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ON WEAKLY NONLINEAR SHALLOW WATER WAVE EQUATIONS

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Abstract.

Using the method of Lie group of transformations and the direct method, solitary wave and explode decay mode solutions of a combined KdV-Boussinesq equation governing weakly nonlinear shallow water waves have been derived.

1 Introduction

Nonlinearity and dispersion are two of the most fundamental concepts in wave motions in nature. The nonlinear shallow water equations governed by the hyperbolic type lead to the wave breaking phenomena. These equations which neglect dispersion develop a vertical slope and a multiple-valued wave profile. Breaking is prevented by including dispersive effects into the shallow water theory. In 1895, Korteweg and de Vries showed that long waves, in water of relatively shallow depth, could be described by a nonlinear equation which is now known as Korteweg de Vries equation[1)]. Periodic solutions of this equation were found in terms of Jacobian elliptic functions cn, known as cnoidal waves. The infinite period counterpart of this solution is the single hump solitary wave solution moving with no change in shape, size and speed. Another example of an equation governing long waves on the surface of shallow water is the Boussinesq equation which include waves moving to both left and right[2)].

In this paper, we consider a combined KdV and Boussinesq equation governing long waves in shallow water. Considering travelling wave solutions, we reduce the basic equations to a second order ordinary differential equation. In section 3, using the method of Lie group of transformations[3,4)], we reduce it to a first order ordinary differential equation. In section 4, we derive using the direct method[5,6)], periodic solutions in terms of elliptic functions and the corresponding solitary wave and the explode decay mode solutions. In section 5, we plot the solitary wave and the explode decay mode solutions.

2 Weakly nonlinear shallow water equations

We consider an inviscid and incompressible fluid of constant depth h. We take the $(x, y)-$ plane as the undisturbed free surface with the $z-$ axis positive upward. The free surface elevation above the undisturbed depth h is $z = \eta(x, y, t)$, so that the free surface is at $z = h + \eta$ and $z = 0$ is the horizontal rigid bottom.

If $\phi(x, y, z, t)$ is the velocity potential of an unbounded fluid lying between the rigid bottom $z = 0$ and the free surface $z = \eta(x, y, t)$ as shown in the figure below, then the nonlinear system of equations for the classical water waves is,

$$\bigtriangledown^2 \phi \;=\; 0, \quad 0 < z < \eta + h, \quad -\infty < x,\, y < \infty, \tag{2.1}$$

$$\phi_t + \frac{1}{2}(\bigtriangledown \phi)^2 + g\eta \;=\; 0, \quad z \;=\; h + \eta, \tag{2.2}$$

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z \;=\; 0, \quad z \;=\; h + \eta, \tag{2.3}$$

$$\phi_x h_x + \phi_y h_y + \phi_z \;=\; 0, \quad z \;=\; 0, \tag{2.4}$$

In the linear theory of surface water waves, two parameters $\epsilon \;=\; \dfrac{a}{h}$, and $\kappa \;=\; ak$, where a is the surface wave amplitude and k is the wave number, must be small. To characterize nonlinear shallow water waves, we introduce two fundamental parameters

$$\epsilon \;=\; \frac{a}{h}, \quad \delta \;=\; \frac{h^2}{l^2} \tag{2.5}$$

where, l is a typical horizontal length like wavelength λ and the following non-dimensional variables:

$$(x^*, y^*) \;=\; \frac{1}{l}(x, y), \;\; z^* \;=\; \frac{z}{h}, \;\; t^* \;=\; \frac{ct}{l}, \;\; \eta^* \;=\; \frac{\eta}{a}, \;\; \phi^* \;=\; \frac{h}{alc}\phi, \tag{2.6}$$

where $c \;=\; \sqrt{gh}$ is the shallow water wave speed. Using (2.5) and (2.6), (2.1) – (2.4) can be written, dropping the asterisks as,

$$\delta(\phi_{xx} + \phi_{yy}) + \phi_{zz} \;=\; 0, \tag{2.7}$$

$$\phi_t + \frac{\epsilon}{2}(\phi_x^2 + \phi_y^2) + \frac{\epsilon}{2\delta}\phi_z^2 + \eta \;=\; 0, \;\; z \;=\; 1 + \epsilon\eta, \tag{2.8}$$

