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Symbolic dynamics

SYMBOLIC ANALYSIS OF THE CHAIN RECURRENT TRAJECTORIES OF DYNAMICAL SYSTEMS

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Abstract.

An application of a new constructive method for the study of global dynamical system structure is presented. The dynamical system is associated with an oriented graph called a symbolic image of the system. The symbolic image can be considered as a finite approximation of the dynamical system. An investigation of the symbolic image gives an opportunity to get a neighborhood of the chain recurrent trajectories set. An algorithm for localization of chain recurrent set is described. A periodic shadowing theorem is proved.

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Key words: dynamical system, symbolic image, chain recurrent point, oriented graph, periodic trajectory.

1 Introduction

At present the methods of symbolic dynamics have wide applications. In this paper one of the variants of such application is presents. Our purpose is to substantiate an algorithm for localization of the chain recurrent set without any preliminary information about the system. All needed estimates can be obtained by the traditional numerical methods of differential equations. The common scheme of our method is the following. At first, the dynamical system is associated with an oriented graph called **Symbolic Image** of the system. Valuable information about the global structure of the system may come from the analysis of this symbolic image. By investigating the symbolic image we can separate the points through which periodic trajectories may pass from those through which periodic trajectories do not pass. This permits the construction of a neighborhood of the chain recurrent set, which includes all the types of returning trajectories (e.g., periodic, almost-periodic, recurrent, homoclinic and so on). Using a special sequence of symbolic images we obtain a sequence of embedded neighborhoods which converges to the chain recurrent set. The proposed construction of the symbolic image is close to the Cell-to-Cell Mappings of C.S.Hsu [6].

We will consider a discrete dynamical system governed by a homeomorphism X defined on a compact manifold M. To describe the continuous version of dynamic system consider a shift operator along trajectories of the system of differential equations which is generated as follows. Let $\dot{x} = f(t,x)$ be a system of ordinary differential equations, where $x \in M$, $f(t,x)$ is a C^1 vector field periodic in t with period ω. Denote its general solution by $F(t,t_0,x)$, $F(t_0,t_0,x) = x$. For investigation of the global evolution of the system, it usually is sufficient to examine a Poincare map $X(x) = F(\omega,0,x)$ which is the shift operator along the trajectories of the system on the period ω. If the system of differential equations is autonomous (i.e., the vector field f does not depend on t), we fix an arbitrary $H \neq 0$ and consider a shift operator of the form $X(x) = F(H,x)$, where $F(t,x)$ is the solution of the autonomous system, $F(0,x) = x$.

2 Chain recurrent set

Let us consider a homeomorphism $X: M \to M$ on a smooth compact C^∞-manifold M.

Definition 1 *Let $\varepsilon > 0$ be given. An infinite sequence $\{x_k : k \in N\}$ is called an ε-trajectory of X if for any k the distance between the image $X(x_k)$ and x_{k+1} is less than ε :*

$$\rho(X(x_k), x_{k+1}) < \varepsilon,$$

where $\rho(x, y)$ is the distance between points x and y on M.

In the majority of cases an exact trajectory is seldom known, and in fact we find only ε-trajectories for sufficiently small positive ε.

Definition 2 *If an ε-trajectory $\{x_k\}$ is periodic, i.e., $x_{k+p} = x_k$ for some $p \in N$, then it is called a p-periodic ε-trajectory and each points $x = x_k$ is called (p, ε)-periodic.*

For example, on the plane R^2 consider a map of the form: $X(x, y) = (y,\ 0.05(1 - x^2)y - x)$. Take the sequence $x_1 = (2, 0)$, $x_2 = (0, -2)$, $x_3 = (-2, 0)$, $x_4 = (0, 2)$, $x_{k+4} = x_k$. This is 4-periodic ε-trajectory with any $\varepsilon > 0.1$.

A p-periodic sequence $\{x_k\}$ is defined by its periodic part $\{x_1,\ x_2,\ ...,\ x_p\}$ putting $x_{k+p} = x_k$. We will say about ε-periodic point x if its period p is not fixed. Denote the set of all ε-periodic points by $Q(\varepsilon)$. It is not difficult to prove the following proposition describing the properties of $Q(\varepsilon)$.

Proposition 1 *(i) The sets $Q(\varepsilon)$, $\varepsilon > 0$ is open.*

(ii) If $\varepsilon_1 > \varepsilon_2$ then $Q(\varepsilon_2) \subset Q(\varepsilon_1)$.

