DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 3, 1999
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Dynamical systems in medicine, biology, ecology, and chemistry

An Exact Solution for Fluid Convection near an Infinite Vertical Plate in a Rotating System

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Abstract

The unsteady convective flow of a viscous, heat conducting fluid near an infinite vertical plate has been considered in a system rotating with a constant angular velocity. The governing partial differential equations have been solved exactly for the special case of Prandtl number equal to unity. Analytical expressions for the primary and secondary velocity profiles in the boundary layer

as well as the corresponding skin friction components have been presented. The effects of rotation on the boundary layer flow and the skin friction have been discussed using the exact solution.

1 Introduction

Fluid convection at vertical plates resulting from buoyancy forces finds applications in several industrial and technological fields such as nuclear reactors, heat exchangers, electronic cooling equipments and crystal growth, among others. It is also known that convection flows are greatly influenced by rotation. However, the interaction between rotation and buoyancy forces are so complex that true predictions of the dynamical features of the fluid flow are possible only for some idealized models. The convective flow problems are mostly non-linear, and are described by (systems of) non-linear partial differential equations. These equations with the corresponding initial and boundary conditions are usually solved by numerical methods. On the other hand, linear problems are sometimes amenable to exact analytical treatments as occurs, for instance, in flow past flat plates [1 – 4]. The degree of complexity of the solution procedure will still be dictated by the nature of the initial and boundary conditions of the problem.

Recently, Ker and Lin [5] carried out convection studies in a differentially heated rotating cubic cavity and obtained numerical solutions of the governing equations. They also did experimental investigations of the same problem and used the experimental data for verifying the numerical solutions. Our aim in this paper is to obtain an exact solution for a special case of their work when a fluid of Prandtl number equal to unity is bounded on only one side by a rotating infinite vertical plate and when the rotational buoyancy force is neglected. The relevant equations governing the motion are given in Section 2. It has been shown that an exact solution for the resulting initial-boundary-value problem can be obtained using the method of Laplace transforms. The usefulness of Laplace transforms in finding exact solutions of fluid dynamical problems is well-known [1, 4, 6, 7]. In the same vein, we have presented in Section 3 analytical solutions for the boundary layer flow variables for fluids of Prandtl number equal to unity. Some numerical results showing the influence of rotation on the boundary layer flow and on the stress at the plate have been presented in Section 4.

2 Governing Equations

The physical situation consists of the unsteady flow of an incompressible viscous fluid near an infinite vertical plate rotating in its own plane. With respect to an arbitrary origin O on this planar wall, the axes Ox' and Oy' are fixed on it in mutually perpendicular directions. The axis Oz' is taken perpendicular to the plate into the fluid. Initially, the plate and the fluid are at rest and at the same temperature T'_∞. Subsequently ($t' > 0$), the plate starts rotating in its own plane with a constant angular velocity Ω', and the plate temperature is raised to T'_w. The fluid flow near the plate thus is driven by the Coriolis force and the thermal buoyancy. Under this physical situation, the field quantities will be functions of the space coordinate z' and time t' only. The governing equations of the resulting unsteady flow subject to rotation and under the Boussinesq approximation can be obtained from [5] in the form

$$\frac{\partial u'}{\partial t'} = \nu\frac{\partial^2 u'}{\partial z'^2} + 2\Omega' v' + g\beta(T' - T'_\infty)\cos\Omega' t' \qquad (1)$$

$$\frac{\partial v'}{\partial t'} = \nu\frac{\partial^2 v'}{\partial z'^2} - 2\Omega' u' - g\beta(T' - T'_\infty)\sin\Omega' t' \qquad (2)$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho c_p}\frac{\partial^2 T'}{\partial z'^2} \qquad (3)$$

where u', v' are the velocity components in the x', y' directions, respectively, T' is the temperature of the fluid, Ω' the constant angular velocity, g the acceleration due to gravity, β the volumetric coefficient of thermal expansion, ν the kinematic viscosity, ρ the density and c_p is the specific heat of the fluid at constant pressure.