$$\delta\{\eta_t + \epsilon(\phi_x\eta_x + \phi_y\eta_y)\} - \phi_z \;=\; 0, \;\; z \;=\; 1 + \epsilon\eta, \tag{2.9}$$

$$\phi_z \;=\; 0, \;\; z \;=\; 0. \tag{2.10}$$

We expand ϕ in terms of δ and write

$$\phi \;=\; \phi_0 + \delta\phi_1 + \delta^2\phi_2 + ... \tag{2.11}$$

and then substitute in $(2.7)-(2.9)$. Retaining terms upto order δ, ϵ in (2.8) and δ^2, ϵ^2 and $\delta\epsilon$ in (2.9), we get

$$\phi_{0t} - \frac{\delta}{2}(u_{tx} + v_{ty}) + \eta + \frac{1}{2}\epsilon(u^2 + v^2) \;=\; 0, \tag{2.12}$$

$$\delta[\{\eta_t + \epsilon(u\eta_x + v\eta_y)\} + (1+\epsilon\eta)(u_x + v_y)] \;=\; \frac{\delta^2}{6}[(\bigtriangledown^2 u)_x + (\bigtriangledown^2 v)_y] \tag{2.13}$$

Differentiation of (2.12) with respect to x and then with respect to y gives

$$u_t + \epsilon(uu_x + vv_x) + \eta_x - \frac{1}{2}\delta(u_{txx} + v_{txy}) \;=\; 0, \tag{2.14}$$

$$v_t + \epsilon(uu_y + vv_y) + \eta_y - \frac{1}{2}\delta(u_{txy} + v_{tyy}) \;=\; 0, \tag{2.15}$$

(2.13) can be simplified as

$$\eta_t + \{u(1+\epsilon\eta)\}_x + \{v(1+\epsilon\eta)\}_y \;=\; \frac{\delta}{6}\{(\bigtriangledown^2 u)_x + (\bigtriangledown^2 v)_y\}. \tag{2.16}$$

$(2.14)-(2.16)$ are the non-dimensional nonlinear shallow water equations. Considering the one-dimensional case, $(2.14)-(2.16)$ reduce to

$$u_t + \epsilon\, uu_x + \eta_x - \frac{1}{2}\delta\, u_{txx} \;=\; 0, \tag{2.17}$$

$$\eta_t + \{u(1+\epsilon\eta)\}_x - \frac{1}{6}\delta\, u_{xxx} \;=\; 0. \tag{2.18}$$

An alternative system equivalent to the nonlinear evolution equations (2.17)–(2.18) known as Boussinesq equations can be derived from the nonlinear shallow water theory, retaining both ϵ and δ order terms with $\delta < 1$. This system, in dimensional variables, is given by

$$\eta_t + \{(h+\eta)u\}_x \;=\; 0, \tag{2.19}$$

$$u_t + uu_x + g\eta_x \;=\; \frac{1}{3}h^2 u_{xxt}. \tag{2.20}$$

(2.19) – (2.20) describe the evolution of long water waves that travel in both positive and neagative x directions. The corresponding weakly nonlinear shallow water equations are given by

$$\eta_t + (hu)_x \;=\; 0, \tag{2.21}$$

$$u_t + uu_x + g\eta_x \;=\; \frac{1}{3}h^2 u_{xxt}. \tag{2.22}$$

3 Group invariant solutions

We define

$$hu \;=\; \frac{\partial V}{\partial t},\; \eta \;=\; -\frac{\partial V}{\partial x}, \tag{3.1}$$

so that,

$$\frac{\partial^2 V}{\partial t^2} - gh\,\frac{\partial^2 V}{\partial x^2} + \frac{1}{2h}\,\frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial t}\Big)^2 \;=\; \frac{h^2}{3}\,\frac{\partial^4 V}{\partial x^2\,\partial t^2}. \tag{3.2}$$

Travelling wave solutions can be found by assuming $z \;=\; x - Ut$ which reduces (3.2) to

$$(U^2 - gh)\frac{d^2V}{dz^2} + \frac{U^2}{2h}\frac{d}{dz}\Big(\frac{dV}{dz}\Big)^2 - \frac{U^2h^2}{3}\frac{d^4V}{dz^4} \;=\; 0. \tag{3.3}$$

