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Bifurcation

GLOBAL BIFURCATIONS OF LIMIT CYCLES

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Abstract

Two-dimensional polynomial dynamical systems are considered. We develop Erugin's two-isocline method for the global analysis of such systems, construct canonical systems with field-rotation parameters and study limit cycle bifurcations. Using the canonical systems, cyclicity results and Wintner–Perko termination principle, we outline a global approach to the solution of Hilbert's Sixteenth Problem.

1 Introduction

We consider two-dimensional dynamical systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of real variables x, y with real coefficients. The main problem of the qualitative theory of such systems is Hilbert's Sixteenth Problem:

Problem. *Find the maximum number and relative position of limit cycles of* (1).

This is the most difficult problem in the qualitative theory of polynomial systems. There are a lot of methods and results on the study of limit cycles. But the Problem has not been solved completely even for the case of simplest (quadratic) systems. It is known only that a quadratic system has at least four limit cycles in (3 : 1) distribution (see [1–4]).

There are three bifurcations of limit cycles: 1) Andronov–Hopf bifurcation (from a singular point of the center or focus type); 2) separatrix cycle bifurcation (from a homoclinic or heteroclinic orbit); 3) multiple limit cycle bifurcation. The first bifurcation was studied completely only for quadratic systems. N. N. Bautin proved that the number of limit cycles bifurcating from a singular point (its cyclicity) was equal to three [5]. Recently H. Żołądek found out that for cubic systems the cyclicity of a singular point was not less than eleven [6]. The second bifurcation has been intensively studying by F. Dumortier, R. Roussarie and C. Rousseau. Now we have the classification of separatrix cycles, know the cyclicity of the most of them (of elementary graphics) and have got some global results [7–9]. The last bifurcation is the most complicated. Multiple limit cycles were considering, for instance, by J.-P. Françoise, C. C. Pough [10] and L. M. Perko [11–13]. All mentioned bifurcations can be generalized for higher-dimensional dynamical systems and can be used for various applications [14–19].

However all these bifurcations of limit cycle are local bifurcations. We consider only a neighborhood of either the point or the separatrix cycle, or the multiple limit cycle studying only the corresponding sufficiently small neighborhood in the parameter space. It needs a qualitative investigation on the whole (both on the whole phase plane and on the whole parameter space), i.e., it needs a global bifurcation theory. This is the first idea introduced for the first time by N. P. Erugin in [20]. Then we should connect all limit cycle bifurcations. This idea came from the theory of higher-dimensional dynamical systems. It was contained in Wintner's principle of natural termination [21] and was used by L. M. Perko for the study of multiple limit cycles in two-dimensional case [11–13]. At last, we must understand how to control the limit cycle bifurcations. The best way to do it is to use field-rotation parameters considered for the first time by G. F. D. Duff in [22]. All these ideas were considered in [23–27] and will be developed in this paper.

2 Previous results

In [25] we showed how to apply the two-isocline method to the global qualitative investigation of polynomial systems (1). This method was developed by N. P. Erugin for two-dimensional systems [20] and then was generalized by V. A. Pliss for the three-dimensional case [14]. An isocline portrait is the most natural construction in the corresponding polynomial equation. It is enough to have only two isoclines (isoclines of zero and infinity) to obtain principal information on the original system, because these two isoclines correspond to the right-hand sides of the system. We know geometric properties of isoclines (conics, cubics, etc.) and can easily get all isocline portraits. By means of them we can obtain all topologically different qualitative pictures of integral curves to within a number of limit cycles and distinguishing center and focus. Hence we are able to carry out the rough topological classification of the phase portraits for the polynomial systems.