The initial and boundary conditions relevant to the problem are:

$$\begin{gathered} u' = 0, \quad v' = 0, \quad T' = T'_\infty \quad \text{for} \quad z' \geq 0 \quad \text{and} \quad t' \leq 0, \\ u' = 0, \quad v' = 0, \quad T' = T'_w \quad \text{at} \quad z' = 0 \quad \text{for} \quad t' > 0, \\ u' \to 0, \quad v' \to 0, \quad T' \to T'_\infty \quad \text{as} \quad z' \to \infty \quad \text{for} \quad t' > 0. \end{gathered} \qquad (4)$$

We now introduce the non-dimensional quantities

$$\begin{aligned} z &= z'/L, \quad t = \nu t'/L^2, \quad (u, v) = (u', v')L/\nu, \\ T &= (T' - T'_\infty)/(T'_w - T'_\infty), \quad \Omega = L^2\Omega'/\nu, \quad P = \nu\rho c_p/k. \end{aligned} \qquad (5)$$

where L is a characteristic length. Using these non-dimensional quantities, eqs (1) - (3) can be expressed in the dimensionless forms

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + 2\Omega v + T\cos\Omega t \tag{6}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial z^2} - 2\Omega u - T\sin\Omega t \tag{7}$$

$$P\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} \tag{8}$$

with the initial and boundary conditions

$$\begin{aligned} u = 0, \quad v = 0, \quad T = 0 \quad \text{for} \quad z \geq 0 \quad \text{and} \quad t \leq 0, \\ u = 0, \quad v = 0, \quad T = 1 \quad \text{at} \quad z = 0\ ; \quad \text{for} \quad t > 0, \\ u \to 0, \quad v \to 0, \quad T \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad t > 0. \end{aligned} \tag{9}$$

The above non-dimensionalization process has revealed that the characteristic length L introduced in eq(5) can be defined as

$$L = \left\{ \frac{\nu^2}{g\beta(T'_w - T'_\infty)} \right\}^{1/3}. \tag{10}$$

It should be noted that the coupled equations (6) and (7) can be combined into a single equation by introducing the complex velocity $q = u + iv$. This yields

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial z^2} - 2i\Omega q + Te^{-i\Omega t} \tag{11}$$

with the initial and boundary conditions

$$\begin{aligned} q = 0 \quad \text{for} \quad z \geq 0 \quad \text{and} \quad t \leq 0, \\ q = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad q \to 0 \quad \text{as} \quad z \to \infty \quad \text{for} \quad t > 0. \end{aligned} \tag{12}$$

In the following, we shall obtain the exact solution of the problem for the special case of fluids whose Prandtl number $P = 1$. This corresponds to those fluids whose velocity and thermal boundary layer thicknesses are of the same order of magnitude. There are several fluids of practical interest that belong to this category.

3 Method of Solution

As stated in Section 1, we shall use the method of Laplace transforms to find the exact solutions of eqs (8) and (11) describing the unsteady flow subject to rotation. Writing

$$\bar{T}(z,s) = \int_0^\infty T(z,t)e^{-st}dt, \quad \bar{q}(z,s) = \int_0^\infty q(z,t)e^{-st}dt, \tag{13}$$

eqs (8) and (11) can be transformed, respectively, to

$$\frac{\partial^2 \bar{T}(z,s)}{\partial z^2} - s\bar{T}(z,s) = 0 \tag{14}$$

$$\frac{\partial^2 \bar{q}(z,s)}{\partial z^2} - (s+2i\Omega)\bar{q}(z,s) + \bar{T}(z,s+i\Omega) = 0 \tag{15}$$

with the boundary conditions

$$\bar{T}(0,s) = 1/s, \quad \bar{q}(0,s) = 0 \quad \text{and} \quad \bar{T} \to 0, \quad \bar{q} \to 0 \quad \text{as} \quad z \to \infty. \tag{16}$$

Equations (14) and (15) can be solved as ordinary differential equations, and we obtain the solutions

$$\bar{T}(z,s) = \frac{\exp(-z\sqrt{s})}{s} \tag{17}$$

$$\bar{q}(z,s) = \frac{\exp\left(-z\sqrt{s+i\Omega}\right) - \exp\left(-z\sqrt{s+2i\Omega}\right)}{i\Omega(s+i\Omega)}. \tag{18}$$

The temperature and velocity distributions can be obtained by taking the inverse transforms of eqs (17) and (18). The solutions can be written as [8, 9]

$$T(z,t) = \operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right) \tag{19}$$

$$q(z,t) = \frac{-i}{\Omega}\left[q_1(z,t) - q_2(z,t)\right] \tag{20}$$

where

$$\begin{aligned} q_1(z,t) &= \exp(-i\Omega t)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right), \\ 2q_2(z,t) &= \exp\left(-\sqrt{i\Omega}\,z - i\Omega t\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}} - \sqrt{i\Omega t}\right) \\ &\quad + \exp\left(\sqrt{i\Omega}\,z - i\Omega t\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + \sqrt{i\Omega t}\right) \end{aligned}$$

