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Symbolic dynamics

SPECTRUM OF A DYNAMICAL SYSTEM AND APPLIED SYMBOLIC DINAMICS

George Osipenko

Laboratory of Nonlinear Analysis and Mathematical Modelling
State Technical University
St.Petersburg, 195251, Russia
e-mail: math@math.hop.stu.neva.ru

Abstract.

The paper introduces a constructive method for localization of the Morse spectrum of a dynamical system on a vector bundle. The Morse spectrum is a limit set of Lyapunov exponents of periodic pseudo-trajectories. The proposed method does not demand any preliminary information on a system. An induced dynamical system on the projective bundle is associated with a directed graph called Symbolic Image. The symbolic image can be considered as a finite discrete approximation of a dynamical system. Valuable information about the system may come from the analysis of a symbolic image. In particular, a neighborhood of the Morse spectrum can be found. A special sequence of symbolic images is considered to obtain a sequence of embedded neighborhoods which converges to the Morse spectrum. The main results of this article were announced in [20].

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1 Linear extension of a homeomorphism

Let $f : M \to M$ be a homeomorphism of the compact manifold M. Let (E, M, π) be a vector bundle over M, E be a total space, and π be a projector from E onto the base M. Assume that for each $x \in M$ a fiber $E(x) = \pi^{-1}(x)$ is d-dimension linear space isomorphic to R^d.

Definition 1 *A homeomorphism F of the total space E is said to be a linear extension of f, if F takes fibers to fibers: $f \circ \pi = \pi \circ F$, i.e., the diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

commutes, and the restriction $F|_{E(x)} : E(x) \to E(f(x))$ on each fiber $E(x)$ is a linear isomorphism.

The investigation of linear extensions has been motivated by the study of the tangent mapping of a diffeomorphism on the tangent bundle of manifold [23, 24, 7, 8, 25, 27]. As an example of a linear extension we will keep in mind a diffeomorphism $f : M \to M$ and its differential $Df = F$ on the tangent bundle $TM = E$. In particular, by locating of the Morse spectrum, one can constructively recognize the hyperbolicity or the normal hyperbolicity of a dynamical system. In this section we briefly show that the vector bundle E is associated with a projective bundle P and with an one-dimension bundle L over P, in this process the linear extension $F : E \to E$ induces a mapping $PF : P \to P$ and its linear extension $LF : L \to L$. Recall that a projective manifold P^{d-1} is a set of one-dimensional subspaces in R^d. For a nonzero vector v we denote by $[v] = y$ a point from P^{d-1} corresponding to the space spanned over v. Let $(P, M, P\pi)$ be a bundle over M such that each fiber $(P\pi)^{-1}(x)$ is a projective manifold $P^{d-1}(x)$ associated with the fiber $E(x)$ of E. The bundle $(P, M, P\pi)$ is called the projective bundle associated with the linear bundle (E, M, π). The linear space R^d can be considered as a collection of one-dimensional subspaces $L = \{L(y),\ y \in P^{d-1}\}$, where $L(y)$ is a subspace spanned over y. So L is an one-dimensional bundle over P^{d-1}. Let us fix a fiber $E(x)$ of E. As above, each linear space $E(x)$ is considered as an one-dimensional vector bundle $L(x)$ over $P^{d-1}(x)$ with fibers $L(x, y)$ spanned over $y \in P^{d-1}(x)$. Thus, the bundle (E, M, π) is associated with the one-dimensional linear bundle $(L, P, LP\pi)$ over the projective space P, where $L = \{L(x, y),\ x \in M,\ y \in P^{d-1}(x)\}$, and $LP\pi$ is the induced projector from L onto P. The bundles $E,\ P$ and L are nontrivial, in general. However there is a manner of trivialization of them. More precise, each vector bundle can be included in trivial one [2]. Hence, without a loss of generality, we can use the following coordinates: (x, v) on E, (x, y) on P, and (x, y, l) on L, where $x \in M$, $v \in E(x)$, $y \in P^{d-1}(x)$, $l \in L(y)$. In this coordinates, the projectors $LP\pi$ and $P\pi$ are of the form:

$$LP\pi(x, y, l) = (x, y), \qquad P\pi(x, y) = x.$$

The linear extension $F : E \to E$ of the homeomorphism $f : M \to M$ is of the form

$$F(x, v) = (f(x), A(x)v).$$

The mapping F induces the mapping $PF : P \to P$ on the projective bundle and its linear extension $LF : L \to L$ on the one-dimensional linear bundle, so the diagram

$$\begin{array}{ccc} L & \stackrel{LF}{\to} & L \\ \downarrow_{LP\pi} & & \downarrow_{LP\pi} \\ P & \stackrel{PF}{\to} & P \\ \downarrow_{P\pi} & & \downarrow_{P\pi} \\ M & \stackrel{f}{\to} & M \end{array}$$

commutes. In the coordinates (x, y, l), these mappings take the forms:

$$PF(x, y) = (f(x), pF(x, y)), \; pF(x, y) = [A(x)e(y)],$$

$$LF(x, y, l) = (f(x), pF(x, y), a(x, y)l) = (PF(x, y), a(x, y)l),$$

where $e(y)$ is a basis vector in the fiber $L(y)$, $a(x, y)l$ is one-dimensional linear mapping, i.e., $a(x, y)l$ is a multiplication by the function $a(x, y)$ as basis vectors are fixed, and $|a(x, y)| = |A(x)e(y)|$.

2 Chain-recurrent points

Let $\rho(*, *)$ be a Riemannian metric on M.

Definition 2 *An infinite in both directions sequence of points $\{x_i\} \subset M$ is called an ε-trajectory of f if for any i the distance between the image $f(x_i)$ and x_{i+1} is less than ε, i.e.,*

$$\rho(f(x_i), x_{i+1}) < \varepsilon.$$

If the sequence $\{x_i\}$ is periodic then it is called a periodic ε-trajectory and the points x_i are called ε-periodic.

In most cases, an exact trajectory of system is seldom known, and in fact we ought to deal with an ε-trajectory for a sufficiently small $\varepsilon > 0$. Denote the set of all ε-periodic points by $Per(\varepsilon)$. The set $Per(\varepsilon)$ is open. It is clear that if $\varepsilon_1 > \varepsilon_2$ then every ε_2-trajectory is an ε_1-trajectory, hence

$$Per(\varepsilon_2) \subset Per(\varepsilon_1).$$

Thus, the sets $Per(\varepsilon),\ \varepsilon > 0$, are embedded one inside the other.

Definition 3 *A point x is called chain recurrent if x is ε-periodic for every positive ε. The set of chain recurrent points is called a chain recurrent set.*

Let us denote the chain recurrent set by CR. It is well known that the chain recurrent set is invariant, closed and contains the returning trajectories of all types such as periodic, almost periodic, recurrent, homoclinic and others. It should be noted that if a chain recurrent point x is not periodic then there exists an arbitrarily small perturbation of the mapping f in C^0-topology for which x is periodic [22]. One may say that a chain recurrent point is either periodic or becomes periodic under a C^0-perturbation. From definition of the chain recurrent set it follows that

$$CR = \lim_{\varepsilon \to 0} Per(\varepsilon) = \bigcap_{\varepsilon > 0} Per(\varepsilon).$$

In other words, the sets $\{Per(\varepsilon),\ \varepsilon > 0\}$ forms a base of neighborhoods of the chain recurrent set CR.

For $\varepsilon > 0$ a finite ε-chain ξ is defined as a finite sequence $x_0, ..., x_m \in M$ of length m with $\rho(f(x_i), x_{i+1}) < \varepsilon,\ i = 0, ...m-1$. The same way one can define an ε-semi-trajectory.

3 Morse spectrum

Let us consider the linear extension $F : E \to E$ and the mapping $PF : P \to P$ on the projective bundle associated with F. Let $\xi = \{(x_0, y_0), ..., (x_m, y_m)\}$ be a finite ε-chain on the projective bundle P for the mapping PF. Define the exponential growth rate of ξ by

$$\lambda(\xi) = \frac{1}{m} \sum_{i=0}^{m-1} \ln |F(x_i, e(y_i))|,$$

where $|F(x, v)| = |A(x)v|,\ e(y_i)$ is basis vector in $L(y_i)$, $|e(y_i)| = 1$. One can say that $|A(x)e(y)|$ is a change coefficient of vector length. By using the induced mapping $LF : L \to L$ the exponential growth rate of ξ can be rewritten in the form

$$\lambda(\xi) = \frac{1}{m}\sum_{i=0}^{m-1} \ln |a(x_i, y_i)|.$$

Recall that if $\xi = \{(x_0, y_0), (x_1, y_1), ...\}$ is an ε-semi-trajectory, then

$$\lambda(\xi) = \overline{\lim_{m\to\infty}}\frac{1}{m}\sum_{i=0}^{m-1} \ln |A(x_i)e(y_i)|$$

is the characteristic or Lyapunov exponent of the ε-semi-trajectory ξ. If $\xi = \{(x_0, y_0), ..., (x_p, y_p) = (x_0, y_0)\}$ is a periodic ε-trajectory of a period p, then for the Lyapunov exponent of ξ we have

$$\lambda(\xi) = \overline{\lim_{m\to\infty}}\frac{1}{m}\sum_{i=0}^{m-1} \ln |A(x_i)e(y_i)| = \lim_{k\to\infty}\frac{1}{m}\sum_{i=0}^{m-1} \ln |A(x_i)e(y_i)| = \frac{1}{p}\sum_{i=0}^{p-1} \ln |A(x_i)e(y_i)|.$$