Using Erugin's two-isocline method, we can give, for example, a geometric interpretation of all four cases of center for the corresponding quadratic equation

$$\frac{dy}{dx} = \frac{x + ax^2 + bxy + cy^2}{-y + mxy + ny^2}. \tag{2}$$

1) Axial symmetry:

$$\frac{dy}{dx} = \frac{x + ax^2 + cy^2}{-y + mxy}. \tag{3}$$

2) Local symmetry (zero divergence) on the whole phase plane (Hamiltonian case):

$$\frac{dy}{dx} = \frac{x + ax^2 + cy^2}{-y - 2cxy + ny^2}. \tag{4}$$

3) Orthogonality of asymptotes of hyperbolas forming the family of isoclines (Lotka–Volterra case):

$$\frac{dy}{dx} = \frac{x + ax^2 + bxy - ay^2}{-y + mxy}. \tag{5}$$

4) Orthogonality of asymptotes of saddles at infinity:

$$\frac{dy}{dx} = \frac{x - x^2 - 5nxy + 2(n^2+1)y^2}{-y + (6n^2+1)xy - ny^2}. \tag{6}$$

Studying contact and rotation properties of isoclines we can also construct the simplest (canonical) systems containing limit cycles.

Theorem 1 [25]. *Any quadratic system with limit* (*separatrix*) *cycles can be reduced to one of the systems either*

$$\dot{x} = -y(1+x) + \alpha Q(x,y), \quad \dot{y} = Q(x,y) \tag{7}$$

or

$$\dot{x} = -y(1+\nu y), \quad \dot{y} = Q(x,y), \quad \nu = 0,1, \tag{8}$$

where

$$Q(x,y) = x + \lambda y + ax^2 + \beta y(1+x) + cy^2.$$

The advantage of systems (7) and (8) is that they contain the minimal number of the essential parameters and some of these parameters rotate the vector field. More precisely, it is true

Lemma [25]. *Parameters α and β rotate the vector field of systems* (7) *and* (8) *on the whole phase plane*: *when any of these parameters increases, the field is rotated in negative direction* (*clockwise*); *when they decrease, the field is rotated in positive direction* (*counterclockwise*). *Parameter λ rotates the field in the half-planes $x > -1$ and $x < -1$ in opposite directions*: *when it increases* (*decreases*), *the field is rotated in negative direction in the half-plane $x > -1$* ($x < -1$) *and in positive direction in the half-plane $x < -1$* ($x > -1$).

Two groups of parameters can be distinguished in such systems: static (a, c) and dynamic (α, β, λ). Static parameters determine the behavior of the phase trajectories in principle, since they control the number, position and type of singular points in finite part of the plane (finite singularities). Parameters from the first group determine also a possible behavior of separatrices and singular points at infinity (infinite singularities) under the variation of parameters from the second group. Dynamic parameters are rotation parameters. They typically do not change the number, position and index of finite singularities and involve a directional rotation in the vector field (in general, finite singular points can move under the variation of some such parameters). The rotation parameters allow to control infinite singularities, the behavior of limit cycles and separatrices. The cyclicity of singular points and separatrix cycles, the behavior of semi-stable and other multiple limit cycles can be studied by means of these parameters as well. Obviously, the number of limit cycles depends on the number of rotation parameters. Thus with the help of rotation parameters we can control all limit cycle bifurcations, i.e., we can solve the finest qualitative problems and carry out the global qualitative investigation of the polynomial systems.

Basing on the center cases and applying field-rotation parameters, we developed a new approach to the classification of separatrix cycles [24, 25]. The classification was carried out according to the number and the type of finite singularities of the original reversible systems and with the help of the successive variation of rotation parameters. We considered the following cases of singular points: 1) three saddles and one antisaddle; 2) two saddles and two antisaddles; 3) one saddle and three antisaddles; 4) simple saddle and antisaddle; 5) two simple antisaddles (nondegenerate cases) and 6) degenerate cases.

That approach allowed not only to define all possible types of separatrix cycles, but also to control their cyclicity and relative position, to keep the track of limit cycles (including multiple limit cycles), to obtain both the corresponding phase portraits and the corresponding division in the parameter space.

Earlier, in [23], we studied limit cycle bifurcations of various codimensions for a similar quadratic system with field-rotation parameters and introduced so-called a function of limit cycles: a cross-section of the Andronov–Hopf manifold formed by the limit cycles and the corresponding values of a rotation parameter. Using numerical and analytical methods, we constructed concrete examples of systems with different number and relative position of limit cycles. In particular, an example of the system with at least four limit cycles in $(3:1)$ distribution was constructed. In that work we considered the case of two singular points and two field-rotation parameters and showed that in such two-parameter families semi-stable limit cycles always moved either to the origin or to the separatrix cycle under the variation of the rotation parameters. Their termination was indicated either by vanishing the first focus quantity at the origin or by vanishing the divergence (or the equivalent value) at the saddle (the saddle points) lying on the separatrix cycle.