Integrating (3.4) once and setting $\dfrac{dV}{dz} \;=\; w$, we get

$$\frac{d^2w}{dz^2} \;=\; \alpha\, w^2 + \beta\, w + \gamma, \tag{3.4}$$

where,

$$\alpha \;=\; \frac{3}{2h^3}, \tag{3.5a}$$

$$\beta = \frac{3(U^2 - gh)}{U^2 h^2}\, w, \tag{3.5b}$$

$$\gamma = -\frac{3B}{U^2 h^2}. \tag{3.5c}$$

We write (3.4) in the form

$$w_2 = \alpha\, w^2 + \beta\, w + \gamma, \tag{3.6}$$

where,

$$w_2 = \frac{d^2 w}{dz^2}. \tag{3.7}$$

Let $X = \xi(z,w)\dfrac{\partial}{\partial z} + \tau(z,w)\dfrac{\partial}{\partial w}$ be the infinitesimal generator of the one-parameter Lie-group of transformations, so that the invariance criterion for (3.6) is

$$\tau^{(2)} = \alpha\tau^2 + \beta\tau + \gamma, \tag{3.8}$$

where,

$$\tau^{(2)} = \tau_{zz} + (2\tau_{zw} - \xi_{zz})w_1 + (\tau_{ww} - 2\xi_{zw})w_1^2$$

$$-\xi_{ww}\, w_1^3 + (\tau_w - 2\xi_z)(\alpha\, w^2 + \beta\, w + \gamma)$$

$$-3\xi_w w_1(\alpha\, w^2 + \beta\, w + \gamma). \tag{3.9}$$

Here, subscripts denote differentiation with respect to the corresponding variables and $w_1 = \dfrac{dw}{dz}$. The resulting set of determining equations for $\xi(z,w)$ and $\tau(z,w)$ are:

$$\xi_{ww} = 0, \tag{3.10a}$$

$$\tau_{ww} - 2\,\xi_{zw} = 0, \tag{3.10b}$$

$$\xi_w = 0, \tag{3.10c}$$

$$-3\xi_w \gamma + 2\tau_{zw} - \xi_{zz} = 0, \tag{3.10d}$$

$$\tau_{zz} + \gamma(\tau_w - 2\xi_z) = \alpha\tau^2 + \beta\tau + \gamma. \tag{3.10e}$$

ξ is a function of z alone from (3.10c) and (3.10a) is automatically satisfied. From (3.10b), $\tau_{ww} = 0$ which gives rise to

$$\tau = K_1 w + K_2. \tag{3.11}$$

(3.10d) and (3.10e) give

$$\xi = \frac{1}{2} K_1 z + K_3. \tag{3.12}$$

Here, K_1, K_2 and K_3 are arbitrary constants. Also, (3.10e) gives rise to the trivial infinite parameter Lie group.

(3.11) and (3.12) give a nontrivial three-parameter Lie group of transformations acting on $(z, w)-$space with infinitesimal generators given by

$$X_1 = \frac{1}{2}\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \; X_2 = \frac{\partial}{\partial w}, \; X_3 = \frac{\partial}{\partial z}. \tag{3.13}$$

Thus the differential equation (3.6) has a 3-dimensional solvable Lie-algebra.

Now, we reduce the second order ordinary differential equation from invariance under translation (K_3). Obvious invariants of the first extension of $z^* = z + \epsilon, \; w^* = w$, are

$$R(z, w) = w, \; S(z, w, w_1) = w_1. \tag{3.14}$$

Hence, (3.6) becomes

$$S\, dS = (\alpha R^2 + \beta R + \gamma)\, dR. \tag{3.15}$$

Integrating (3.15),

$$\frac{S^2}{2} = \frac{\alpha R^3}{3} + \frac{\beta R^2}{2} + \gamma R + \mu, \tag{3.16}$$

or,

$$(\frac{dw}{dz})^2 = \frac{2\,\alpha}{3}\,w^3 + \beta\,w^2 + 2\gamma\,w + 2\mu. \tag{3.17}$$

4 Solitary wave and explode decay mode solutions

We shall consider two cases when the shallow water wave speed is the same as the travelling wave speed and when they are unequal.