and erfc is the complementary error function defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Equations (19) and (20) are exact solutions. As these have been obtained formally by Laplace transforms, it may be desirable and interesting to verify the solutions. Since $\operatorname{erfc}(0) = 1$, and $\operatorname{erfc}(\infty) = 0$, it can be easily verified that the initial and boundary conditions are satisfied by $T(z,t)$ and $q(z,t)$. The verification of $T(z,t)$ given by eq(19) as the solution of eq(8) (with $P = 3D1$) is straightforward, and is not done here. We shall, however, show that eq(20) represents the exact solution of eq(11). To this end, we observe that the relevant partial derivatives of $q(z,t)$ can be written in the forms

$$\frac{\partial q}{\partial t} = q_2 - q_1 + r, \quad \frac{\partial^2 q}{\partial z^2} = r - q_2, \tag{21}$$

where

$$r = \frac{iz \exp\left(-\dfrac{z^2}{4t}\right)}{2\sqrt{\pi}\Omega t^{3/2}} \left[\exp(-2i\Omega t) - \exp(-i\Omega t)\right].$$

We thus get

$$\frac{\partial q}{\partial t} - \frac{\partial^2 q}{\partial z^2} = 2q_2 - q_1. \tag{22}$$

Using eq(22) in eq(11), and noting that $q_1 = T\exp(-i\Omega t)$, it can be seen that eq(11) is identically satisfied.

It is of practical importance to analyze the effect of rotation on the st ress at the boundary. This can be done by evaluating the skin friction co mponents. Using the complex velocity function given by eq(20), the compon ents of skin friction τ_x and τ_y in the x and y directions can be obtained as

$$\begin{aligned} \tau_x + i\tau_y &= -\left(\frac{\partial q}{\partial z}\right)_{z=0} \\ &= \frac{i\exp(-i\Omega t)}{\Omega\sqrt{\pi t}} \left[\exp(-i\Omega t) + \sqrt{i\Omega\pi t}\, \operatorname{erf}\left(\sqrt{i\Omega t}\right) - 1\right]. \end{aligned} \tag{23}$$

Velocity and Skin Friction Components

The expressions for $q(z,t)$ and the skin friction given by eq(20) and eq(23), respectively, are complex functions, and therefore require these to be separated into real and imaginary parts so as to yield analytical expressions for the primary and secondary velocity distributions and the respective skin friction components. For this we write [10]

$$\operatorname{erfc}(x+iy) = f(x,y) + ih(x,y) \tag{24}$$

where

$$\begin{aligned}
f(x,y) &= \sum_{n=0}^{\infty} \left[(xy)^{2n} g_n(x)\cos 2xy - (n+1)g_{n+1}(x)\sin 2xy\right],\\
h(x,y) &= -\sum_{n=0}^{\infty} \left[(n+1)g_{n+1}(x)\cos 2xy + (xy)^{2n} g_n(x)\sin 2xy\right],\\
g_{n+1}(x) &= \frac{2}{2n+1}\left[\frac{\exp\left(-x^2\right)}{\sqrt{\pi}x^{2n+1}/(n+1)} - \frac{g_n(x)}{n+1}\right],\\
g_0(x) &= \operatorname{erfc}(x).
\end{aligned}$$

After extensive algebraic derivations, it can be shown that the primary and secondary velocity components u and v are given by

$$u(z,t) = u_1(z,t) + u_2(z,t) + u_3(z,t) \tag{25}$$

$$v(z,t) = v_1(z,t) + v_2(z,t) + v_3(z,t) \tag{26}$$

where

$$\begin{aligned}
u_1(z,t) &= \frac{-\exp(\Omega_0 z)}{2\Omega}\left[\varphi_1 \sin(\Omega_0 z - \Omega t) + \psi_1 \cos(\Omega_0 z - \Omega t)\right],\\
u_2(z,t) &= \frac{\exp(-\Omega_0 z)}{2\Omega}\left[\varphi_2 \sin(\Omega_0 z + \Omega t) - \psi_2 \cos(\Omega_0 z + \Omega t)\right],\\
u_3(z,t) &= -\frac{1}{\Omega}\sin(\Omega t)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right)\\
v_1(z,t) &= \frac{\exp(\Omega_0 z)}{2\Omega}\left[\varphi_1 \cos(\Omega_0 z - \Omega t) - \psi_1 \sin(\Omega_0 z - \Omega t)\right],\\
v_2(z,t) &= \frac{\exp(-\Omega_0 z)}{2\Omega}\left[\varphi_2 \cos(\Omega_0 z + \Omega t) + \psi_2 \sin(\Omega_0 z + \Omega t)\right],\\
v_3(z,t) &= -\frac{1}{\Omega}\cos(\Omega t)\operatorname{erfc}\left(\frac{z}{2\sqrt{t}}\right),
\end{aligned}$$

$$\varphi_{1,2} = f\left(\frac{z}{2\sqrt{t}} \pm \Omega_0\sqrt{t},\ \pm\Omega_0\sqrt{t}\right),$$
$$\psi_{1,2} = h\left(\frac{z}{2\sqrt{t}} \pm \Omega_0\sqrt{t},\ \pm\Omega_0\sqrt{t}\right), \quad \Omega_0 = \sqrt{\Omega/2}.$$