The Morse spectrum of F on the chain recurrent set CR of the associated projective mapping PF is defined as

$$\begin{aligned}\Sigma(F) &= \{\lambda \in R : \textit{ there are } \varepsilon_k \to 0 \textit{ and finite } \varepsilon_k - \textit{chains } \xi_k \textit{ of lenths } m_k \\ &\quad \textit{in } CR \textit{ with } m_k \to \infty \textit{ and } \lambda(\xi_k) \to \lambda \textit{ as } k \to \infty\}.\end{aligned}$$

F. Colonius and Kliemann [7] showed that the Morse spectrum coincides with the periodic Morse spectrum which is defined as

$$\begin{aligned}\Sigma_{per}(F) &= \{\lambda \in R : \textit{ there are } \varepsilon_k \to 0 \textit{ and periodic } \varepsilon_k - \textit{trajectories } \xi_k \\ &\quad \textit{with } \lambda(\xi_k) \to \lambda \textit{ as } k \to \infty\}.\end{aligned}$$

Thus, to study the Morse spectrum it is sufficient to investigate the behavior of Lyapunov exponents of periodic ε-trajectories on the projective bundle as $\varepsilon \to 0$.

4 Symbolic Image [14]

Now we recall a construction of a symbolic image. Let $f : M \to M$ be a homeomorphism of manifold M and $C = \{M(1), \cdots, M(s)\}$ be a finite covering of M by closed sets. The sets $M(i)$ are called cells of the covering.

Definition 4 *Let G be a directed graph having s vertices where each vertex i corresponds to the cell $M(i)$. The vertices i and j are connected by a directed edge $i \to j$ if and only if $M(j) \cap f(M(i)) \neq \emptyset$. The graph G is called a symbolic image of f with respect to the covering C.*

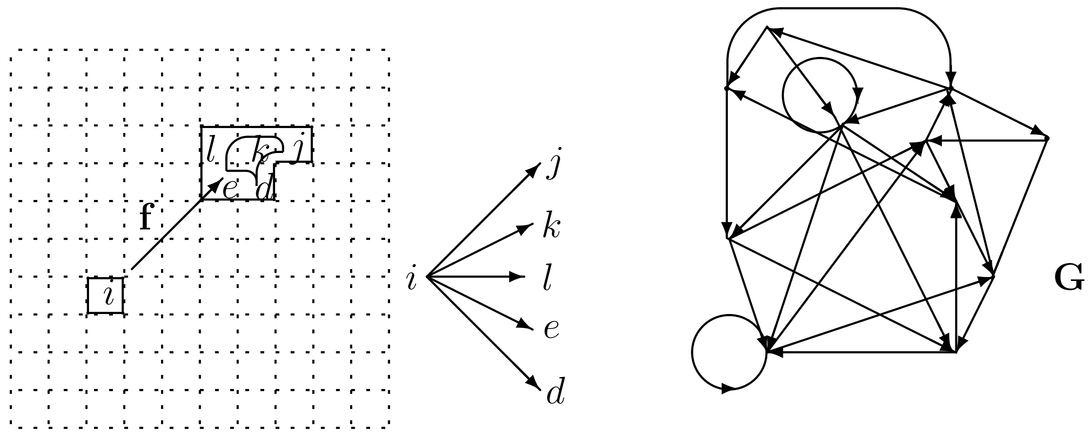

Figure 1: **Construction of a Symbolic Image.**

Denote by Ver the set of vertices of G. The graph G can be considered as a multi-correspondence $G : Ver \to Ver$ between the vertices. Graph G is uniquely determined by its $s \times s$ matrix of transitions $\Pi = (\pi_{ij})$: $\pi_{ij} = 1$ if and only if there is the directed edge $i \to j$, otherwise $\pi_{ij} = 0$. Much of an effective information of a dynamical system may come from the investigation of a symbolic image. It is easily seen that the symbolic image depends on the covering C. Vary the covering C one changes the symbolic image of the mapping f. It is natural to consider the symbolic image as a finite discrete approximation of the mapping f. This approximation is more precise if the mesh of the covering is smaller. Let

$$diam M(i) = max(\rho(x_1, x_2) : x_1, x_2 \in M(i))$$

be a diameter of the cell $M(i)$. Let d be the largest diameter of the cells $M(i)$ of the covering C. Denote by q the largest diameter of the images $f(M(i))$, $i = 1, ..., s$. We define the number r as follows. If a cell $M(k)$ does not belong to the covering $C(i)$ then the distance

$$r_{ik} = \rho(f(M(i)), M(k)) = min(\rho(x, y) : x \in f(M(i)), y \notin M(k)))$$

between the cell $M(k)$ and the image $f(M(i))$ is positive. Let r be a minimum of such r_{ik}. Since the number of the pairs (i, k) described above is finite then r is positive.

Definition 5 *A sequence $\{z_k\}$ of vertices of the graph G is called an admissible path or simply a path if for each k the graph G contains the edge $z_k \to z_{k+1}$. If the sequence $\{z_k\}$ is periodic, then $\{z_k\}$ is called a periodic (admissible) path.*

There is a natural connection between admissible paths on the symbolic image G and ε-trajectories of the homeomorphism f. It can be said that an admissible path is a trace of ε-trajectory and vice versa. However, there are some relationships between the parameters d, q, r of a symbolic image and the number ε for which the connections take place.

Proposition 1 [16]

1. *If a sequence $\{z_k\}$ is a path on the symbolic image G and $x_k \in M(z_k)$ then the sequence $\{x_k\}$ is an ε-trajectory of the homeomorphism f for any $\varepsilon > q + d$. In particular, if the sequence $\{z_k\}$ is a periodic path on the symbolic image then the sequence $\{x_k\}$ can be chosen a periodic ε-trajectory.*

2. *If a sequence $\{x_k\}$ is an ε-trajectory of the homeomorphism f , $\varepsilon < r$ and $x_k \in M(z_k)$ then the sequence $\{z_k\}$ is an admissible path on the symbolic image G. In particular, if the sequence $\{x_k\}$ is periodic ε-trajectory then the sequence $\{z_k\}$ can be chosen a periodic path on the symbolic image G.*

Definition 6 *A vertex of the symbolic image is called recurrent if there is a periodic path passing through it. The set of recurrent vertices is denoted by RV. A pair of recurrent vertices i, j are called equivalent if there is a periodic path through i and j.*

The recurrent vertices $\{i\}$ are uniquely defined by the nonzero diagonal elements $\pi_{ii} \neq 0$ of the powers of the transitions matrix Π^m, $m \leq s$, where s is the number of the covering cells. By Definition 6, the set of recurrent vertices RV decomposes into several classes $\{H_k\}$ of equivalent recurrent vertices. It is evident that each periodic path ω is in a certain class $H_k = H(\omega)$ determined uniquely by ω.

5 Symbolic image of the projective mapping

Now we apply the symbolic image construction to the mapping PF. Let us consider a symbolic image $G(f)$ of the mapping $f : M \to M$ with respect to a covering $C(M) = \{m(1), \cdots, m(q)\}$. To construct a symbolic image of the induced mapping $PF : P \to P$ it is convenient to choose a covering $C(P) = \{M(z)\}$ of the projective space P agreed with the covering $C(M)$ such that the projection of each cell is a cell: $P\pi(M(z)) = m(j)$. The agreed covering generates a natural mapping h from $G(PF)$ onto $G(f)$ taking the vertices z on the vertex j: $h(z) = j$. Since $PF(M(z_1)) \cap M(z_2) \neq \emptyset$ and $P\pi(M(z_{1,2})) = m(j_{1,2})$ implies $f(m(j_1)) \cap m(j_2) \neq \emptyset$, the directed edge $z_1 \to z_2$ on $G(PF)$ is mapped by h on the directed edge $j_1 \to j_2$ on $G(f)$. Hence, the mapping h takes the directed graph $G(PF)$ on the directed graph $G(f)$ so that the diagram

$$\begin{array}{ccc} Ver & \stackrel{G(PF)}{\to} & Ver \\ \downarrow h & & \downarrow h \\ ver & \stackrel{G(f)}{\to} & ver \end{array}$$

commutes, where Ver and ver are the vertices of $G(PF)$ and $G(f)$, respectively.

