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KINK-ANTIKINK SOLUTIONS OF A NERVE CONDUCTION EQUATION

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Abstract

We have solved a nerve conduction equation in terms of hyperbolic functions and found the restrictions on the coefficients of the governing equation and the solution parameters for the kink-antikink solutions to exist.

1. Introduction

The governing equation for wave front propagation in non-uniform axon cables is

$$u_{xx} + r(x)\,u_x - u_t - p(u) \;=\; 0, \tag{1.1}$$

where $r(x)$ is a real function of cable diameter, and

$$p(u) \;=\; q_0 + q_1\,u + q_2\,u^2 + q_3\,u^3, \;\; q_i \;\in\; R. \tag{1.2}$$

The infinitesimal operators of the symmetry group of the partial differential equation (1.1) have already been found [1, 2] using Sophus Lie's theory of symmetry groups [3]. The equation (1.1) has a wide literature and applications [4, 5, 6, 7].

The aim of this paper is to derive kink-antikink solutions of equation (1.1) with $r(x) \;=\;$ constant, using a direct method [4,5] using the properties of hyperbolic functions. The method is to balance the highest nonlinear term and the derivative term for some combination of hyperbolic functions and then to equate the like powers of the functions on both sides. For example, our solution will be of the form

$$u(z) \;=\; \sum_{i=0}^{m} a_i\, tanh^i\, \mu z, \tag{1.3}$$

where the expansion coefficient $\{a_i\}_0^m$, order of expansion m, and wave number μ are to be determined. It is easy to note that a derivative term u_{nz} of n^{th} order has highest power in $tanh\,\mu z$ of $m+n$. Therefore, for equations in u possessing a highest derivative term of order d and highest nonlinear term u^h, we have,

$$m \;=\; \frac{d}{h-1} \tag{1.4}$$

2. Nerve Conduction Equation

We consider the equation

$$u_{xx} + A\,u_x - u_t - (q_0 + q_1\,u + q_2\,u^2 + q_3\,u^3) \;=\; 0, \;\;\; q_i \;\in\; R, \tag{2.1}$$

where, A is a constant, which is a model equation for wavefront propagation in uniform axon cables. A change of variable $u = u^* + \alpha$, where α is a constant, transvfers the equation (2.1) to the form

$$u^*_{xx} + Au^*_x - u^*_t - (q^*_1 u^* + q^*_2 u^{*2} + q^*_3 u^{*3}) + P_3(\alpha) = 0,$$

where q^*_1, q^*_2, q^*_3 are the constant depending on α and $P_3(\alpha)$ is a polynom of degree 3. If α is a root of the polynom we get an equation of the form (2.1) with $q_0 = 0$. So without losss of generality we can put $q_0 = 0$ in (2.1). We look for travelling wave solutions of equation (2.1) in the form

$$u(z) = u(x - vt), \tag{2.2}$$

where, v is the wave velocity.

Using (2.2) in (2.1) and rearranging the terms, we get

$$u_{zz} = (v - A)\, u_z + (q_1\, u + q_2\, u^2 + q_3\, u^3). \tag{2.3}$$

Equation (2.3) is an ordinary differential equation of second order involving a third degree polynomial in u. Consistent with equation (1.4), we assume a solution of (2.3) in the form

$$u(z) = a_0 + a_1\, tanh\, \mu z, \tag{2.4}$$

where, a_0, a_1 and μ are parameters to be determined. Substituting (2.4) into (2.3) and equating like powers of $tanh\, \mu z$ on both sides, we get the following equations:

$$2\, a_1\, \mu^2 = a_1^3\, q_3, \tag{2.5}$$

$$0 = (A - v)\, a_1\, \mu + q_2\, a_1^2 + 3\, a_0\, a_1^2\, q_3, \tag{2.6}$$

$$-2\, a_1\, \mu^2 = q_1\, a_1 + 2\, a_0\, a_1\, q_2 + 3\, a_0^2\, a_1\, q_3, \tag{2.7}$$