Definition 3 *A point x is called chain recurrent if x is ε-periodic for each positive ε, i.e., there exists a periodic ε-trajectory passing through x. A chain recurrent set, denoted Q, is the set of all the chain recurrent points*

It is know [3] that the chain recurrent set is invariant, closed, and contains periodic, nonwandering and other singular trajectories. It should be remarked that if a chain recurrent point is not periodic then there exists as small as one likes perturbation of X in C^0-topology for which this point is periodic [3]. One may say that a chain recurrent point may become periodic under small C^0-perturbations of the map X. From the definition of the chain recurrent set it immediately follows that

$$Q = \lim_{\varepsilon \to 0} Q(\varepsilon) = \bigcap_{\varepsilon > 0} Q(\varepsilon) \tag{1}$$

Thus the family of open sets $\{Q(\varepsilon),\ \varepsilon > 0\}$ forms a fundamental system of neighborhoods of the chain recurrent set Q.

3 Construction of the Symbolical Image [9]

Let $C = \{M(1), ..., M(n)\}$ be a finite covering of the domain M by closed sets. The set $M(i)$ is named a cell of the covering. For each cell $M(i)$ we define the covering $C(i),\ i = 1, ..., n$, of the image $X(M(i))$ consisting of cells $M(j) \in C$ whose intersections with $X(M(i))$ are not empty :

$$C(i) = \{M(j) : M(j) \cap X(M(i)) \neq \emptyset\}.$$

Set

$$c(i) = \{j : M(j) \cap X(M(i)) \neq \emptyset\}.$$

The cells of the covering $C(i)$ are called the image cells of $M(i)$ under X. Let us construct the oriented graph G (see Fig.1) associating to each cell $M(i)$ the vertex i . The vertices i and j are connected by the oriented edge $i \to j$ if and only if $j \in c(i)$, i.e., the cell $M(j)$ is included in the covering of the cell $M(i)$.

Definition 4 *The constructed graph G is called the symbolic image of X with respect to the covering C.*

The oriented graph G is uniquely determined by its $n \times n$ matrix of transitions $\Pi = (\pi_{ij})$, where $\pi_{ij} = 1$ if and only if there is the oriented edge $i \to j$, otherwise $\pi_{ij} = 0$. It is easily seen that the symbolic image depends on the covering C. By varying C one can change the symbolic image of the mapping X. We can consider the symbolical image as a finite approximation of the mapping X. It is natural to expect this approximation to be more precise provided the mesh of the covering is made smaller.

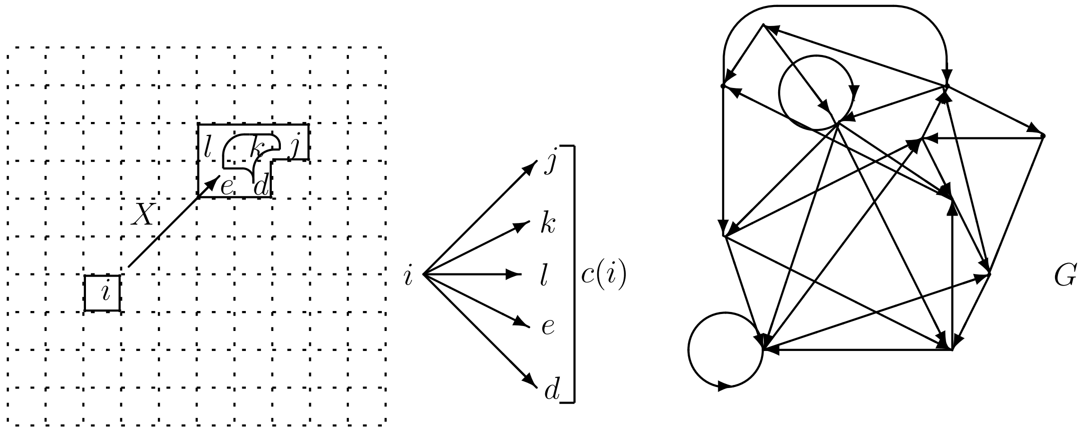

Figure 1: **Construction of Symbolic Image.**

Let

$$diam M(i) = \max(\rho(x, y) : x, y \in M(i))$$

be the diameter of the cell $M(i)$, and d be the largest diameter of the cells from C. Denote a union of the cells $M(j)$ belonging to the covering $C(i)$ by R_i :

$$R_i = \bigcup_{j \in c(i)} M(j)$$

By construction, R_i contains the image $X(M(i))$ of the cell $M(i)$ and is contained in the closed d-neighborhood of the image $X(M(i))$:

$$X(M(i)) \subset R_i \subset \{x : \rho(x, y) \leq d,\ y \in X(M(i))\}.$$

Let q be the largest diameter of the images $X(M(i))$, $i = 1, 2, ..., n$. We define the number r as follows. If a cell $M(k)$ does not belong to the covering $C(i)$ then the distance

$$r_{ik} = \rho(X(M(i)), M(k)) = \min(\rho(x, y) : x \in X(M(i)), y \in M(k))$$

between the cell $M(k)$ and the image $X(M(i))$ is positive. Let r be minimum of such r_{ik}. Since the number of pairs (i, k) described above is finite then $r > 0$. So the number r is the smallest distance between the images $X(M(i))$ and the cells $M(k)$ whose intersections are empty. The number r is called the lower bound of the symbolic image G. The lower bound depends on the covering C. By varying C one can construct a covering for which r is arbitrarily small. The next propositions describe some properties of lower bound.