6 Linear extension of symbolic image and its spectrum

Let $F : E \to E$ be a linear extension of the homeomorphism $f : M \to M$, $PF : P \to P$ be the associated projective mapping and $LF : L \to L$ be the linear extension of PF on the one-dimension bundle. In the coordinates (x, y, l) these mappings take the forms:

$$LF(x, y, l) = (PF(x, y), a(x, y)l) = (f(x), pF(x, y), a(x, y)l).$$

At first, we construct a linear extension of the symbolic image by fixing a linear mapping $a[ji]$ to each edge $i \to j$. Let $G(PF)$ be the symbolic image of PF. The existence of an edge $i \to j$ on $G(PF)$ guarantees the existence of a point (x, y) in the cell $M(i)$ such that the image $PF(x, y)$ is in the cell $M(j)$. Obviously, such point is not unique. By setting $a[ji]l = a(x, y)l$ we fix a linear mapping to the edge $i \to j$. The value $|a[ji]| = |A(x)e(y)|$ is a change coefficient of a vector length. We suppose each cell of the covering $C(P)$ small enough to ensure the existence of a continuous basis vectors $e(x, y) \in L(x, y)$, $(x, y) \in M(i)$. Fixing this basis we can identify the one-dimension linear spaces $L(x, y)$ for $(x, y) \in M(i)$. The obtained linear space is denoted by R_i. Let $(x^*, y^*) \in M(i)$ be a point such that $PF(x^*, y^*) \in M(j)$. We have the estimate

$$|a(x^*, y^*) - a(x, y)| < \eta(d),$$

where $\eta(d)$ is a modulus of continuity of $a(x, y)$ and d is the maximal diameter of cells from the covering $C(P)$. The structure consisting of the symbolic image $G(PF)$ and the linear maps $\{a[ji] : R_i \to R_j\}$ is said to be an one-dimensional linear extension LG of a symbolic image $G(PF)$.

Each periodic path $\omega = \{z_0, z_1, ..., z_p = z_0\}$ on $G(PF)$ of period p induces a liner map

$$a(\omega) = a[z_p z_{p-1}]...a[z_2 z_1]a[z_1 z_0] : R_{z_0} \to R_{z_0}.$$

The number $\sigma(\omega) = (|a(\omega)|)^{\frac{1}{p}}$ is called the multiplicator and

$$\lambda(\omega) = \frac{1}{p} \sum_{k=1}^{p} \ln |a[z_k z_{k-1}]| = \ln |\sigma(\omega)|$$

is called the characteristic or the Lyapunov exponent of a periodic path ω.

Definition 7 *The spectrum of the linear extension LG on the set of recurrent vertices RV is defined as*

$$\Sigma = \{\lambda \in R : \textit{ there are periodic paths } \omega_k \textit{ on } G(PF) \textit{ with } \lambda(\omega_k) \to \lambda \textit{ as } k \to \infty\}.$$

The first aim is to discover a constructive method for a computation of the spectrum of the linear extension LG. The second aim is to compare the spectrum of LG and the Morse spectrum of a dynamical system.

7 Computation of spectrum of the symbolic image linear extension

Let us consider some class H of equivalent recurrent vertices. A periodic path $\omega = \{z_1, ..., z_p = z_0\}$ is called simple if the vertices $z_1, ..., z_p$ are different, i.e., $z_i \neq z_j$ as $i \neq j$; $i, j = 1, ..., p$. Let $\omega = \{z_1, ..., z_p\}$ be a periodic path. If ω is not simple, there is a vertex z^* such that $z^* = z_l = z_{l+p_1}$. Consider two finite sequences $\omega^* = \{z_1, ..., z_{l-1}, z_{l+p_1}, ..., z_p\}$ and $\omega^{**} = \{z_{l+1}, ..., z_{l+p_1}\}$. Since there are the edges $z_{l-1} \to z_l = z_{l+p_1}$ and $z_{l+p_1} = z_l \to z_{l+1}$, the sequences ω_1 and ω_2 are periodic admissible paths of periods p_1 and $p_2 = p - p_1$, respectively. Obviously, $p_1, \ p_2 < p$. We will say that the path ω is a sum of the periodic paths ω^* and ω^{**} and write

$$\omega = \omega^* + \omega^{**}.$$

By repeating this decomposition of periodic paths we come to the decomposition of ω in a sum of periodic paths $\omega_1, ..., \omega_q$ of periods $p_1, ..., p_q$, $p_1 + ... + p_q = p$. Because the periods are positive integers, and the maximal period of components decreases, the decomposition process finishes. The final decomposition $\omega = \phi_1 + \phi_2 + ... + \phi_r$ consists of simple periodic paths. Note, the simple periodic paths $\phi_1, ..., \phi_r$ may coincide. When a periodic path ω^* repeats k times in ω, we write

$$\omega = k\omega^* + \omega^{**}.$$

Since a symbolic image has a finite number of vertices, the number of simple period paths is finite. For a class H let $\phi_1, ..., \phi_q$ be the all simple periodic paths of periods $p_1, ..., p_q$, respectively. Let

$$\lambda(\phi_j) = \frac{1}{p_j} \sum_{k=1}^{p_j} \ln |a[z_k^j z_{k-1}^j]|$$

be the characteristic exponent of periodic path $\phi_j = \{z_1^j, ..., z_{p_j}^j\}$. Suppose that $\omega \subset H$ is a periodic path on $G(PF)$, and $\omega = k_1\phi_1 + ... + k_q\phi_q$ is a decomposition of ω with the period $p = k_1p_1 + ... + k_qp_q$. Without loss of generality, we can consider each simple periodic path ϕ_j be contained in ω with the coefficient $k_j \geq 0$ (the case $k_j = 0$ means that, actually, ω does not pass through the simple periodic path ϕ_j). If so, we will say that the simple periodic path ϕ_j is contained in ω with the weight $\mu_j = \frac{k_j \ p_j}{p}$. Obviously, $\sum_{j=1}^{q} \mu_j = 1$.

Proposition 2 *The characteristic exponent of each periodic path* $\omega = k_1\phi_1 + ... + k_q\phi_q$ *is given by the formula*

$$\lambda(w) = \sum_{j=1}^{q} \mu_j \lambda(\phi_j),$$

where $\mu_j = \frac{k_j \ p_j}{p}$.

Proof. Let ω be a periodic path on the symbolic image $G(PF)$. Suppose that $\omega = k_1\phi_1 + ... + k_q\phi_q$ is a decomposition of the periodic path ω of the period $p = k_1p_1 + ... + k_qp_q$, where $p_1, ..., p_q$ are periods of the simple periodic paths $\phi_1, ..., \phi_q$. For the characteristic exponent of $\omega = \{z_1, ..., z_p\}$ we have

$$\lambda(w) = \frac{1}{p} \sum_{k=1}^{p} \ln |a[z_k z_{k-1}]| = \frac{1}{p} \sum_{j=1}^{q} k_j \sum_{k=1}^{p_j} \ln |a[z_k^j z_{k-1}^j]| = \frac{1}{p} \sum_{j=1}^{q} k_j p_j \lambda(\phi_j) = \sum_{j=1}^{q} \mu_j \lambda(\phi_j).$$

Thus, the characteristic exponent of ω is an arithmetic mean of characteristic exponents of simple periodic paths with the weights μ_j, $\sum_{k=1}^{q} \mu_j = 1$.

□

Let

$$\begin{aligned} \lambda_{\min}(H) &= \min\{\lambda(\phi_j), \ j = 1, ..., q\}, \\ \lambda_{\max}(H) &= \max\{\lambda(\phi_j), \ j = 1, ..., q\} \end{aligned}$$

be the minimum and the maximum of characteristic exponents of simple periodic paths of the class H. From Proposition 2 it follows

Proposition 3 *Characteristic exponent* $\lambda(\omega)$ *of each periodic path* ω *of class* H *satisfies the inequality*

$$\lambda_{\min}(H) \leq \lambda(\omega) \leq \lambda_{\max}(H).$$

Proposition 4 *For every $\lambda \in [\lambda_{\min}(H), \lambda_{\max}(H)]$ there is a sequence of periodic paths $\{\omega_m\}$ in H such that the characteristic exponents $\lambda(\omega_m) \to \lambda$ as $m \to \infty$.*

Proof. Without loss of generality we can assume that $\lambda_{\min}(H) < \lambda < \lambda_{\max}(H)$. Let $\phi_{\min} = \{z_1^*, ..., z_l^*\}$ and $\phi_{\max} = \{z_1^{**}, ..., z_e^{**}\}$ be simple periodic paths realizing the characteristic exponents $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$, respectively. Since the vertices z_l^* and z_1^{**} are equivalent recurrent vertices, there is a periodic path ψ through z_1^* and z_1^{**}. Fix such a periodic path $\psi = \{z_1 = z_1^*, ..., z_j = z_1^{**}, ..., z_q\}$. In H, there is a periodic admissible path ω of the form

$$\omega = \{\overbrace{\overbrace{z_1^*, ..., z_l^*}, ..., \overbrace{z_1^*, ..., z_l^*}}^{k^*-times}, z_1 = z_1^*, ..., z_j = \overbrace{\overbrace{z_1^{**}, ..., z_e^{**}}, ..., \overbrace{z_1^{**}, ..., z_e^{**}}}^{k^{**}-times},$$

