$$\dot{y} = f(y + x_d, x_d) - \dot{x}_d$$

If one return to the symbols introduced in the astatism definition, i.e. ($y \to x$, $x_d \to w$), one has

$$\dot{x} = f(x+w,w) - \dot{w} \qquad (f(w,w) \equiv 0)$$

One can see now that the right–hand side depends not only on w as in (1) but on $\dot{w}$ also.

Note that the output coincides here with the state of the system and hence the notion of internal astatism coincides with the notion of input–output astatism.

In more general case if one desires to estimate the tracking errors for the closed–loop system described by the differential equations of n-th order

$$x^{(n)} = F(x,\dots,x^{(n-1)},k(x,\dots,x^{(n-1)},x_d)) \equiv f(x,\dots,x^{(n-1)},x_d)$$

one has to consider behavior of solutions to the equations of the following type

$$x^{(n)} = f(x+w,\dots,x^{(n-1)} + w^{(n-1)},w) - w^{(n)}$$

where x, w stand now for the errors and the desired trajectory.

The examples shown above demonstrate that the generalized description

$$\dot{x} = f_0(x,w) + f_1(x,w,w',\dots,w^{(k)}) \tag{7}$$

where $\quad f_0(0,w) \equiv 0 \quad$ and $\quad f_1(x,w,0,\dots,0) \equiv 0$
be useful in applications.

Note that the astatism property is defined by $f_0(x,w)$ only. However the system reaction depend on f_1 too.

Theorem 3. *Let the system*

$$\dot{x} = f_0(x,w) \tag{8}$$

be astatic and moreover there exists a radially unbounded Lyapunov function $V(x,w)$ such that

$$\left\|\frac{\partial V}{\partial x}\right\| \le c_x\|x\|, \qquad \left\|\frac{\partial V}{\partial w}\right\| \le c_w\|x\|, \qquad w \in W, \quad \|x\| \le R$$

and its time derivative along trajectories of the system (8) under $w = const \in W$ satisfy

$$\dot{V} \le -Q(x), \qquad \text{under} \quad \|x\| \le R$$

where $Q(x) > 0, \|x\| \neq 0$, $Q(0) = 0$, $\frac{Q(x)}{\|x\|} \to \infty$ under $\|x\| \to \infty$.

Let f_1 be Lipschitzian in all arguments.

Then there exist $\delta > 0, R_0 > 0$, such that if

$$\|w', \dots w^{(k)}\| \le \delta, \qquad w(t) \in W, \qquad \|x_0\| \le R_0,$$

then for any solution to the system (7)

$$\overline{\lim_{t\to\infty}}\|x(t)\| \le \Delta,$$

under any given $\Delta > 0$.

Remark. If for any $w \in W$ and $\|x\| \le R$:

$$\|f_1(x,w,w',\dots,w^{(k)})\| \le (L + \beta_2\|x\|)f_2(w',\dots,w^{(k)}),$$

where f_2 is a function such that there exists a continuous function $\bar{f}_2(\delta)$ vanishing under $\delta \to 0$ and satisfying the inequality

$$\|w', \dots w^k\| \le \delta \implies \|f_2(w',\dots,w^{(k)})\| \le \bar{f}_2(\delta),$$

$f_1(x,w,\dots,w^{(k)})$ is not necessary to be Lipschitzian, and $Q(x) = \beta\|x\|^2$, then $\overline{\lim\limits_{t\to\infty}}\|x(t)\| \le \Delta = \frac{\bar{f}_2(\delta)L}{\beta - \beta_2\bar{f}_2(\delta)} \to 0$ under $\delta \to 0$. This remark was used in [4].

4 Integral feedback and astatism

Consider the following system

$$\dot{z} = f(z, u, w), \qquad z(0) = z_0 \tag{9}$$

where u is a control. Choose u in the form of integral feedback

$$\dot{u} = -\mu\psi(z); \qquad u(0) = 0 \tag{10}$$

where $\mu > 0$, $\psi(0) = 0$, $z^T\psi(z) \neq 0$ under $z \neq 0$, .

Under $w = const$ the closed–loop system (9), (10) has the unique equilibrium

$$z = 0, \qquad u = u_\infty(w) = const.$$

The system is astatic if the equilibrium is asymptotically stable. Let under zero control and $w = const \in W$ the system (9) have an asymptotically stable solution

$$z = z_\infty(w), \qquad f(z_\infty(w), 0, w) \equiv 0.$$

The integral feedback (10) is introduced to shift that equilibrium to the desired position $z = 0$ independently on any $w = const \in W$.

First of all, as in the linear theory, one has to know if the stability is not destroyed.

Theorem 4. *Let f and ψ be twice differentiable functions. Let, under any $w \in W$ and $u \in U$, U be bounded and the equation*

$$f(z, u, w) = 0$$

have a root $z_\infty(u, w)$ which is locally exponentially stable equilibrium of the system (9) under $w = const \in W$, $u = const \in U$. Let the system

$$\dot{u} = -\mu\psi(z_\infty(u, w)) \tag{11}$$

have a locally exponentially stable solution.

Then there exists $\bar{\mu} > 0$, such that under any $\mu \in (0; \bar{\mu}]$, the closed–loop system (9), (10) is input–output astatic with input w and output z.

Proof to the theorem 4 is based on classical results by Tikhonov (see, f.e. [5, 1]) and Klimushev [6]. In fact, let us introduce a "slow" time $\tau = \mu t$. Then (9), (10) can be rewritten in a singular perturbed form

$$\frac{du}{d\tau} = -\psi(z), \qquad \mu\frac{dz}{d\tau} = f(z, u, w) \tag{12}$$

Under a small μ, the control u is a "slow" variable in reference to the object state $z(t)$. The system (11) is the reduced one in reference to (12) and it gives an appropriate description of the "slow" variables. The conditions of the Theorem ensure that existence of the exponentially stable solution to the reduced system yields the convergence of $u(t)$ defined by the original system to a constant value. However, due to the properties of $\psi(z)$ it is possible only if $z(t) \to 0$.