Proposition 2 *If a point $x \in M(j)$ and $\rho(x, X(M(i))) < r$ then the cell $M(j)$ belongs to the covering $C(i)$.*

Proof. Let $x \in M(j)$. Since the distance $\rho(x, X(M(i))) < r$ then the distance between the cell $M(j)$ and the image $X(M(i))$ is less than r. By the definition of the lower bound, the number r is the smallest distance between the images $X(M(i))$ and cells $M(k)$ which do not intersect. Since $\rho(M(j), X(M(i))) < r$ then the cell $M(j)$ has to intersect the image $X(M(i))$. Consequently, the cell $M(j)$ belongs to the covering $C(i)$. ■

Corollary 1 *The set $R_i = \{\cup M(j) : \ j \in c(i)\}$ contains r-neighborhood of the image $X(M(i))$:*

$$\{x : \ \rho(x, X(M(i)) < r\} \subset R_i.$$

Proposition 3 *The lower bound r satisfies the inequality $r \leq d$.*

Proof. The number of pairs (i, k) such that $r_{ik} = \rho(X(M(i)), M(k)) > 0$ is finite. Hence there exists a pair (i, m) for which $r = r_{im}$ and the segment $[x_j, x_m]$ of the geodesic realizes the value r . This means that there exist points $x_j \in X(M(j))$ and $x_m \in M(m)$, $X(M(j)) \cap M(m) = \emptyset$ such that the length of the segment $[x_j, x_m]$ is equal to r. Since the number r is the smallest distance between the non intersecting images $X(M(i))$ and cells $M(k)$, all points of the open segment $(x_j, x_{m)}$ do not belong to the cell $M(m)$, and belong to some cells from the covering $C(j)$. Since the cells are closed, the point x_m belongs to some cell $M(q)$ from $C(j)$. We have $r = \rho(x_m, X(M(j)))$, $x_m \in M(q)$, $M(q) \cap X(M(j)) \neq \emptyset$. Hence, there is a point $x_q \in M(q) \cap X(M(j))$. From this it follows that $r = \rho(x_m, X(M(j))) \leq \rho(x_m, x_q) \leq diam M(q) \leq d$. ■

4 Structure of the Symbolic Image

Definition 5 *A sequence $\{z_k\}$ of vertices of the graph G is called an admissible path (or simply a path) if for each k the graph G contains the oriented edge $z_k \longrightarrow z_{k+1}$.*

A path $\{z_k\}$ is said to be p-periodic if $z_k = z_{k+p}$ for each $k \in N$. There is a natural connection between the admissible paths on the symbolic image G and the ε -trajectories of the homeomorphism X. It can be said that an admissible path is the trace of an ε-trajectory and vice versa. However, there are some relationships between the parameters d, q, r of the symbolic image and the number ε for which the connections take place.

Proposition 4 *If a sequence $\{z_k\}$ is a path on the symbolic image G and a sequence $\{x_k\}$ is such that $x_k \in M(z_k)$, then the sequence $\{x_k\}$ is an ε-trajectory of X for any $\varepsilon > q + d$.*

Proof. Let $\{z_k\}$ be an admissible path on G. Consequently, there exists an oriented edge $z_k \to z_{k+1}$ for every number k. This means that the cell $M(z_{k+1})$ belongs to the covering $C(z_k)$. Hence, the image $X(M(z_k))$ intersects the cell $M(z_{k+1})$ and

$$\rho(X(x_k), x_{k+1}) \leq diam X(M(z_k)) + diam M(z_{k+1}) \leq q + d.$$

Therefore, the sequence $\{x_k\}$ is an ε-trajectory of X for each $\varepsilon > q + d$. Note that the point x_k is defined by the inclusion $x_k \in M(z_k)$. Hence the constructed sequence $\{x_k\}$ is non uniquely determined by the path $\{z_k\}$ ■

Corollary 2 *If a sequence $\{z_1, z_2, ..., z_p\}$ of vertices is a p-periodic path and a sequence $\{x_1, x_2, ..., x_p\}$ is such that $x_k \in M(z_k)$, then the sequence $\{x_1, x_2, ..., x_p\}$ is a p-periodic ε-trajectory for each $\varepsilon > d + q$.*