$$0 = (v - A)\, a_1\, \mu + q_1\, a_0 + q_2\, a_0^2 + q_3\, a_0^3. \tag{2.8}$$

Equations (2.5) to (2.7) give rise to

$$\mu = \pm \sqrt{\frac{q_3}{2}}\, a_1, \tag{2.9}$$

$$a_0 = \frac{1}{3q_3} \left\{ \pm \sqrt{\frac{q_3}{2}}\, (v - A) - q_2 \right\}, \tag{2.10}$$

$$a_1^2 = \frac{1}{3q_3^2}\left\{q_2^2 - \frac{q_3}{2}(v-A)^2\right\} - \frac{q_1}{q_3}. \tag{2.11}$$

Equation (2.8) gives a constraint equation involving the parameters a_0, a_1 and μ. It is quite clear that solution in the form (2.4) is possible only when q_3 is positive.

We set

$$q_1 = \frac{q_2^2 - 3\,q_3}{3\,q_3} \tag{2.12}$$

so that

$$a_1^2 = \frac{6-(v-A)^2}{6\,q_3}, \tag{2.13}$$

which restrict the solution to be valid only when

$$|v-A| < \sqrt{6}. \tag{2.14}$$

Since we are looking for kink-antikink solutions, we put $a_1 = sa_0$, where s is a real number. Using expressions (2.10) and (2.13), we get

$$a_0 = \frac{-3\,q_2 \pm \sqrt{(27+9s^2)q_3 - 3s^2q_2^2}}{2(9+3s^2)q_3}. \tag{2.15}$$

Since q_3 is always positive, we take the positive sign in (2.15) when q_2 is positive and the negative sign when q_2 is negative. Therefore, for a_0 to be positive, q_2 should be less than $\sqrt{3q_3}$ when it is positive and q_2 should be greater than $-\sqrt{3q_3}$ when it is negative. From (2.12) it follows that for a_0 to be positive, q_1 should be negative for a positive q_2 and q_1 should be positive for a negative q_2.

With $a_1 = sa_0$, our solution $u(z)$ can be written in the form

$$u(z) = a_0\,(1 + s\,tanh\,\mu z). \tag{2.16}$$

Case 1. $s > 0,\ \mu > 0$

In this case, if a_0 is positive, a_1 will be positive and so $\mu = \sqrt{\frac{q_3}{2}}\,a_1$ which gives rise to a kink solution. If a_0 is negative, a_1 becomes negative and then $\mu = -\sqrt{\frac{q_3}{2}}\,a_1$ which leads to an anti-kink solution. These solutions are shown in Fig.1 and Fig.4.

Case 2. $s < 0, \ \mu > 0$

Here, if a_0 is positive, a_1 has negative sign and so $\mu = -\sqrt{\frac{q_3}{2}}\, a_1$ which is the case of an anti-kink solution. If a_0 is negative, a_1 becomes positive and so $\mu = \sqrt{\frac{q_3}{2}}\, a_1$ which gives rise to a kink solution. These solutions are represented by Fig.2 and Fig.3.

Case 3. $s > 0, \ \mu < 0$

In this case, if a_0 is positive, a_1 will be positive and so, $\mu = -\sqrt{\frac{q_3}{2}}\, a_1$ which leads to an anti-kink solution. If a_0 is negative, a_1 becomes negative and then $\mu = \sqrt{\frac{q_3}{2}}\, a_1$ which is the case of a kink solution. These solutions are again shown in Fig.2 and Fig.3.

Case 4. $s < 0, \ \mu < 0$

Here, if a_0 is positive, a_1 will be negative and so, $\mu = \sqrt{\frac{q_3}{2}}\, a_1$ which gives rise to a kink solution. If a_0 is negative, a_1 becomes positive and $\mu = -\sqrt{\frac{q_3}{2}}\, a_1$ which leads to an anti-kink solution. These solutions are again represented by Fig.1 and Fig.4.