$$z_1^{**} = z_j, ..., z_q\}$$

with the decomposition

$$\omega = k^*\phi_{\min} + k^{**}\phi_{\max} + \psi.$$

Let $p^*,\ p^{**}$ and q be periods of $\phi_{\min},\ \phi_{\max}$ and ψ, respectively. For the weights of $\phi_{\min},\ \phi_{\max}$ and ψ we have $\mu(\phi_{\min}) = \frac{k^*p^*}{p},\ \mu(\phi_{\max}) = \frac{k^{**}p^{**}}{p}$ and $\mu(\psi) = \frac{q}{p}$, where $p = k^*p^* + k^{**}p^{**} + q$ is period of ω. If k^* and $k^{**} \to \infty$ then $\mu(\psi) = \frac{q}{p} \to 0$. Since $\lambda_{\min}(H) < \lambda < \lambda_{\max}(H)$ there is $\delta,\ 0 < \delta < 1$, so that $\lambda = \delta\lambda(\phi_{\min}) + (1-\delta)\lambda(\phi_{\max})$. Let us choose a sequence of integers $\{V_m\}$ and $\{W_m\}$ so that $V_m,\ W_m \to \infty$ and $\frac{V_m}{W_m} \to \delta$ as $m \to \infty$. Since $0 < \delta < 1$ then $0 < V_m < W_m$. We construct a sequence of integers $\{k_m^*\}$ and $\{k_m^{**}\}$ so that

$$\frac{k_m^*p^*}{k_m^*p^* + k_m^{**}p^{**}} = \frac{V_m}{W_m},$$

i.e.,

$$k_m^{**} = p^{**}V_m, \quad k_m^* = p^*(W_m - V_m).$$

Let us consider the described above sequence of periodic paths $\{\omega_m\}$ with the decompositions

$$\omega_m = k_m^*\phi_{\min} + k_m^{**}\phi_{\max} + \psi.$$

We have

$$\begin{aligned}
\mu_m(\phi_{\min}) &= \frac{k_m^*p^*}{k_m^*p^* + k_m^{**}p^{**} + q} = \frac{V_m}{W_m}\frac{k_m^*p^* + k_m^{**}p^{**}}{k_m^*p^* + k_m^{**}p^{**} + q} \to \delta \text{ as } m \to \infty,\\
\mu_m(\phi_{\max}) &= \frac{k_m^{**}p^{**}}{k_m^*p^* + k_m^{**}p^{**} + q} = (1 - \frac{V_m}{W_m})\frac{k_m^*p^* + k_m^{**}p^{**}}{k_m^*p^* + k_m^{**}p^{**} + q} \to 1 - \delta \text{ as } m \to \infty,\\
\mu_m(\psi) &= \frac{q}{k_m^*p^* + k_m^{**}p^{**} + q} \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Thus, the characteristic exponent of $\omega_m = k_m^* \phi_{\min} + k_m^{**} \phi_{\max} + \psi$ is

$$\lambda(\omega_m) = \mu_m(\phi_{\min})\lambda(\phi_{\min}) + \mu_m(\phi_{\max})\lambda(\phi_{\max}) + \mu_m(\psi)\lambda(\psi).$$

So,

$$\lambda(\omega_m) \to \delta\lambda(\phi_{\min}) + (1-\delta)\lambda(\phi_{\max}) = \lambda \ as \ m \to \infty.$$

□

From Propositions 3 and 4 it follows

Theorem 1 *The spectrum of the linear extension LG consists of the intervals* $[\lambda_{\min}(H_k), \lambda_{\max}(H_k)]$, *where* $\{H_k\}$ *is the full family of classes of equivalent recurrent vertices of the symbolic image* $G(PF)$.

One has to emphasize that the intervals $[\lambda_{\min}(H_k), \lambda_{\max}(H_k)]$ may intersect (see [26, 7]).

8 Spectrum of symbolic image

By construction, the linear extension LG depends on the choice of the linear mappings $\{a[ji] : R_i \to R_j\}$ for the edges $\{i \to j\}$ of $G(PF)$. The mapping $a[ji]$ is determined by the choice of a point $(x, y) \in M(i)$, $(x, y) \in M(j)$ since $a[ji]l = A(x)e(y)l$, where $e(y) \in L(y)$, $|e(y)| = 1$. It follows from here that the characteristic exponent depends on the choice as well. Let us examine the variation of exponent under admissible variations of the linear mappings $\{a[ji]; R_i \to Rj\}$.

Let $\omega = \{z_0, z_1, ..., z_p = z_0\}$ be a periodic path on $G(PF)$. By definition, the characteristic exponent of the path ω for a linear extension LG is of the form

$$\lambda(w) = \frac{1}{p}\sum_{k=1}^{p} \ln |a[z_k z_{k-1}]|,$$

where each $|a[z_k z_{k-1}]| = |a[ji]|$ is determined by the edge $i \to j$ rather then the number k. In other words, if the path ω passes through the edge $i \to j$ twice, i.e. $z_{k-1} = i,\ z_k = j$ and $z_{l-1} = i,\ z_l = j$, then $a[z_k z_{k-1}] = a[z_l z_{l-1}] = a[ji]$. Let us consider a more general case and define other exponent

$$\sigma(w) = \frac{1}{p}\sum_{k=1}^{p} \ln |\alpha[z_k z_{k-1}]|,$$

where each $\alpha[z_k z_{k-1}]$ depends on k. In other words, if a path ω passes twice through an edge $i \to j$, i.e., $z_{k-1} = i,\ z_k = j\ z_{l-1} = i,\ z_l = j$, and $k \neq l$, then

$\alpha[z_k z_{k-1}]$ and $\alpha[z_l z_{l-1}]$ can be different. Moreover, a value $\alpha[z_k z_{k-1}]$ is defined as

$$\alpha[z_k z_{k-1}] = a(x, y),\ where\ (x, y) \in M(z_{k-1}),$$

i.e., we do not require that the image $F(x, y)$ is in $M(z_k)$.

Definition 8 *Let $\omega = \{z_0, z_1, ..., z_p = z_0\}$ be an admissible periodic path on the symbolic image $G(PF)$. A nonstationary exponent $\sigma(w)$ of the path ω is defined by the equality*

$$\sigma(w) = \frac{1}{p} \sum_{k=1}^{p} \ln |\alpha[z_k z_{k-1}]|,$$

where

$$\alpha[z_k z_{k-1}] = a(x, y),\ (x, y) \in M(z_{k-1}),\ k = 1, ...p.$$

It is clear that the nonstationary exponent $\sigma(\omega)$ admits more variation than $\lambda(\omega)$.

Definition 9 *The set*

$$\Sigma(G(PF)) = \{\sigma \in R:\ there\ is\ a\ sequence\ of\ periodic\ paths\ \omega_k\ on\ G(PF)$$
$$with\ nonstationary\ exponents\ \sigma(\omega_k)\ such\ that\ \sigma(\omega_k) \to \sigma\ as\ k \to \infty\}$$

is called a spectrum of the symbolic image $G(PF)$ on the set of recurrent vertices RV.

It is evident that the spectrum $\Sigma(G(PF))$ does not depend on linear extension LG. To find the spectrum of a symbolic image we introduce the following notations

$$\alpha(i) = \min_{(x,y)\in M(i)} |a(x, y)|, \quad \beta(i) = \max_{(x,y)\in M(i)} |a(x, y)|.$$

Since each cell $M(i)$ is compact, there are points (x^*, y^*) and $(x^{**}, y^{**}) \in M(i)$ such that $\alpha(i) = |a(x^*, y^*)|$ and $\beta(i) = |a(x^{**}, y^{**})|$. The maximal and the minimal nonstationary exponents

$$\Lambda_{min}(\omega) = \frac{1}{p} \sum_{k=1}^{p} \ln \alpha(z_k), \quad \Lambda_{max}(\omega) = \frac{1}{p} \sum_{k=1}^{p} \ln \beta(z_k)$$

are defined for each periodic path $\omega = \{z_0, z_1, ..., z_p = z_0\}$. By picking $\alpha[z_k z_{k-1}]$ such that $|\alpha[z_k z_{k-1}]| = \alpha(z_{k-1})$, we realize $\Lambda_{min}(\omega)$ as a nonstationary exponent $\sigma(w) = \frac{1}{p} \sum_{k=1}^{p} \ln \alpha(z_{k-1})$ of the path ω. The same way, $\Lambda_{max}(\omega)$ is realized as a nonstationary exponent of the path ω. Therefore, we obtain the estimates

$$\Lambda_{min}(\omega) \le \sigma(\omega) \le \Lambda_{max}(\omega), \tag{1}$$

$$\Lambda_{min}(\omega) \leq \lambda(\omega) \leq \Lambda_{max}(\omega), \tag{2}$$

where $\sigma(\omega)$ is any nonstationary exponent, and $\lambda(\omega)$ is any characteristic exponent of a linear extension LG. Let $\phi_1, ..., \phi_q$ be a full collection of simple periodic paths of class H with periods $p_1, ..., p_q$, respectively. Set

$$\Lambda_{min}(H) = \min\{\Lambda_{min}(\phi_j),\ j = 1, ..., q\},$$

$$\Lambda_{max}(H) = \max\{\Lambda_{max}(\phi_j),\ j = 1, ..., q\}.$$

Theorem 2 *The spectrum $\Sigma(G(PF))$ of the symbolic image $G(PF)$ consists of the intervals $\{[\Lambda_{min}(H_k), \Lambda_{max}(H_k)]\}$, where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices on the symbolic image $G(PF)$.*

Proof. We fix a class H of equivalent recurrent vertices on the symbolic image. As indicated above, $\Lambda_{min}(\phi_j)$ and $\Lambda_{max}(\phi_j)$ are realized as some nonstationary exponents under a corresponding choice of $\alpha[z_k z_{k-1}]$. Since the number of simple periodic paths is finite, the exponents $\Lambda_{min}(H)$ and $\Lambda_{max}(H)$ are realized as well. By repeating the proof of Proposition 2, one can show that each nonstationary exponent of a periodic path is an arithmetic mean of nonstationary exponents of simple periodic paths with corresponding weights. From this it follows that the spectrum $\Sigma(G(PF))$ of the symbolic image $G(PF)$ is in $\cup_k[\Lambda_{min}(H_k), \Lambda_{max}(H_k)]$, where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices on the symbolic image $G(PF)$. By repeating the proof of Proposition 4, one can show that each $\lambda \in [\Lambda_{min}(H_k), \Lambda_{max}(H_k)]$ belongs to the spectrum of a symbolic image. Thus,

$$\Sigma(G(PF)) = \bigcup_k [\Lambda_{min}(H_k), \Lambda_{max}(H_k)],$$

where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices of the symbolic image $G(PF)$.