Proposition 5 *If a sequence $\{z_k\}$ is a path on the symbolic image G then there exists a sequence $\{x_k\}$, $x_k \in M(z_k)$ which is an ε-trajectory of X for each $\varepsilon > d$.*

Proof. Let $\{z_k\}$ be an admissible path on G. Consequently, there is the oriented edge $z_k \to z_{k+1}$ for each k. This means the intersection $X(M(z_k)) \bigcap M(z_{k+1}) \neq \emptyset$. Hence, there is a point $x_k \in M(z_k)$ such that its image $X(x_k) \in M(z_{k+1})$, i.e.,

$$x_k \in M(z_k) \bigcap X^{-1}(M(z_{k+1})). \tag{2}$$

We can say the pair $x_k \to X(x_k)$ realizes the oriented edge $z_k \to z_{k+1}$. Fix such x_k for each $k \in N$, and check that the sequence $\{x_k\}$ is ε-trajectory for any $\varepsilon > d$. In fact, the image $X(x_k)$ and the point x_{k+1} belong to the cell $M(z_{k+1})$ by construction. Hence, the distance

$$\rho(X(x_k), x_{k+1}) \leq diam M(z_{k+1}) \leq d < \varepsilon.$$

Note that in the considered case the sequence $\{x_k\}$ is non unique as well, however the point x_k is determined by the inclusion (2). ■

Corollary 3 *If a sequence $\{z_1,\ z_2, ...,\ z_p\}$ is a p-periodic path on the symbolic image G then there exists a sequence $\{x_1,\ x_2, ...,\ x_p\}$, $x_k \in M(z_k)$ which is p-periodic ε-trajectory for each $\varepsilon > d$.*

Proposition 6 *If a sequence $\{x_k\}$ is an ε-trajectory of X, $\varepsilon < r$, and $x_k \in M(z_k)$, then the sequence $\{z_k\}$ is an admissible path on the symbolic image G.*

Proof. Let the hypotheses of the proposition hold. Fix some integer $k \in N$. Since $\rho(X(x_k), x_{k+1}) < r$ and $x_k \in M(z_k)$ then $\rho(X(M(z_k)),M(z_{k+1})) < r$. Because r is the smallest distance between $X(M(i))$ and $M(k)$ such that $M(k) \cap X(M(i)) = \emptyset$, the cell $M(z_{k+1})$ has to intersect $X(M(z_k))$. Thus there exists the oriented edge $z_k \to z_{k+1}$ for each $k \in N$, and the sequence $\{z_k\}$ is an admissible path on the symbolic image G. ■

Corollary 4 *If a sequence $\{x_1, x_2, ..., x_p\}$ is a p-periodic ε-trajectory, $\varepsilon < r$ and a sequence $\{z_1, z_2, ..., z_p\}$ is such that $x_k \in M(z_k)$, then the sequence $\{z_1, z_2, ..., z_p\}$ is an admissible p-periodic path on the symbolic image G.*

Definition 6 *A vertex of the symbolic image is called recurrent if a periodic path passes through it.*

Denote by $P(d)$ the union of the cells $M(i)$ for which the vertex i is recurrent, i.e.,

$$P(d) = \{\cup M(i) : i \text{ is recurrent}\}, \tag{3}$$

where d is the largest diameter of the cells $M(i)$. It should be noted that P depends generally on the covering C. However, in what follows we need only to consider the dependence of P on the largest diameter d. Denote by T the union of the cells $M(k)$ for which the vertex k is non recurrent, i.e.,

$$T = \{\cup M(k) : \ k \text{ is non recurrent}\}.$$

The following theorem describes the properties of the sets $P(d)$ and T.

Theorem 1 *(i) The set $P(d)$ is a closed neighborhood of the chain recurrent set. Moreover, $P(d)$ is a subset of ε-periodic point set for each $\varepsilon > q + d$, i.e.,*

$$Q \subset P(d) \subset Q(\varepsilon), \ \varepsilon > q + d. \tag{4}$$

(ii) If the maximum diameter d is small enough then this neighborhood is sufficiently small, i.e., for any neighborhood H of Q there exists $d > 0$ such that $P(d) \subset H$.