<u>Case1</u> ($U^2 = gh$)

When $U^2 = gh,\ \beta = 0$ and the equation (3.17) reduces to

$$(\frac{dw}{dz})^2 = \frac{2\alpha}{3}\,w^3 + 2\gamma w + 2\mu. \tag{4.1}$$

We assume a solution in the form

$$w = A\,\wp(z), \tag{4.2}$$

where, $\wp$ is the Weierstrass elliptic function which satisfies the well known differential equation

$$(\frac{d\wp}{dz})^2 = 4\,\wp^3 - g_2\wp - g_3. \tag{4.3}$$

g_2 and g_3 are known as the invariants of the Weierstrass elliptic function satisfying the condition

$$g_2^3 - 27\,g_3^2 > 0. \tag{4.4}$$

Substituting (4.2) in (4.1) and equating the coefficients of equal powers of $\wp(z)$, we get

$$4\,A^2 = \frac{2\alpha}{3}\,A^3, \tag{4.5}$$

$$-A^2\,g_2 = 2\,\gamma\,A, \tag{4.6}$$

$$-A^2\,g_3 = 2\,\mu. \tag{4.7}$$

(4.5) – (4.7) give rise to

$$A = \frac{6}{\alpha}, \tag{4.8}$$

$$g_2 = -\frac{\alpha\gamma}{3}, \tag{4.9}$$

$$g_3 = -\frac{\alpha^2\mu}{18}. \tag{4.10}$$

Since the two invariants g_2 and g_3 satisfy (4.4), g_2 must always be positive and hence α and γ should be of opposite signs. Thus the constant of integration B is a positive quantity.

The condition (4.4) puts the restriction on μ as

$$4\,\gamma^3 + 9\,\alpha\,\mu^2 \,<\, 0. \tag{4.11}$$

Therefore, the solution is

$$w(z) = \frac{6}{\alpha}\,\wp(z + \epsilon\,;\, g_2\,, g_3), \tag{4.12}$$

where, ϵ is an integration constant of (4.3).

Thus the exact bounded periodic solution is

$$w(z) = \frac{6}{\alpha}[e_3 + (e_2 - e_3)sn^2(\sqrt{e_1 - e_3}\,z + \epsilon')], \tag{4.13}$$

where ϵ' is an arbitrary real constant and e_1, e_2, e_3 are real roots of the equation

$$4y^3 - g_2 y - g_3 = 0, \tag{4.14}$$

with $e_1 > e_2 > e_3$. Here, sn is the Jacobian sine elliptic function where it is related to the Weierstrass elliptic function by

$$\wp(z) = e_3 + (e_2 - e_3)\,sn^2(\sqrt{e_1 - e_3}\,z + \epsilon'). \tag{4.15}$$

The solitary wave limits are obtained when the period is infinite which occurs when the modulus of the Jacobian elliptic function is equal to unity. The modulus m and the e's are related by

$$m = \frac{e_2 - e_3}{e_1 - e_3}, \tag{4.16}$$

and so in the solitary wave limit, $e_1 = e_2$. Since, e_1, e_2, e_3 are roots of the cubic $4y^3 - g_2 y - g_3 = 0,\ e_1 + e_2 + e_3 = 0$. So, the solitary wave solution is,

$$w(z) = \frac{6}{\alpha}[e_3 + (e_2 - e_3)\, sn^2(\sqrt{e_1 - e_3}\, z + \epsilon')]. \tag{4.17}$$

Using (3.1), we have,

$$u(x,t) = -4h^2\, U\, [e_1 - (e_1 - e_3)\, sech^2\{\sqrt{e_1 - e_3}\,(x - Ut) + \epsilon'\}], \tag{4.18}$$

$$\eta(x,t) = -4h^3\, [e_1 - (e_1 - e_3)\, sech^2\{\sqrt{e_1 - e_3}\,(x - Ut) + \epsilon'\}], \tag{4.19}$$

The corresponding explode-decay mode solutions are given by

$$u(x,t) = -4h^2\, U\, [e_1 + (e_1 - e_3)\, cosech^2\{\sqrt{e_1 - e_3}\,(x - Ut) + \epsilon'\}], \tag{4.20}$$

$$\eta(x,t) = -4h^3\, [e_1 + (e_1 - e_3)\, cosech^2\{\sqrt{e_1 - e_3}\,(x - Ut) + \epsilon'\}], \tag{4.21}$$

<u>*Case*2</u> ($U^2 \neq gh$)

In this case we have the equation

$$(\frac{dw}{dz})^2 = \frac{2\alpha}{3}\, w^3 + \beta\, w^2 + 2\gamma\, w + 2\mu. \tag{4.22}$$

We assume a solution in the form

$$w(z) = C\, tanh^2(\lambda\, z), \tag{4.23}$$

where, C and λ are to be determined in terms of the coefficients in (4.22).