In [26, 27] we used all this information and by means of the field-rotation parameters and functions of limit cycles we tried to control semi-stable limit cycles changing the rotation parameters so that to push the semi-stable limit cycles either to a singular point of focus (centre) type or to some separatrix cycle and to obtain a contradiction with their cyclicity.

3 Local bifurcation surfaces of multiple limit cycles

Let us first rewrite system (1) in the vector form

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{\mu}), \tag{9}$$

where $\boldsymbol{x} \in \boldsymbol{R}^2, \boldsymbol{\mu} \in \boldsymbol{R}^n, \boldsymbol{f} \in \boldsymbol{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function), then recall a few basic facts about multiple limit cycles and formulate Perko's theorems [13] on the local existence of fold, cusp, swallow-tail and multiplicity-m limit cycle bifurcation surfaces for polynomial system (9).

Assume that system (9) has a limit cycle

$$L_0 : \boldsymbol{x} = \boldsymbol{\phi}_0(t)$$

of minimal period T_0 at some parameter value $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$. Let l be the straight line normal to $\boldsymbol{L}_0$ at the point $\boldsymbol{p}_0 = \boldsymbol{\phi}_0(0)$ and let r denote the coordinate along l with r positive on the exterior of L_0. It then follows from the implicit function theorem for analytic functions that there is a $\delta > 0$ such that the Poincaré map $h(r, \boldsymbol{\mu})$ is defined and analytic for $|r| < \delta$ and $\|\boldsymbol{\mu}-\boldsymbol{\mu}_0\| < \delta$ [1]. The displacement function for (9) along the normal line l to L_0 is then defined as the function

$$d(r, \boldsymbol{\mu}) = h(r, \boldsymbol{\mu}) - r.$$

In term of the displacement function, a limit cycle L_0 of (9) is *a multiple limit cycle* iff $d(0, \boldsymbol{\mu}_0) = d_r(0, \boldsymbol{\mu}_0) = 0$ and it is *a simple limit cycle* (or hyperbolic limit cycle) if it is not a multiple limit cycle; furthermore, L_0 is a limit cycle of multiplicity m iff

$$d(0, \boldsymbol{\mu}_0) = d_r(0, \boldsymbol{\mu}_0) = \ldots = d_r^{(m-1)}(0, \boldsymbol{\mu}_0) = 0, \quad d_r^{(m)}(0, \boldsymbol{\mu}_0) \neq 0.$$

The multiplicity of L_0 is independent of the point $\boldsymbol{p_0} \in \boldsymbol{L_0}$ through which we take the normal line l [2].

The following formulas, which determine the derivatives of the displacement function in terms of integrals of the vector field $\boldsymbol{f}$ along the periodic orbit $\boldsymbol{\phi}_0(t)$, are classical [2]:

$$d_r(0, \boldsymbol{\mu}_0) \;=\; e^{\int_0^{T_0} \boldsymbol{\nabla}\cdot \boldsymbol{f}(\boldsymbol{\phi}_0(t), \boldsymbol{\mu}_0)\, dt} - 1$$

and

$$d_{\mu_j}(0, \boldsymbol{\mu}_0) \;=\; \frac{-\omega_0}{\|\boldsymbol{f}(\boldsymbol{\phi}_0(0), \boldsymbol{\mu}_0)\|} \int_0^{T_0} e^{-\int_0^t \boldsymbol{\nabla}\cdot \boldsymbol{f}(\boldsymbol{\phi}_0(s), \boldsymbol{\mu}_0)\, ds} \boldsymbol{f} \wedge \boldsymbol{f}_{\mu_j}(\boldsymbol{\phi}_0(t), \boldsymbol{\mu}_0)\, dt$$