Remark 1. If the system (9) has globally asymptotically stable and locally exponentially stable equilibrium (i.e. globally asymptotically stable hyperbolic equilibrium) under $u = const \in U$ and the system (11) also has globally asymptotically stable and locally exponentially stable solution under $w = const \in W$ then the closed–loop system (9), (10) has globally stable equilibrium under positive μ, small enough. It follows from Hoppensteadt theorem [7].

Remark 2. Under some properties of the functions f, ψ ensuring the uniqueness of zero equilibrium point, the condition $z^T\psi(z) \neq 0$ under $z \neq 0$ may be omitted. Moreover all results are true if f, ψ smoothly depend on μ.

5 Astatism of controlled Lagrangian system and a slow tracking

Consider a dynamical system described by the Lagrangian equation

$$A(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Q \tag{13}$$

where, by definition, q is n-vector of generalized coordinates, $A(q)$ is a positive definite inertia matrix, $C(q,\dot{q})\dot{q}$ is a vector of centrifugal and Coriolis forces such that

$$\dot{A}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$$

and $g(q)$ is a vector of gravitational forces,

Let

$$Q = w + v$$

where v is a control and w is an external disturbance. Introduce a proportional–differential (PD) feedback, i.e.

$$v = -K_p(q - q_d) - K_v\dot{q} + \hat{u} \tag{14}$$

where K_p, K_v are positive definite matrix of gains and $q_d = const$ defines a desired position.

Under $\hat{u} = const$, one can show [8] that the system (13), (14) has globally asymptotically stable (GAS) equilibrium $q = \bar{q}, \quad \dot{q} = 0$, satisfying the condition

$$g(\bar{q}) = w + \hat{u} - K_p(\bar{q} - q_d) \tag{15}$$

if only

$$K_p > \alpha I, \tag{16}$$

where α is a Lipschitz constant for $g(q)$.

Introduce now $\hat{u}$ as an additional integral feedback

$$\hat{u} = -F(u), \qquad \dot{u} = \mu(q - q_d) \tag{17}$$

where $F(u) = [F_1(u_1), \dots, F_n(u_n)]^T$ and $F_i(u_i)$ are twice continuously differentiable functions such that

$$\begin{gathered} u_i F_i(u_i) > 0 \qquad \text{under} \quad u_i \neq 0, \quad i = 1 \dots n \\ |F_i(u_i)| \geq \bar{F} \qquad \text{under} \quad |u_i| > \bar{U} \qquad \text{and} \qquad \frac{dF_i(u_i)}{du_i} > 0 \quad \text{under} \quad |u_i| \leq \bar{U} \\ \bar{F} > \max_{1 \leq i \leq n} \{|w_i| + |g_i(q_d)|\}, \qquad w \in W \quad \forall i = 1 \dots n \end{gathered} \tag{18}$$

Theorem 5. *Under the conditions (16), (18) and $\mu > 0$ small enough the closed–loop system (13), (14), (17) has GAS equilibrium $(q, u) = (q_d, u_\infty) \equiv (q_d, -F^{-1}[g(q_d) - w])$.*

The result follows from the Theorem 4 and Remark 1 if one uses special types of Lyapunov functions (energy like one as in [9, 10, 11] and Lur'e – Postnikov like one: $\left[V(u, w) = \int_0^{u - u_\infty} \{F(s + u_\infty) - F(u_\infty)\}^T ds\right]$) for the fast and reduced systems corresponding to (13), (14) and (17) with $q = \bar{q}$ given by (15) and for their first approximation systems.

It proves the astatism of the Lagrangian system with PID feedback in reference to constant external forces.

For the particular case of linear feedback that result was shown early in [12].

Using the Theorem 3 and the Lyapunov function presented in [4] one can show now that PID feedback ensures a tracking of a desired trajectory $q_d(t)$ with a bounded error if $q_d(t)$ is changed slowly enough.

Theorem 6. *Let $q_d(t)$ be twice differentiable and there exist constants*

$$a_1 > 0, \qquad \alpha, a_2, c_1, c_2, d_1, d_2 \geq 0$$

such that for all $x, y, z \in \Re^n$

$$a_1 I \leq A(x) \leq (a_2 + c_2\|x\| + d_2\|x\|^2)I,$$
$$\|C(x,y)z\| \leq (c_1 + d_1\|x\|)\|y\|\|z\|, \qquad \|g(x) - g(y)\| \leq \alpha\|x - y\|;$$

and either the desired motion is bounded or $d_1, c_2, d_2 = 0$.

Moreover, let there exist $\varepsilon > 0$ *and* $R > 0$ *such that*

$$K_v > \varepsilon \max_{\|x\| \leq R} \{A(x+q_d) + (c_1+d_1\|x+q_d\|)\|x\|I\}, \qquad \varepsilon(K_p - \alpha I) > K_i > 0$$

and there exist $\delta_1, \delta_2 > 0$ *such that*

$$\|\dot{q}_d\| \leq \delta_1, \qquad \|\ddot{q}_d\| \leq \delta_2$$

Then all solutions to the system (13), (14), (17) are uniformly ultimately bounded.

6 Conclusion

The results presented in the paper allow to extend the useful notion of astatism to nonlinear systems. It is necessary to have in mind that in non–linear systems the astatism does not ensure a boundedness of tracking errors if the desired velocities are very large. However, some numerical simulations for robotic manipulators show that the upper level of admissible velocities may have practically reasonable high value.

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