(iii) The chain recurrent set Q coincides with the intersection of the sets $P(d)$ for all positive d:

$$Q = \bigcap_{d>0} P(d). \tag{5}$$

(iv) The points of T are not chain recurrent, i.e., $T \cap Q = \emptyset$. Moreover, if $\varepsilon < r$ then there is no periodic ε-trajectory passing through $x \in T$, i.e.,

$$Q(\varepsilon) \cap T = \emptyset, \ \varepsilon < r.$$

Proof. (i) Suppose we are given $\varepsilon_1, \varepsilon_2$, such that $\varepsilon_1 < r < q + d < \varepsilon_2$. At first we prove that

$$Q(\varepsilon_1) \subset P(d) \subset Q(\varepsilon_2). \tag{6}$$

In fact, if a point x belongs to $Q(\varepsilon_1)$, then there exists a periodic ε_1-trajectory $\{x_1, ..., x_p\}$ passing through $x = x_1$. Consider a finite sequence of cells $\{M(z_i) : i = 1, ..., p\}$ such that $x_i \in M(z_i)$. Because $\varepsilon_1 < r$, according to Corollary 4 the sequence $\{z_i\}$ is an admissible periodic path passing through the vertex z_1. Thus the vertex z_1 is recurrent. Hence the cell $M(z_1)$ is contained in $P(d)$. From this it follows that $Q(\varepsilon_1) \subset P(d)$.

Consider a point x belonging to $P(d)$. There exists a cell $M(z)$ such that $x \in M(z)$ and the vertex z is recurrent. In other words on the symbolic image G there exists a periodic path $\{z_1, z_2, ..., z_p\}$, $z_1 = z$. Let us construct a periodic sequence $\{x_1, ..., x_p\}$, such that $x_1 = x$ and $x_i \in M(z_i)$. By Corollary 2, the sequence $\{x_i\}$ is a periodic ε_2-trajectory for any $\varepsilon_2 > q + d$. Hence the point $x = x_i$ lies in $Q(\varepsilon_2)$, i.e. $P(d) \subset Q(\varepsilon_2)$. Thus (4) and (6) hold. From the inclusions $Q \subset Q(\varepsilon_1) \subset P(d)$ it follows that $P(d)$ is a closed neighborhood of the chain recurrent set.

(ii) Let H be arbitrary neighborhood of Q. Since X is a continuous mapping and M is compact, the largest diameter q of the images $X(M(i))$ tends to 0 as the largest diameter of cells $d \to 0$. Set $\varepsilon_2 = \frac{3}{2}(q + d)$. We have $\varepsilon_2 \to 0$ as $d \to 0$. Because $\{Q(\varepsilon), \varepsilon > 0\}$ is a fundamental system of neighborhoods of Q, there is $\varepsilon^* > 0$ such that $Q(\varepsilon^*) \subset H$. Take d such that $\varepsilon_2 = \frac{3}{2}(q + d) \leq \varepsilon^*$. By Proposition 1 and the inclusion (6) we have $P(d) \subset Q(\varepsilon_2) \subset Q(\varepsilon^*) \subset H$.

(iii) As is easily seen, equality (5) follows from the inclusions (6) and the equality (1). Note that generally $P(d_1)$ is not contained in $P(d_2)$ even though $d_1 < d_2$. As we shall see later, there is an algorithm for constructing a sequence of imbedded neighborhoods $P_1 \supset P_2 \supset ...$ of Q.

(iv) We prove this proposition by contradiction. Let $x \in M(k)$ where k is a non recurrent vertex. Let $\{x_1, ..., x_p\}$ be a periodic ε -trajectory passing through $x = x_1$ and $\varepsilon < r$. Consider a sequence $\{z_1, ..., z_p\}$ such that $z_1 = k$ and $x_i \in M(z_i)$. As $\varepsilon < r$, according to Corollary 4, the sequence $\{z_i\}$ is periodic path on the symbolic image G. Because $z_1 = k$, the vertex k is recurrent. The resulting contradiction completes the proof of (iv) and the theorem. ■

Theorem 1 leads us to the following inclusions

$$Q \subset Q(\varepsilon_1) \subset M\backslash T = P(d)\backslash T \subset P(d) \subset Q(\varepsilon_2), \tag{7}$$

where $\varepsilon_1 < r < q + d < \varepsilon_2$. By definition, the set T is closed and $T \cap Q = \emptyset$. Hence, the set $P(d)\backslash T$ is an open neighborhood of the chain recurrent set.

Theorem 1 permits to localize the chain recurrent set without any preliminary information about dynamic system.

5 Localization of chain recurrent set

In this section the algorithm localizing the chain recurrent set is described.

1. Starting with an initial covering C, the symbolic image G of the map X is found. The cells of the initial covering may have arbitrarily large diameter d_0 .

2. The recurrent vertices $\{i_k\}$ of the graph G are recognized. Using the recurrent vertices, a closed neighborhood $P = \{\cup M(i_k) : \ i_k \ is \ recurrent\}$ of the chain recurrent set is found.

3. The cells corresponding to the recurrent vertices $\{M(i_k) : i_k \ is \ recurrent\}$ are partitioned. For example, the largest diameter of the cells may be divided by 2. Thus the new covering is defined.