□

Theorem 3 *The spectrum $\Sigma(G(PF))$ offers the properties:*

1) spectrum of any linear extension $\Sigma(LG)$ is in $\Sigma(G(PF))$,

2) Morse spectrum $\Sigma(F)$ is in $\Sigma(G(PF))$.

Proof. 1) From the definition of a nonstationary exponent it follows, that each characteristic exponent $\lambda(\omega)$ of the periodic path $\omega = \{z_0, z_1, ...z_p = z_0\}$ can be realized as a nonstationary exponent $\sigma(\omega)$ setting $\alpha[z_k z_{k-1}] = a[z_k z_{k-1}]$, $k = 1, ...p$. From this it follows that the characteristic exponent $\lambda(\omega)$ of any linear extension LG is in the spectrum of a symbolic image.

2) Let us show that for any periodic ε-trajectory

$$\xi = \{(x_0, y_0), (x_1, y_1), ..., (x_p, y_p) = (x_0, y_0)\},\ \varepsilon < r,$$

where r is the lower bound of a symbolic image, there is an admissible periodic path $\omega = \{z_0, z_1, ...z_p = z_0\}$ such that the characteristic exponent $\lambda(\xi) = \frac{1}{p}\sum_{k=0}^{p-1} \ln |a(x_k, y_k)|$ of ξ is realized as a nonstationary exponent $\sigma(\omega)$ of ω. Let a vertex z_k be such that $(x_k, y_k) \in M(z_k)$. Since $\varepsilon < r$, according to Proposition 1 the path $\omega = \{z_0, z_1, ..., z_p = z_0\}$ is an admissible periodic path on the symbolic image. By setting $\alpha[z_k z_{k-1}] = a(x_{k-1}, y_{k-1})$, we obtain

$$\sigma(\omega) = \frac{1}{p}\sum_{k=1}^{p} \ln |\alpha[z_k z_{k-1}]| = \frac{1}{p}\sum_{k=0}^{p-1} \ln |a(x_k, y_k)| = \lambda(\xi).$$

Hence, a characteristic exponent of any periodic ε-trajectory, $\varepsilon < r$, is in $\Sigma(G(PF))$. Since the Morse spectrum is a limit set of periodic ε-trajectories as $\varepsilon \to 0$, it is in the spectrum of symbolic image $\Sigma(G(PF))$.

□

9 Estimate of Morse spectrum and symbolic image spectrum

In this section we find a family of intervals containing the desired spectra under the supposition that a spectrum $\Sigma(LG)$ of a linear extension of symbolic image is known. Since M is compact, the mapping $A(x)$ has a modulus of continuity $\eta_A(\rho)$ on x. Set

$$\eta(\rho) = \eta_A(\rho) + \max_{x\in M} |A(x)|\rho,$$

$$\theta = (\min_{x\in M,\ |e|=1} |A(x)e|)^{-1} = \max_{x\in M} |A^{-1}(x)|.$$

Proposition 5 *Let ω be an admissible periodic path on a symbolic image $G(PF)$, $\lambda(\omega)$ be a characteristic exponent for a linear extension LG, and $\sigma(\omega)$ be a nonstationary exponent of the same periodic path ω. Then*

$$|\lambda(\omega) - \sigma(\omega)| \leq \theta\eta(d),$$

where d is a maximal diameter of cells of covering $C(P)$.

Proof. Let (x, y) and (x^*, y^*) be two points from a cell $M(i)$, $a = a(x, y)$, and $a^* = a(x^*, y^*)$. We have the estimate

$$|a^* - a| \leq |A(x^*)e(y^*) - A(x)e(y)| \leq |A(x^*) - A(x)| + |A(x)||e(y^*) - e(y)| \leq$$

$$\eta_A(\rho(x,x^*)) + \max_x |A(x)|\rho_1(y^*,y) \le \eta(d),$$

where $\rho_1(*)$ is a distance on the projective manifold, d is a maximal diameter of cells of the covering $C(P)$.

Let LG be a linear extension of the symbolic image $G(PF)$ and $\omega = \{z_0, z_1, ..., z_p = z_0\}$ be a periodic path on $G(PF)$. Denote by $\lambda = \frac{1}{p}\sum_{k=1}^{p} \ln|a[z_k z_{k-1}]|$ and $\sigma = \frac{1}{p}\sum_{k=1}^{p} \ln|\alpha[z_k z_{k-1}]|$ a characteristic exponent and a nonstationary exponent of the path ω, respectively. Since $a[z_k z_{k-1}] = a(x_{k-1}, y_{k-1})$ and $\alpha[z_k z_{k-1}] = a(x^*_{k-1}, y^*_{k-1})$, where the points (x_{k-1}, y_{k-1}) and (x^*_{k-1}, y^*_{k-1}) are in the cell $M(k-1)$, we have the estimate

$$|a[z_k z_{k-1}] - \alpha[z_k z_{k-1}]| \le \eta(d).$$

From this it follows the estimate

$$|\lambda - \alpha| \le \frac{1}{p}\sum_{k=1}^{p} |\ln|a[z_k z_{k-1}]| - \ln|\alpha[z_k z_{k-1}]|| \le (\max_P \frac{1}{|a(x,y)|})\eta(d) =$$

$$\frac{\eta(d)}{\min_P |a(x,y)|} = \theta\eta(d). \quad (3)$$

□

According to Proposition 1, each ε-trajectory with $\varepsilon < r$ is realized as an admissible path on the symbolic image $G(PF)$. From this and Proposition 5 it follows

Proposition 6 *Let $\xi = \{(x_0, y_0), (x_1, y_1), ..., (x_p, y_p) = (x_0, y_0)\}$ be a periodic ε-trajectory, $\varepsilon < r$, and the vertices z_k are such that $(x_k, y_k) \in M(z_k)$. Then $\omega = \{z_0, z_1, ..., z_p = z_0\}$ is an admissible periodic path, and for any linear extension LG characteristic exponent $\lambda(\omega)$ offers the inequality*

$$|\lambda(\omega) - \lambda(\xi)| \le \theta\eta(d).$$

According to Theorem 1, the spectrum of the linear extension

$$\Sigma(LG) = \bigcup_k [\lambda_{\min}(H_k), \lambda_{\max}(H_k)],$$

where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices of the symbolic image $G(PF)$, $\lambda_{\min}(H_k)$ and $\lambda_{\max}(H_k)$ are the maximum and minimum of characteristic exponents of simple periodic paths of the class H_k.

Theorem 4 *The Morse spectrum $\Sigma(F)$ of the linear extension $F : E \to E$ and the spectrum $\Sigma(G(PF))$ are in*

$$\bigcup_k [\lambda_{\min}(H_k) - \theta\eta(d), \lambda_{\max}(H_k) + \theta\eta(d)],$$

where $\{H_k\}$ is the full family of classes of equivalent recurrent vertices on symbolic image $G(PF)$, d is a maximal diameter of cells of covering $C(P)$.

Proof. According to Theorem 3, the Morse spectrum is in the spectrum of a symbolic image. Thus, it is sufficient to prove that the last is in the described family of intervals. Let σ be in the spectrum of a symbolic image. This means that there is a sequence of periodic paths $\omega_m = \{z_0^m, z_1^m, ..., z_{p_m}^m = z_0^m\}$ and their nonstationary exponents

$$\sigma_m = \sigma(\omega_m) = \frac{1}{p_m}\sum_{i=1}^{p_m} \ln|\alpha[z_i^m z_{i-1}^m]|,$$

such that

$$\sigma_m \to \sigma \ as \ m \to \infty.$$

Let a periodic path ω_m be in the class H and LG be a linear extension of the symbolic image $G(PF)$. According to Theorem 1 the characteristic exponent

$$\lambda(\omega_m) = \frac{1}{p_m}\sum_{i=1}^{p_m} \ln|a[z_i^m z_{i-1}^m]|$$

of the path ω_m is in the interval $I = [\lambda_{\min}(H), \lambda_{\max}(H)]$. In addition, we have the estimate

$$|\alpha[z_i^m z_{i-1}^m] - a[z_i^m z_{i-1}^m]| \le \eta(d).$$

Since all exponents $\lambda(\omega_m)$ are in the interval $[\lambda_{min}(H), \lambda_{max}(H)]$, by (3) each nonstationary exponent

$$\sigma(\omega_m) = \frac{1}{p_m}\sum_{i=1}^{p_m} \ln|\alpha[z_i^m z_{i-1}^m]|$$

is in the closed interval $[\lambda_{\min}(H) - \theta\eta(d), \lambda_{\max}(H) + \theta\eta(d)]$. Hence, the limit

$$\lambda = \lim_{m\to\infty} \lambda_m$$

is in this interval as well.