Furthermore, the components of skin friction are given by

$$\tau_x = \frac{1}{\sqrt{2\Omega}}\left[\tau_{x1} + \sqrt{\frac{2}{\pi\Omega t}}\,\tau_{x2}\right] \tag{27}$$
$$\tau_y = \frac{1}{\sqrt{2\Omega}}\left[\tau_{y1} + \sqrt{\frac{2}{\pi\Omega t}}\,\tau_{y2}\right] \tag{28}$$

where

$$\tau_{x1} = 1 - f\left(\Omega_0\sqrt{t}, \Omega_0\sqrt{t}\right) + h\left(\Omega_0\sqrt{t}, \Omega_0\sqrt{t}\right),$$
$$\tau_{x2} = \sin 2\Omega t - \sin \Omega t,$$
$$\tau_{y1} = 1 - f\left(\Omega_0\sqrt{t}, \Omega_0\sqrt{t}\right) - h\left(\Omega_0\sqrt{t}, \Omega_0\sqrt{t}\right),$$
$$\tau_{y2} = \cos 2\Omega t - \cos \Omega t.$$

4 Numerical Results

The numerical values of the primary and secondary velocities u and v, respectively, have been presented in Table 1. Table 1a shows the effect of the parameter Ω when $t = 0.1$ while Table 1b shows the temporal variations for $\Omega = 0.5$. It may be noted that Ω is a measure of the relative influence of the rotation and thermal buoyancy. For relatively small values of Ω, the primary velocity u increases to a maximum near the boundary and then decreases to zero. Within this range of values of Ω, the secondary velocity v decreases to a minimum and then approaches its stationary value. However, it was seen during our computations that for much higher values of Ω, which correspond to the dominance of Coriolis force over thermal buoyancy, the velocity components undergo oscillations in the boundary layer. Similar oscillations of velocity have also been noted in the rotating cavity flow considered in [5]. In the present unsteady problem, the oscillatory flow was evident even for small values of Ω when t is large. We also note that u increases with t, and v decreases with it.

The components of skin friction, τ_x and τ_y, are shown in Table 2. Here also, the monotonic influence of the rotation parameter Ω on the stress at the

Table 1: Velocity components u, v.

Table 1a: $t = 0.1$			
Ω	z	$u(\times 10^2)$	$v(\times 10^2)$
0.1	0.5	1.480	-0.020
	1.0	1.197	-0.003
	1.5	0.007	0.0000
1.0	0.5	1.467	-0.196
	1.0	0.195	-0.028
	1.5	0.007	-0.001
10.0	0.5	0.347	-1.400
	1.0	0.032	-0.188
	1.5	0.001	-0.007

Table 1b: $\Omega = 0.5$			
t	z	$u(\times 10^2)$	$v(\times 10^2)$
0.1	0.5	1.477	-0.098
	1.0	0.197	-0.014
	1.5	0.007	-0.001
0.3	0.5	5.949	-1.152
	1.0	3.520	-0.720
	1.5	1.142	-0.241
0.5	0.5	9.394	-3.048
	1.0	7.871	-2.690
	1.5	4.137	-1.465

boundary is evident when it takes small values. In particular, τ_x increases with increasing Ω while the reverse occurs in the case of τ_y. It was seen from computations that for very high values of Ω, τ_x and τ_y are subject to oscillations as in the case of the velocity variables. However, for a fixed value of Ω, both τ_x and τ_y increase with t.

Table 2: Skin frictions τ_x, τ_y.

Table 2a: $t = 0.1$		
Ω	τ_x	τ_y
0.1	0.1796	0.3541
0.3	0.1818	0.3488
0.5	0.1838	0.3434
0.7	0.1857	0.3380
1.0	0.1882	0.3298
3.0	0.1955	0.2763
5.0	0.1876	0.2277
7.0	0.1674	0.1881
10.0	0.1228	0.1521

Table 2b: $\Omega = 0.5$		
t	τ_x	τ_y
0.1	0.1838	0.3434
0.2	0.2662	0.4664
0.3	0.3318	0.5478
0.4	0.3878	0.6055
0.5	0.4365	0.6472
0.6	0.4789	0.6768
0.7	0.5156	0.6972
0.8	0.5468	0.7102
1.0	0.5933	0.7199

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