4. The symbolic image G is constructed for the new covering. It should be noted that the new symbolic image may be constructed on the set $P = \{\cup M(i_k) : \ i_k \ is \ recurrent\}$. In other words, the cells corresponding to non recurrent vertices do not participate in the construction of the new covering and the new symbolic image.

5. Then one goes back to the second step.

Repeating this partitioning process we obtain a sequence of neighborhoods $P_0, P_1, P_2, \ldots$ of the chain recurrent set Q and a sequence of the largest diameters $d_0, d_1, d_2, \ldots$ of cells. The following theorem substantiates the described algorithm for localization of the chain recurrent set.

Theorem 2 *The sequence of sets $P_0, P_1, P_2, \ldots$ offers the following properties:*

(i) the neighborhoods P_k are imbedded one inside the other, i.e.,

$$P_0 \supset P_1 \supset P_2 \supset \ldots \supset Q,$$

(ii) if the largest diameters $d_k \to 0$ as k becomes infinite then

$$\lim_{k\to\infty} P_k = \bigcap_k P_k = Q. \tag{8}$$

Proof. (i) Let $C = \{M(i)\}$ be a covering of M and G be the symbolic image for the covering C. Suppose a new covering NC is produced by taking a partition of C. Denote by NG the symbolic image for NC. For convenience the cells of the new covering NC are designated as $m(i,k)$ so that $\bigcup_k m(i,k) = M(i)$, i.e., the cells $\{m(i,k) : k = 1, 2, \ldots\}$ of the new covering form a partition of the cell $M(i)$. In this case the vertices of NG are designated as (i,k). Since $X(m(i,k)) \subset X(M(i))$ and $m(j,l) \subset M(j)$, the intersection $X(M(i)) \cap M(j) \neq \emptyset$ provided $X(m(i,k)) \cap m(j,l) \neq \emptyset$. This means that there exists a mapping from the graph NG to the graph G such that the vertex (i,k) is mapped to the vertex i and the oriented edge $(i,k) \to (j,l)$ to the oriented edge $i \to j$. From this it follows that each admissible path on the new graph NG is transformed to some admissible path on the graph G. Particularly, the periodic paths on the graph NG are mapped to the periodic paths of the graph G. Hence, the recurrent vertices of the new symbolic image NG are mapped on the recurrent vertices of G. Set

$$P = \{\cup M(i) : i \textit{ is recurrent on } G\},$$

$$NP = \{\cup m(i,k) : (i,k) \textit{ is recurrent on } NG\}.$$

From the previous it follows that NP is a subset of $P : NP \subset P$. Applying this result to each step of the partition, we see that the algorithm gives the sequence of imbedded neighborhoods $P_0 \supset P_1 \supset P_2 \supset \ldots$ of the chain recurrent set.

(ii) Let $d_k \to 0$ as $k \to \infty$. The inclusions 6 are obtained for an arbitrary symbolic image. Hence these inclusions hold for the neighborhoods P_k . It follows $P_k \subset Q(\varepsilon)$ provided $q_k + d_k < \varepsilon$, where q_k is the largest diameter of the images $X(M(i))$. Set $\varepsilon_k = (3\backslash 2)(q_k + d_k)$. Since X is a continuous mapping and M is compact, $q_k \to 0$ as $d_k \to 0$. Thus we have $\varepsilon_k \to 0$ as $k \to \infty$. Because the family $\{Q(\varepsilon),\ \varepsilon > 0\}$ is a fundamental system of neighborhood of Q, $Q \subset P_k \subset Q(\varepsilon_k)$ and $\lim_{k\to\infty} Q(\varepsilon_k) = \bigcap_k Q(\varepsilon_k) = Q$ we obtain $Q \subset \bigcap_k P_k \subset \bigcap_k Q(\varepsilon_k) = Q$, i.e.,

$$\lim_{k\to\infty} P_k = \bigcap_k P_k = Q \blacksquare$$

6 Example

Let us consider the Van-der-Pol system

$x' = y,$

$y' = \varepsilon(1 - x^2)y - x,$

where $\varepsilon = 1,5$. It is well known that the chain recurrent set of the Van-der-Pol system consists of an equilibrium point (0,0) and a periodic orbit. The system is studied numerically in the square $M = [-3,5; 3,5] \times [-3,5; 3,5]$. The initial covering consists of 49 cells, which are $1\times$ 1 squares. The subsequent partitions are sub-divisions into 4 equal squares. Fig.2 presents the neighborhoods $P_1, P_2, P_3, P_4, P_5, P_6$ of the chain recurrent set. According to Theorem 2 these neighborhoods are embedded one inside the other and tend to the equilibrium point and the periodic orbit. Notice that the initial covering does not separate the equilibrium point and the periodic orbit. The separation appears at the first partition of P_0 . As seen from Fig.2, the small enough neighborhoods of the periodic orbit and the equilibrium point are obtained at the sixth step of the algorithm. A computer program realizing the algorithm described above has been made in St. Petersburg Technical University, 1991 by A. Moiseev and the author.