□

We denote by $\Sigma(\widetilde{LG})$ a union of the intervals $\widetilde{I_k} = [\lambda_{\min}(H_k) - \theta\eta(d), \lambda_{\max}(H_k) + \theta\eta(d)] : \Sigma(\widetilde{LG}) = \cup_k \widetilde{I_k},$ where d is a maximal diameter of cells on the projective space and $\{H_k\}$ is the full family of classes of equivalent recurrent vertices. Recall that the Hausdorff distance $H(A, B)$ between sets A and B is defined by

$$H(A, B) = \max\{h(A, B),\ h(B, A)\},$$

where

$$h(A, B) = \sup_{u\in B} \rho(u, A) = \sup_{u\in B} \inf_{v\in A} \rho(u, v).$$

Theorem 5 *Let $\{LG^m\}$ be a sequence of linear extensions of images $\{G^m\}$ of PF with the maximal diameters d^m of cells. If $d^m \to 0$ as $m \to \infty$ then*

$$H(\Sigma(F), \Sigma(LG^m)) \to 0,$$
$$H(\Sigma(F), \Sigma(\widetilde{LG^m})) \to 0,$$
$$H(\Sigma(F), \Sigma(G^m)) \to 0.$$

as $m \to \infty$.

Proof. First we notice that

$$H(\Sigma(\widetilde{LG^m}), \Sigma(LG^m)) \leq \theta\eta(d^m), H(\Sigma(G^m), \Sigma(LG^m)) \leq \theta\eta(d^m),$$

$$H(\Sigma(\widetilde{LG^m}), \Sigma(G^m)) \leq \theta\eta(d^m),$$

and $\theta\eta(d^m) \to 0$ as $m \to \infty$. From this it follows that to prove theorem it is sufficient to establish one of the required limits. For example that $H(\Sigma(F), \Sigma(G^m)) \to 0$ as $m \to \infty$.

We prove by contradiction that $h(\Sigma(F), \Sigma(G^m)) \to 0$ as $m \to \infty$. Let us assume on the contrary that there are $\chi > 0$, a sequence of symbolic images $\{G^m\}$ and $\sigma^m \in \Sigma(G^m)$ such that $\rho(\sigma^m, \Sigma(F)) \geq \chi$. There is a sequence $\{\omega_j^m\}$ of periodic paths on G^m such that their nonstationary exponents $\sigma(\omega_j^m) \to \sigma^m$ as $j \to \infty$. According to Proposition 1, each periodic path ω_j^m generates a periodic ε^m-trajectory ξ_j^m on the projective space, $\varepsilon^m \to 0$ as $m \to \infty$. In addition, we can choose ξ_j^m such that its characteristic exponent λ_j^m coincides with $\sigma(\omega_j^m)$. One can consider that $\rho(\lambda_j^m, \Sigma(F)) \geq \frac{1}{2}\chi > 0$. Since $\{\lambda_j^m\}$ is bounded, there is a convergent subsequence $\lambda_{j(k)}^{m(k)} \to \lambda$ as $k \to \infty$. By definition of the spectrum, the limit λ has to be in $\Sigma(F)$ that contradicts to the inequality $\rho(\lambda_{j(k)}^{m(k)}, \Sigma(F)) \geq \frac{1}{2}\chi$. Thus, $h(\Sigma(F), \Sigma(G^m)) \to 0$ as $m \to \infty$. According to Theorem 3, $\Sigma(F) \subset \Sigma(G^m)$. From this it follows that $h(\Sigma(G^m), \Sigma(F)) = 0$. So, $H(\Sigma(F), \Sigma(G^m)) \to 0$ as $m \to \infty$.

□

Theorem 5 guarantees that a good estimate for Morse spectrum can be obtained if the largest cell diameter d of the covering $C(P)$ is sufficiently small. However, we do not have suitable estimate for the diameter d. In this case an algorithm constructing a monotone sequvence of sets converging to the Morse spectrum is of greatest practical utility. Let us consider one of them.

10 Localization of the chain recurrent set

Denote by $RV(d)$ the union of cells $M(i)$ for which the vertices i are recurrent :

$$RV(d) = \{\bigcup M(i) : i \ are \ recurrent\},$$

where d is the largest diameter of the cells $M(i)$. It should be noted that in fact the constructed set $RV(d)$ depends on the covering $C(P)$. However, in what follows we need only to consider the dependence of $RV(d)$ on d.

Theorem 6 [16]

1. *The set $RV(d)$ is a closed neighborhood of the chain recurrent set CR. Moreover, $RV(d)$ is a subset of ε-periodic points set for any $\varepsilon > q+d$, i.e.,*

$$RV(d) \subset Per(\varepsilon), \ \ \varepsilon > q + d.$$

2. *The chain recurrent set CR coincides with an intersection of the sets $RV(d)$ for all positive d:*

$$CR = \bigcap_{d>0} RV(d).$$

Theorem 6 makes possible to localize the chain recurrent set without preliminary information on a dynamical system [16]. The subdivision is a main step of the construction.

11 Subdivision process

Let $C = \{M(i)\}$ be a covering of P and G be a symbolic image for C. Suppose a new covering NC is produced by taking a subdivision of C, i.e., each cell $M(i)$ is subdivided. Denote by NG the symbolic image for NC. It is a convenience the cells of the new covering to designate as $M(i,k)$. Thus, the cells $M(i,k), \ k = 1, 2, \ldots$, form a partition of the cell $M(i)$:

$$\bigcup_k M(i,k) = M(i).$$

The vertices of the new symbolic image are denoted as (i,k). Some cells are not excluded actually to be not subdivided, i.e., $M(i,1) = M(i)$. The described subdivision generates a natural mapping S from NG on G which takes the vertices $(i,k), \ k = 1, \ldots$, onto the vertex i. Since from $PF(M(i,k)) \cap M(j,l) \neq \emptyset$ it follows $PF(M(i)) \cap M(j) \neq \emptyset$, the directed edge $(i,k) \to (j,l)$ is mapped onto the directed edge $i \to j$. Hence, the mapping S takes the directed graph NG into the directed graph G.

An algorithm localizing the chain recurrent set CR is available in [16]. It consists of the following steps:

1. Starting with an initial covering C, the symbolic image G of the map PF is found. The cells of the initial covering may have arbitrary diameter d_0 .
2. The recurrent vertices $\{i_k\}$ of the graph G are recognized. Using the recurrent vertices, a closed neighborhood $V = \{\cup M(i_k):\ i_k \textit{ is recurrent}\}$ of the chain recurrent set CR is found.
3. The cells corresponding to recurrent vertices $\{M(i_k) : i_k \textit{ is recurrent}\}$ are subdivided. For example, the largest diameter of the cells may be divided by 2. Thus, the new covering is defined.
4. The symbolic image G is constructed for the new covering. It should be noted that the new symbolic image may be constructed on the set $V = \{\cup M(i_k):\ i_k \textit{ is recurrent}\}$. In other words, the cells corresponding to non recurrent vertices do not participate in the construction of the new covering and the new symbolic image.
5. Then one goes back to the second step.

Repeating this subdivision process we obtain a sequence of neighborhoods $V_0,\ V_1,\ V_2,\ ...$ of the chain recurrent set CR and a sequence of the largest diameters $d_0,\ d_1,\ d_2,\ ...$. The following theorem substantiates the described algorithm for localization of the chain recurrent set.

Theorem 7 [16] *The sequence of sets $V_0,\ V_1,\ V_2,\ ...$ offers the following properties:*

(i) the neighborhoods V_k are imbedded one inside the other, i.e.,

$$V_0 \supset V_1 \supset V_2 \supset ... \supset CR,$$

(ii) if the largest diameters $d_k \to 0$ as k becomes infinite then

$$\lim_{k\to\infty} V_k = \bigcap_k V_k = CR. \tag{4}$$

12 Localization of Morse spectrum

Let us modify the subdivision process to localize the Morse spectrum of a dynamical system. To this end we apply the subdivision process to mapping PF and after the second step of the described algorithm we produce the following

2^*. The classes $\{H_m\}$ of equivalent recurrent vertices are found, and the family of simple periodic paths $\{\phi_j^m\}$ is recognized for each class H_m,

2^{**}. The intervals $I_m = [\Lambda_{\min}(H_m), \Lambda_{\max}(H_m)]$ are determined by using the families $\{\phi_j^m\}$.