Substituting (4.23) in (4.22) and equating the coefficients of like powers of $tanh^2(\lambda\, z)$ on both sides of the resulting equation,

$$4C^2\lambda^2 = \frac{2\alpha}{3}\, C^3, \tag{4.24}$$

$$-8C^2\,\lambda^2 \;=\; \beta\, C^2, \tag{4.25}$$

$$4\, C^2\, \lambda^2 \;=\; 2\,\gamma\, C, \tag{4.26}$$

$$2\,\mu \;=\; 0. \tag{4.27}$$

From (4.24), we get $C \;=\; \dfrac{6\,\lambda^2}{\alpha}$ and $\lambda^2 \;=\; -\dfrac{\beta}{8}$ and so β should be negative which leads to

$$U^2 \;<\; gh. \tag{4.28}$$

From (4.26), $\lambda^2 \;=\; \dfrac{\gamma}{2\,C}$ and thus the integration constant B should satisfy

$$B \;=\; -\frac{\beta^2\, U^2\, h^2}{16\,\alpha}. \tag{4.29}$$

In this case B is a negative quantity.

Thus the solitary wave solution of (4.22) is

$$w(z) \;=\; \frac{3}{2}\frac{h(gh-U^2)}{U^2}\, tanh^2\{\sqrt{\frac{3(gh-U^2}{8U^2h^2}}\, z\}. \tag{4.30}$$

Using (3.1), we have

$$u(x,t) \;=\; -\frac{3}{2}\frac{(gh-U^2)}{U}\, tanh^2\{\sqrt{\frac{3(gh-U^2)}{8U^2h^2}}\,(x-Ut)\}. \tag{4.31}$$

$$\eta(x,t) \;=\; -\frac{3}{2}\frac{h(gh-U^2)}{U^2}\, tanh^2\{\sqrt{\frac{3(gh-U^2)}{8U^2h^2}}\,(x-Ut)\}. \tag{4.32}$$

The corresponding explode-decay mode solutions are given by

$$u(x,t) \;=\; -\frac{3}{2}\frac{(gh-U^2)}{U}\, coth^2\{\sqrt{\frac{3(gh-U^2)}{8U^2h^2}}\,(x-Ut)\}. \tag{4.33}$$

$$\eta(x,t) \;=\; -\frac{3}{2}\frac{h(gh-U^2)}{U^2}\, coth^2\{\sqrt{\frac{3(gh-U^2)}{8U^2h^2}}\,(x-Ut)\}. \tag{4.34}$$

5 Numerical Results

In case 1, $\beta = 0$. Taking $h = 1$ and $B = 1$, we have $\alpha = 1.5$, $\gamma \approx -0.3061$ and $g_2 \approx 0.1531$. Now, $g_3 = -\frac{\mu}{8}$ where, μ satisfies (4.11). For solitary wave solution, $e_1 \approx 0.113$, $e_2 \approx 0.113$ and $e_3 \approx -0.226$. Thus the solitary wave solution is given by,

$$u(x,t) = -1.415 + 4.245\, sech^2(0.582x - 1.823t). \tag{5.1}$$

This is plotted in Fig.1.

The corresponding explode decay mode solution is plotted in Fig.2 given by

$$u(x,t) = -1.415 - 4.245\, cosech^2(0.582x - 1.823t). \tag{5.2}$$

In case 2, where $U^2 < gh$, we take $h = 1$ and $U = 3$. Thus the solitary wave solution (See Fig.3) and the corresponding explode decay mode solution (See Fig.4) are given respectively by

$$u(x,t) = -0.4\, tanh^2\{\frac{x-3t}{\sqrt{30}}\}, \tag{5.3}$$

$$u(x,t) = -0.4\, coth^2\{\frac{x-3t}{\sqrt{30}}\}, \tag{5.4}$$

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