Appendix. An application of the Newton method.

This result was obtained by I.A.Komarchev and the author. The described algorithm gives the sequence of neighborhoods of periodic trajectories. Moreover, according to Corollary 2 we can find the periodic ε-trajectories in each step of the algorithm. So the problem of existence of true periodic trajectory near the periodic ε-trajectory is arisen. In this section we apply the Newton method to find a sufficient condition for the existence of a periodic trajectory near a periodic ε-trajectory. Besides that we get an algorithm constructing this trajectory. Remind briefly the Newton method.

Theorem 3 *[7] Let V, W be open sets in a Banach space, $F : V \to W$ be a differential map, and the derivative F' is a Lipschizian map with constant L. Let the operator $F'(x_0)$ be invertible at some point $x_0 \in V$ and $KRL < 1/4$, where $K = \|(F'(x_0))^{-1}\|$, $R = \|(F(x_0))^{-1}F(x_0)\|$. Let the ball $\{x : \|x - x_0\| \leq 2R\}$ lies in V. Then there exists a unique solution x^* of the equation $F(x) = 0$ and $\|x_n - x^*\| \leq \frac{R}{2^{n-1}}$, where*

$$x_{n+1} = x_n - (F'(x_0))^{-1}F(x_n),\ n = 0, 1, 2, \dots .$$

Now we consider our method.

Theorem 4 *Let V, W be open sets in a Banach space, $F : V \to W$ be a differential map and the derivative F' is a Lipschizian map with a constant L. Let $d > 0, \varepsilon > 0$ be some real numbers, $x_0 \in V$ such that*

1. *$\|F(x_0)\| = \varepsilon$; $\|F(x)\| \geq \varepsilon$ for $x \in V \setminus B_d(x_0)$;*

2. *$B_{2d}(x_0) \subset V$;*

3. *The operator $F'(x_0)$ is an invertible map and $KLd < 1/4$, where $K = \|(F'(x_0))\|$.*

Then there exists an unique solution x^* of the equation $F(x) = 0$, which lies in $B_d(x_0)$ and $\|x_n - x^*\| \leq d/2^{n-1}$, where $x_{n+1} = x_n - (F'(x_0))^{-1}F(x_n)$, $n = 0, 1, 2, \dots$.

Proof. We prove that $R = \|(F'(x_0))^{-1}F(x_0)\| \leq d$ and so the conditions of the theorem 1 are true. The proof of it is carried out by contradiction.

Let $R > d$. We define $\delta > 0$ and the vector v by the equalities $\delta = d/R, u = -\|(F'(x_0))^{-1}(\delta F(x_0))\|$. Then $\|u\| = \delta\|(F'(x_0))^{-1}F(x_0)\| = (d/R)R = d$ and the vector $x_0 + u$ lies on the boundary of $B_d(x_0)$. By the condition 1 and the continuity of F we have

$$\|F(x_0 + u)\| \geq \varepsilon. \tag{9}$$

On the other hand,

$$\|F(x_0 + u)\| \leq \|F(x_0) + F'(x_0)u\| + \|F(x_0 + u) - F(x_0) - F'(x_0)u\| \leq$$

$$\|F(x_0) + F'(x_0)u\| + \|F'(x_0 + \theta u) - F'(x_0)\|\|u\| \leq$$

$$\|F(x_0) + F'(x_0)u\| + L\|u\|^2 = \|F(x_0\|(1 - \delta) + L\|u\|^2 \leq \varepsilon(1 - \delta) + Ld^2.$$

From this inequality and (9) we have $\varepsilon \leq \varepsilon(1-\delta)+Ld^2$ or $\varepsilon\delta \leq Ld^2$. Since $R = \|F'(x_0)F(x_0)\| \leq \|F'(x_0)\|\|F(x_0)\| = K\varepsilon$ then $\varepsilon\delta \leq Ld \cdot d \leq LdR\delta \leq Ld\delta K\varepsilon$ or $1 \leq LdK$ and we have the contradiction with 3. From the condition 1 it follows that $x^* \in B_d(x_0)$ ■.

Suppose X be a diffeomorphism and $\{x_1, x_2, ..., x_p\}$ be a p-periodic ε-trajectory of X. Since M is a manifold, there are neighborhoods $V(x_i) \equiv V_i$ which we can identify with the balls $V_i(0)$ of radii a_i in R^n. Set $D = \cup_i V_i$.