Then one goes to the third step. Repeating the subdivision process and the calculation of intervals I_m we obtain

1) a sequence of neighborhoods $\{V_k\}$ of the chain recurrent set,

2) a sequence of the largest diameters of cells $\{d_k\}$,

3) a sequence of families of intervals $\{I_m^k\}$.

Theorem 8 *The constructed sequence of intervals offers the following properties:*

(i) each set $\Sigma^k = \cup_m I_m^k$ contains the Morse spectrum of F,

(ii) the sets Σ^k are embedded one inside the other, i.e.,

$$\Sigma^0 \supset \Sigma^1 \supset \Sigma^2 \supset ... \supset \Sigma(F),$$

(iii) if $d_k \to 0$ as k becomes infinite then

$$\lim_{k\to\infty} \Sigma^k = \bigcap_k \Sigma^k = \Sigma(F). \tag{5}$$

Proof. (i) follows from Theorem 3, and (iii) is a corollary of Theorem 5. To prove the statement (ii) we consider a covering $C = \{M(i)\}$ and a new covering $NC = \{m(ij)\}$ which is a subdivision of C, i.e., $\cup_j m(ij) = M(i)$. Let G and NG be symbolic images for the coverings C and NC, respectively. As indicated above, there is the mapping $S : NG \to G$ taking the directed graphs. In particular, S takes a periodic path on some periodic path. However, an image of simple path may be not simple periodic path. Consider an interval $NI = [\Lambda_{\min}(NH), \Lambda_{\max}(NH)]$ corresponding to some class NH of the new symbolic image. Since the mapping S takes the equivalent periodic paths into the equivalent periodic paths, the image $S(NH)$ is in some class H. Let us prove that the interval $I = [\Lambda_{\min}(H), \Lambda_{\max}(H)]$ contains the interval NI. In fact, according to Theorem 2, there is a simple periodic path $\phi \subset NG$ on which $\Lambda_{min}(NH)$ is realized as a nonstationary exponent, i.e.,

$$\Lambda_{\min}(NH) = \sigma(\phi) = \frac{1}{p}\sum_{k=1}^{p} \ln \alpha(z_k)$$

where $\phi = \{z_0, z_1, ..., z_p = z_0\}$, $\alpha(z_k) = \min\{|a(x,y)|,\ (x,y) \in m(z_k)\}$. Let a periodic path $\omega = \{z_1^*, ..., z_p^*\}$ be an image $S(\phi)$ of the simple periodic path ϕ with the minimal nonstationary exponent

$$\sigma(\omega) = \frac{1}{p}\sum_{k=1}^{p} \ln \alpha^*(z_k^*),$$

where

$$\alpha^*(z_k^*) = \min_{(x,y)\in M(z_k^*)} |a(x,y)|.$$

From the inclusion $m(z_k) \subset M(z_k^*)$ it follows $\alpha^*(z_k^*) \le \alpha(z_k)$ and $\sigma(\omega) \le \sigma(\phi) = \Lambda_{\min}(NH)$. Hence, $\Lambda_{\min}(H) \le \sigma(\omega) \le \Lambda_{\min}(NH)$. The same way, one can prove that $\Lambda_{\max}(NH) \le \Lambda_{\max}(H)$. Thus,

$$\Lambda_{\min}(H) \le \Lambda_{\min}(NH) \le \Lambda_{\max}(NH) \le \Lambda_{\max}(H),$$

i.e., $NI \subset I$.

□

13 Exponential estimates

Let us apply of the obtained results to estimate an action of the mapping F along an ε-trajectory $\xi = \{(x_i, y_i)\}$. According to Proposition 1, if $\varepsilon < r$, r is a lower bound of symbolic image, then there is an admissible path $\omega = \{z_i\}$ corresponding to the ε-trajectory ξ where $z_i :\ (x_i, y_i) \in M(z_i)$. By Theorem 2, the spectrum of the symbolic image $\Sigma(G(PF))$ consists of the intervals $\{[\Lambda_{min}(H_k), \Lambda_{max}(H_k)]\}$, where $\{H_k\}$ is the full family of equivalent recurrent vertices classes. The interval $[\Lambda_{min}(H), \Lambda_{max}(H)]$ is reasonably named a spectrum of class H.

Theorem 9 *If the spectrum of class H is in an interval $[a, b]$ then there exist positive constants K_* and K^* such that for any finite ε-trajectory $\xi = \{(x_0, y_0), (x_1, y_1), ..., (x_p, y_p)\}$, $\varepsilon < r$, r is a lower bound of symbolic image, for which the corresponding admissible path $\omega = \{z_i\}$, $z_i :\ (x_i, y_i) \in M(z_i)$ is in H, the following estimate holds*

$$K_* \exp(pa) \le \prod_{k=0}^{p-1} |F(x_k, e(x_k, y_k))| \le K^* \exp(pb). \qquad (6)$$

Proof. Recall that each directed edge $i \to j$ induces the value $\alpha[ji] = a(x,y),\ |a(x,y)| = |F(x,e(y))|$, where $(x,y) \in M(i),\ e(y)$ is a base vector in one-dimension space $L(y)$.

First we obtain the desired estimates for a simple periodic path. Let $\psi = \{z_0, z_1, ...z_p = z_0\}$ be a simple periodic path in the class H. By fixing the values $\alpha[z_k z_{k-1}],\ k = 1, ..., p$, one can find a nonstationary exponent

$$\sigma(\psi) = \frac{1}{p}\sum_{k=1}^{p} \ln|\alpha[z_k z_{k-1}]|.$$

Since it is in the interval $[a,b]$, we obtain the inequality

$$a \le \frac{1}{p}\sum_{k=1}^{p} \ln|\alpha[z_k z_{k-1}]| \le b.$$

From this it follows the estimate

$$\exp(pa) \le \prod_{k=1}^{p} |\alpha[z_k z_{k-1}]| \le \exp(pb).$$

Now, consider any path $\omega = \{z_0, z_1, ..., z_p\}$ in the class H. Let us decompose ω in a sum of simple paths as it done above. Let the path ω pass through a vertex z^* twice, i.e., $z^* = z_l = z_{l+p_1},\ p_1 > 0$. Consider two finite sequences $\omega_1 = \{z_0, z_1, ..., z_{l-1}, z_l = z_{l+p_1}, ..., z_p\}$ and $\omega_2 = \{z_l, z_{l+1}, ..., z_{l+p_1} = z_l\}$. By construction, the path ω_2 is an admissible periodic path of period p_2. We say that the path ω is a sum of paths ω_1 and ω_2, i.e.,

$$\omega = \omega_1 + \omega_2,$$

where ω_1 is a nonperiodic path, in general. By repeating the decomposition process, we come to the representation

$$\omega = k_1\psi_1 + k_2\psi_2 + ... + k_q\psi_q + \omega_0$$

of the path ω in a sum of simple periodic paths $\{\psi, \psi_2, ..., \psi_q\} \subset H$ plus some path ω_0 with different vertices of the class H. In this case $p = k_1p_1 + k_2p_2 + ...k_qp_q + p_0,\ k_i \ge 0,\ p_0$ is a length of the path ω_0. Clearly, p_0 is less than the maximal period of simple periodic paths of the class H, i.e., $p_0 \le t = \max\{p_j,\ j = 1, ..., q\}$. Let us obtain an estimate of the product

$$\Pi(\omega) = \prod_{k=1}^{p} |\alpha[z_k z_{k-1}]|, \tag{7}$$

where $\alpha[z_k z_{k-1}] = a(x_{k-1}, y_{k-1}),\ (x_{k-1}, y_{k-1}) \in M(z_{k-1})$. To do this we change the position of factors in (7) according to the decomposition

$$\omega = k_1\psi_1 + k_2\psi_2 + ... + k_q\psi_q + \omega_0.$$

We start with a product $\Pi(\omega_0)$ consisting of factors $\alpha^0[ji] = \alpha[ji],\ i \to j \in \omega_0$, corresponding to the path ω_0. Next are k_1 products $\Pi_i(\psi_1),\ i = 1, ..., k_1$ consisting of factors $\alpha^i[ji] = \alpha[ji],\ i \to j \in \psi_1$, corresponding to the simple periodic path ψ_1, k_2 products $\Pi_i(\psi_2)$ corresponding to the simple periodic paths ψ_2, and so on. As a result we get the representation of the product $\Pi(\omega)$ in the form

$$\Pi(\omega) = \prod_{j=1}^{q}(\prod_{i=1}^{k_j}\Pi_i(\psi_j)) \cdot \Pi(\omega_0).$$

For each factor $\Pi_i(\psi_j)$ we have the estimate

$$\exp(p_j a) \le \Pi_i(\psi_j) \le \exp(p_j b). \tag{8}$$

As the product $\Pi(\omega_0)$ has no more than t factors then

$$(K_{min})^t \le \Pi(\omega_0) \le (K_{max})^t, \tag{9}$$

where $K_{min} \le |\alpha[z^* z^{**}]| \le K_{max},\ z^* \to z^{**} \in H$, this we suppose $K_{min} \le 1 \le K_{max}$ without a loss of generality. From inequalities (8) and (9) it follows the estimate

$$(K_{min})^t \exp(a\sum_{j=1}^{q} k_j p_j) \le \Pi(\omega) \le (K_{max})^t \exp(b\sum_{j=1}^{q} k_j p_j).$$

Thus,

$$K_* \exp(pa) \le \Pi(\omega) \le K^* \exp(pb), \tag{10}$$

where

$$\begin{aligned} K_* &= \begin{cases} (K_{min})^t \exp(-at), & \text{fi}\Upsilon\ \ a > 0, \\ (K_{min})^t, & \text{fi}\Upsilon\ \ a \le 0; \end{cases} \\ K^* &= \begin{cases} (K_{max})^t \exp(-bt), & \text{fi}\Upsilon\ \ b < 0, \\ (K_{max})^t, & \text{fi}\Upsilon\ \ b \ge 0. \end{cases} \end{aligned} \tag{11}$$

From this it follows (6).