Thus when s and μ have same signs, (2.16) is a kink solution when a_0 is positive, that is, when $q_1 < 0$ for positive q_2 and $q_1 > 0$ for negative q_2. (2.16) is an anti-kink solution when a_0 is negative.

When s and μ have opposite signs, (2.16) is a kink solution when a_0 is negative, that is, when $q_1 > 0$ for positive q_2 and $q_1 < 0$ for negative q_2. (2.16) is an anti-kink solution for positive a_0.

3. Conclusion

We have derived kink-antikink solutions of a nerve conduction equation explicitly in terms of hyperbolic functions. It is quite interesting to note that by this direct method we could obtain certain conditions involving the constant coefficients of the governing equation and the solution parameters for such solutions to exist. We could find that for a solution in the form (2.16) to exist the coefficients of the highest nonlinear term must be strictly positive and (2.12) should be satisfied.

This is only a preliminary mathematical study of the nerve conduction equations. An investigation about the stability of the solutions and its biological relevance will be done in the future. Solutions of these equations using other methods and comparison of solutions is also a matter of future research.

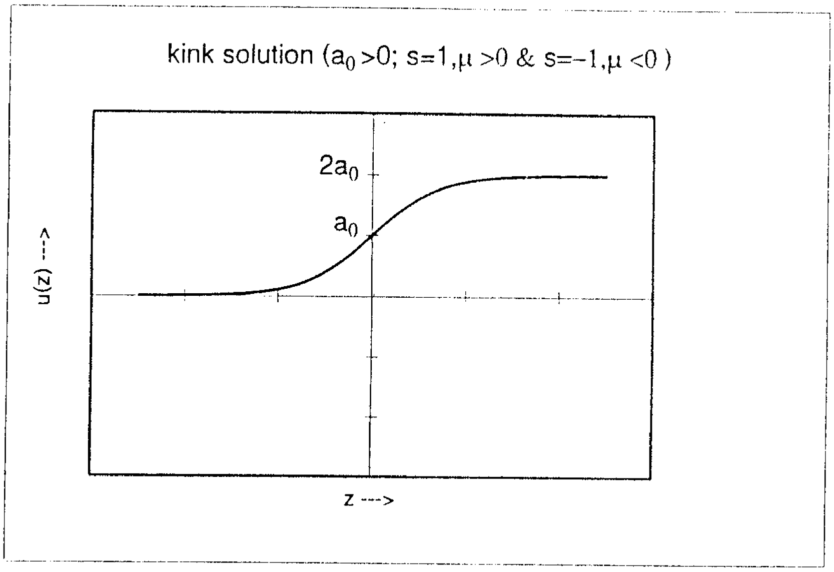

Fig 1.

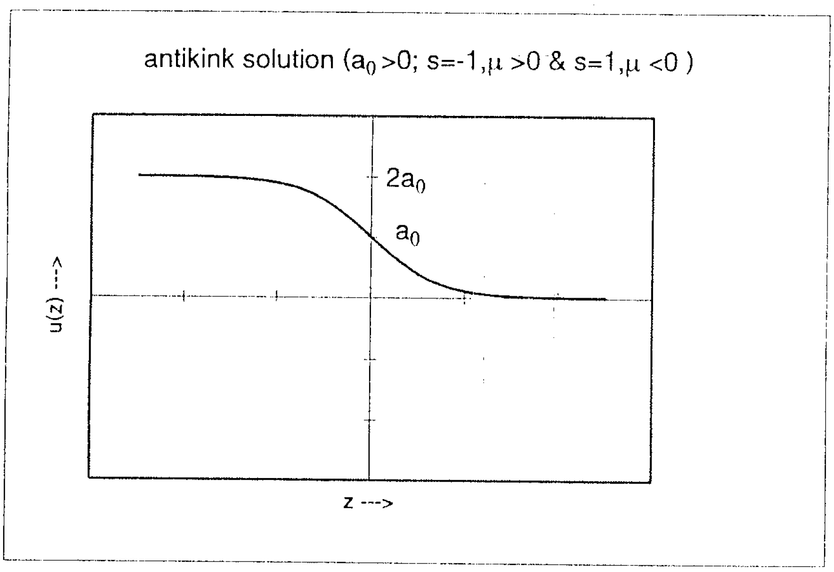

Fig 2.