Theorem 5 *Let the following conditions be satisfied:*

1. *the derivative X' is a Lipschitz map with the constant L in the neighborhoods V_i, $i = 1, 2, \dots, p$;*
2. *the map $C = A_p A_{p-1} \dots A_1 - I$ is invertible, where $A_i = X'(x_i)$, $i = 1, 2, \dots, p$;*
3. *for each p-periodic ε-trajectory $\{z_1, z_2, \dots, z_p\}$ in D, $z_i \in B_d(x_{i_j})$, where $(j_1, \dots, j_p)$ is some permutation of $1, 2, \dots, p$ and $d > 0$ such that $LK\frac{a^p-1}{a-1}d < \frac{1}{4}$, for $K = \max_i |(A_{i-1}A_{i-2}...A_1A_p...A_i - I)^{-1}|$, $a = \max_D |X'(x)|$,*
4. *$B_{2d}(x_i) \subset D$ for each $i = 1, 2 \dots, p$.*

Then there exists the unique periodic trajectory $\{y_1, y_2, ..., y_p\}$ of the diffeomorphism X such that $y_i \in B_d(x_i)$.

Proof. Let us identify each p-periodic ε-trajectory $\{x_k,\ x_k = x_{k+p}\}$ with the finite sequence $\{x_1, x_2, ..., x_p\}$. Let $\oplus_{i=1}^p R^n$ be the Banach space of all finite sequences $v = \{v_1, v_2, ..., v_p\}$, $v_i \in R^n$ with the norm $\|v\| = max_i|v_i|$. Consider the set $D = \oplus_{i=1}^{p} V_i \subset \oplus_{i=1}^{p} R^n$ and the map F on D defined by

$$F(v) = \{X(x_1 + v_1) - (x_2 + v_2), ..., X(x_p + v_p) - (x_1 + v_1)\}.$$

The operator $F'(0)$ is of the form

$$F'(0) = \begin{pmatrix} A_1 & -I & 0 & ... & 0 \\ 0 & A_2 & -I & ... & 0 \\ ... & ... & ... & ... & ... \\ -I & 0 & 0 & ... & A_p \end{pmatrix}. \tag{10}$$

At first we prove that the operator $F'(0)$ is invertible. Let $w = F'(0)u$ where $w = \{w_1, w_2, ...w_p\}$ and $u = \{u_1, u_2, ...u_p\}$. From (10) it follows

$$(A_{i-1}A_{i-2}...A_1A_pA_{p-1}...A_i - I)u_i = v_i$$

where $v_i = (A_{i-1}...A_1A_p...A_{i+1}w_i + ... + A_{i-1}...A_2w_1 + ... + w_{i-1})$.

Since the operators

$$C_i = (A_{i-1}A_{i-2}...A_1A_pA_{p-1}...A_i - I) = A_{i-1}A_{i-2}...A_1CA_1^{-1}A_2^{-1}...A_{i-1}^{-1}$$

$i = 1, 2, ..., p$ are invertible, then

$$u_i = C_i^{-1}v_i, \tag{11}$$

and hence, $F'(0)$ is invertible as well. From (11) we obtain

$$\|u\| \le K(a^{p-1} + a^{p-2} + ... + 1)\|w\|. \tag{12}$$

Because a is an estimation of the derivative norm, we can consider $a \neq 1$. From the inequality (12) we obtain

$$\left\|(F'(0))^{-1}\right\| \le K\frac{a^p - 1}{a - 1}.$$

To finish the proof we apply the Theorem 4.

References

[1] V.M.Alekseev, Symbolic Dynamics, 11th Mathematical School, Kiev, 1976 (in Russian).

[2] R.Bowen, Symbolic Dynamics , Ann.Math.Soc.Providence, R.I., vol.8, 1982.

[3] I.U.Bronshtein, Nonautonomous dynamical system, Kishinev, Shtinisa, 1984 (in Russian).

[4] B.Coomes, H.Kocak, K.Palmer, Periodic Shadowing, Chaotic Numerics, Contemporary Mathematics 172, 1994.

[5] P.Hartman, Ordinary Differential Equations, N.Y., 1964.

[6] C.S.Hsu, Cell-to-Cell Mapping, Springier-Verlag, N.Y.,1987.

[7] A.N.Kolmogorov, S.W.Fomin, Elements of the theory of functions and the functional analysis. Science, Moscow, 1968 (in Russian).

[8] Z. Nitecki, Differentiable Dynamics, London, 1971.

[9] G.S.Osipenko, On the symbolical image of the dynamical system, in Boundary value problems, Perm. 1983, 101-105 (in Russian).

[10] G.S.Osipenko. Verification of the transversality condition by the methods of symbolical dynamics, Differential Equations, N 9, 1990, 1528-1536 (in Russian).

[11] G.S.Osipenko, The periodic points and symbolic dynamics, in Seminar on Dynamical Systems, Birkhauser Verlag AG, 1993, 261-267.