□

Corollary 1 *The constants K_* and K^* of the exponential estimate (6) are found by the formulas (11), where $K_{min} \le |F(x,e)| \le K_{max},\ K_{min} \le 1 \le K_{max},\ t$ is a maximal period of simple periodic paths of class H.*

Corollary 2 *Let $\xi = \{(x_k, y_k) = PF^k(x,y),\ k = 0, 1, ..., p\}$ be a finite part of trajectory of a point $(x,y) = (x_0, y_0) \in P$ such that a path $\omega = \{z_k\} :\ (x_k, y_k) \in M(z_k),\ k = 0, 1, ..., p$ corresponding to ξ is in H, then*

$$K_* \exp(pa)|v| \le |F^p(x,v)| \le K^* \exp(pb)|v|,$$

where $v \in L(x,y)$.

Proof. First, note that the estimate (6) holds for the trajectory ξ. If $(x_1, y_1) = PF(x, y)$ then a basis vector in the space $L(x_1, y_1)$ is of the form

$$e(x_1, y_1) = \frac{A(x)e(x,y)}{|A(x)e(x,y)|}.$$

Hence, we have

$$|F(x_1, e(x_1, y_1))||F(x, e(x,y))| = |A(f(x))e(x_1, y_1)||A(x)e(x,y)| =$$

$$|A(f(x))A(x)e(x,y)| = |F^2(x, e(x,y))|.$$

The same way we obtain the equality

$$\prod_{k=0}^{p-1} |F(x_k, e(x_k, y_k))| = |F^p(x, e(x,y))|,$$

where $e = v/|v|$. Then (6) takes the form

$$K_* \exp(pa)|v| \le |F^p(x, v)| \le K^* \exp(pb)|v|.$$

□

14 Chain-recurrent components in the project bundle

A subset $\Omega \subset CR$ is called a component of the chain-recurrent set if each two points from Ω can be connected by a periodic ε-trajectory for any $\varepsilon > 0$.

Recall some information on attractors. A closed invariant Lyapunov asymptotically stable set is called an attractor. An attractor of the inverse mapping f^{-1} is called a repeller of f. An intersection of an attractor and a repeller is called a Morse set.

Proposition 7 [3, 5] *An invariant set A is an attractor of f if there is a neighborhood U of A such that*

$$f(cl\ U) \subset U,\ A = \bigcap_{n>0} f^n(U),$$

where cl stands for the closure.

The described set U is called a fundamental neighborhood of an attractor A, and the set $W^s(A) = \cup_{n<0} f^n(U)$ is called an attraction domain of A.

Definition 10 [13] *A filtration of f is a finite sequence $\{U_0 = \emptyset,\ U_1, \dots, U_l = M\}$ of open sets such that $U_0 \subset U_1 \subset \dots \subset U_l$ and $f(clU_k) \subset U_k$ for each $k = 0, 1, ..., m$.*

The second condition is a property of fundamental neighborhood of an attractor. The next proposition describes a structure of attractors induced by a filtration.

Proposition 8 [5] *Let $\{U_0 = \emptyset, U_1, ..., U_l\}$ be a filtration. The following properties hold:*

(i) a maxilal invariant set in $U_k,\ k = 0, 1, ..., l$

$$A_k = \{\cap f^n(U_k):\ n \in Z^+\}$$

is an attractor, and

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = M,$$

(ii) a maximal invariant set in $U_k \setminus U_{k-1}$:

$$\Omega_k = \{\cap F^n(U_k \setminus U_{k-1}):\ n \in Z\},\ k = 1, ..., l,$$

is a Morse set, and the chain recurrent set CR is in $\cup_{k=1}^{l} \Omega_k$.

The family $\{\Omega_1, ..., \Omega_l\}$ is called a Morse decomposition.

Theorem 10 [5, 7, 26] *Let F be a linear extension of a homeomorphism $f: M \to M$ on a vector bundle (E, M, π), and PF be an induced by F mapping on the projective bundle $(P, M, P\pi)$. If Ω is a component of the chain-recurrent set of f on the base M then*

(i) the chain-recurrent set of the restriction $PF|_{P\pi^{-1}(\Omega)}$ has l components $\Omega_1, ..., \Omega_l,\ 1 \le l \le \dim E(x),\ x \in M$, which form a Morse decomposition,

(ii) each set Ω_i defines a (continuous, constant dimensional) subbundle E_i over Ω

$$E_i = \{v \in \pi^{-1}(\Omega):\ v \ne 0\ \Rightarrow [v] = y \in \Omega_i\},$$

(iii) the following decomposition into a Whitney sum holds

$$E|_\Omega = E_1 \oplus ... \oplus E_l,$$

(iv) conversely, each chain-recurrent component Ω_* on the projective space P is projected onto a chain-recurrent component Ω of the base M, which is of the form described in (ii).

In particular, from (iv) it follows that any component Ω_* of the chain-recurrent set on the projective bundle P meets each leaf $P\pi^{-1}(x)$ at a some projective manifold which continuously depends on $x \in \Omega = P\pi(\Omega_*)$. J.Selgrade

[26] proved that the described property holds not only for a component of the chain-recurrent set but for each Morse set Ω_* on the projective bundle. The decomposition described in (i)-(iii) is called the finest Morse decomposition on the projective bundle.

Let $C(M) = \{m(j)\}$ be a covering of the manifold M and $C(P) = \{M(z)\}$ be an agreed covering of the projective space P, i.e., a natural projection of each cell is a cell: $P\pi(M(z)) = m(j)$. We denote by $G(f)$ and $G(PF)$ the symbolic images of f and PF, respectively. As indicated above, for the agreed coverings there is a "natural projection" $h(z) = j$ from $G(PF)$ on $G(f)$, where $P\pi(M(z)) = m(j)$. Moreover, the mapping h takes the directed graph $G(PF)$ on the directed graph $G(f)$.

Theorem 11 *Let $G(f)$ and $G(PF)$ be the symbolic images of the mappings f and PF for the agreed coverings $C(M)$ and $C(P)$, respectively. If H is a class of equivalent recurrent vertices on the symbolic image $G(f)$ and $\{H_1, ..., H_l\}$ is a full collection of equivalent recurrent vertices classes on $G(PF)$ which are projected on H , i.e., $h(H_m) = H,\ m = 1, ..., l$, then*

(i) number l of classes $H_m :\ h(H_m) = H$ is less or equal to $\dim E(x),\ x \in M$,

(ii) if Ω_m is a maximal invariant set in $V_m = \{\cup M(z),\ z \in H_m\}$ then the family $\{\Omega_1, ...\Omega_l\}$ is a Morse decomposition,

(iii) each Ω_m defines a (continuous, constant dimensional) subbundle E_m over a component Ω being in $V = \{\cup m(j),\ j \in H\}$:

$$E_m = \{v \in \pi^{-1}(\Omega) :\ v \neq 0 \Rightarrow [v] = y \in \Omega_m\},$$

(iv) the following decomposition into a Whitney sum holds

$$E|_\Omega = E_1 \oplus ... \oplus E_l,$$

(v) if the spectrum of the class H_m is in an interval $[a_m, b_m]$ then for each point $(x, v) \in E_m$ and every integer $p > 0$ the following estimate hold

$$K_* \exp(pa_m)\ |v| \leq |F^p(x, v)| \leq K^* \exp(pb_m)\ |v|,$$

where the constants K_* and K^* are given by (11),

(v) if 1 is not in the spectrum of classes $\{H_m\}$ then the linear extension F is hyperbolic over Ω,

(vi) for each component Ω of the chain-recurrent set on the base M there exists $d_0 > 0$ such that for every covering $C(P)$ with the maximal diameter of cells

$d < d_0$ the full family $\{H_m\}$ of equivalent recurrent vertices on $G(PF)$ induces (see(ii)) a decomposition $\{\Omega_1, ..., \Omega_l\}$ which is the finest Morse decomposition over Ω.

Proof. The statement (i) follows from (ii)-(iv). The statement (ii) follows from the results of the paper [18]. The statements (iii) and (iv) are proved in [5], p. 117. Corollary 2 guarantees the validity of (v). The statement (vi) follows from Theorem 6.

□

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