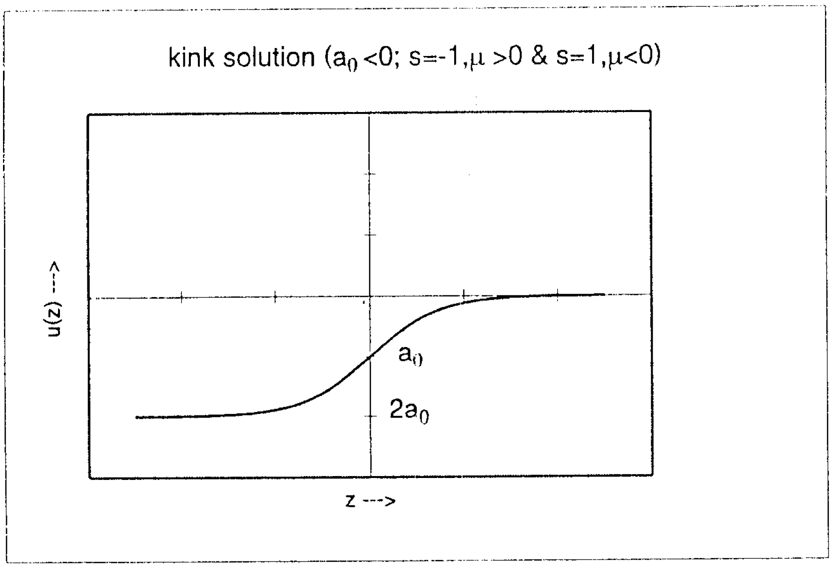
kink solution (a0 <0; s=-1,μ >0 & s=1,μ<0)
u(z) --->
a0
2a0
z --->

Fig 3

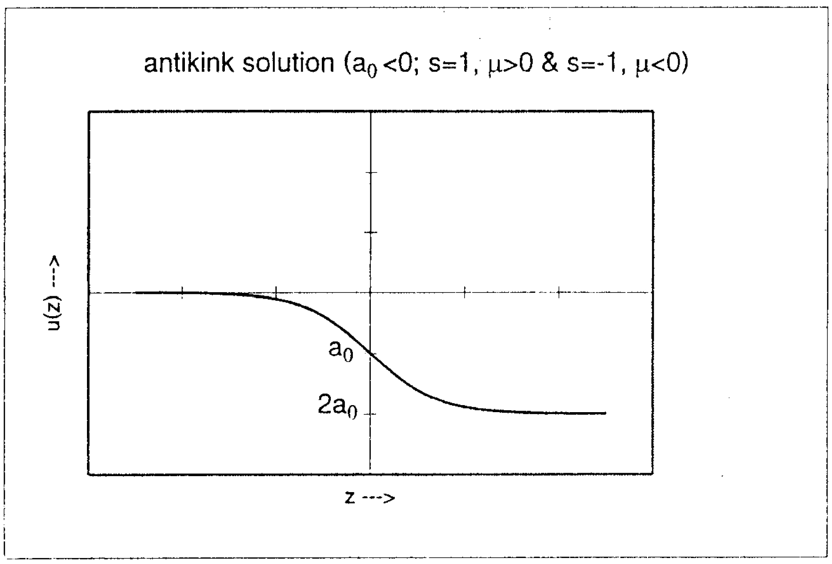
antikink solution (a0 <0; s=1, μ>0 & s=-1, μ<0)
u(z) --->
a0
2a0
z --->

Fig 4.

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