for $j = 1, \dots, n$, where $\omega_0 = \pm 1$ according to whether L_0 is positively or negatively oriented, respectively, and where the wedge product of two vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\boldsymbol{R}^2$ is defined as $\boldsymbol{x} \wedge \boldsymbol{y} = x_1 y_2 - x_2 y_1$. Similar formulas for $d_{rr}(0, \boldsymbol{\mu}_0)$ and $d_{r\mu_j}(0, \boldsymbol{\mu}_0)$ can be derived in terms of integrals of the vector field $\boldsymbol{f}$ and its first and second partial derivatives along $\boldsymbol{\phi}_0(t)$. The hypotheses of the theorems in this section will be stated in terms of conditions on the displacement function $d(r, \boldsymbol{\mu})$ and its partial derivatives at $(0, \boldsymbol{\mu}_0)$.

In this section we formulate Perko's theorems on the local existence of $(n-m+1)$-dimensional surfaces, C_m, of multiplicity-m limit cycles for polynomial system (9) with $\boldsymbol{\mu} \in \boldsymbol{R}^n$ and $n \geq m \geq 2$. These results describe the topological structure of the codimension $(m-1)$ bifurcation surfaces C_m. For $m = 2, 3$, and 4, C_2, C_3, and C_4 are the familiar fold, cusp, and swallow-tail bifurcation surfaces; for $m \geq 5$, the topological structure of the surfaces C_m is more complex. For instance, C_5 and C_6 are butterfly and wigwam bifurcation surfaces respectively. Since the proofs of the theorems in this section, describing the universal unfolding near a multiple limit cycles of (9), parallel the proofs in elementary catastrophe theory, they are not included in this paper (see also [13] for more details).

Theorem 2. *Suppose that $n \geq 2$, that for $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$ system* (9) *has a multiplicity-two limit cycle L_0, and that $d_{\mu_1}(0, \boldsymbol{\mu}_0) \neq 0$. Then given $\epsilon > 0$, there is a $\delta > 0$ and a unique function $g(\mu_2, \dots, \mu_n)$ with $g(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = \mu_1^{(0)}$, defined and analytic for $|\mu_2 - \mu_2^{(0)}| < \delta, \dots, |\mu_n - \mu_n^{(0)}| < \delta$, such that for $|\mu_2 - \mu_2^{(0)}| < \delta, \dots, |\mu_n - \mu_n^{(0)}| < \delta$,*

$$C_2 : \mu_1 = g(\mu_2, \dots, \mu_n)$$

is an $(n-1)$-dimensional, analytic fold bifurcation surface of multiplicity-two limit cycles of (9) *through the point $\boldsymbol{\mu}_0$.*

Theorem 3. *Suppose that $n \geq 3$, that for $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$ system* (9) *has a multiplicity-three limit cycle L_0, that $d_{\mu_1}(0, \boldsymbol{\mu}_0) \neq 0$, $d_{r\mu_1}(0, \boldsymbol{\mu}_0) \neq 0$ and for $j = 2, \dots, n$,*

$$\Delta_j \equiv \frac{\partial(d, d_r)}{\partial(\mu_1, \mu_j)}(0, \boldsymbol{\mu}_0) \neq 0.$$

Then given $\epsilon > 0$, there is a $\delta > 0$ and constants $\sigma = \pm 1$ for $j = 2, \dots, n$, and there exist unique functions $h_1(\mu_2, \dots, \mu_n)$, $h_2(\mu_2, \dots, \mu_n)$ and $g^{\pm}(\mu_2, \dots, \mu_n)$ with $h_1(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = \mu_1^{(0)}$, $h_2(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = \mu_1^{(0)}$ and $g^{\pm}(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = \mu_1^{(0)}$, where h_1 and h_2 are defined and analytic for $|\mu_j - \mu_j^{(0)}| < \delta$, $j = 2, \dots, n$, *and* $g^{\pm}$ *are defined and continuous for* $0 \le \sigma_j(\mu_j - \mu_j^{(0)}) < \delta$ *and analytic for* $0 < \sigma_j(\mu_j - \mu_j^{(0)}) < \delta$, $j = 2, \dots, n$ *such that*

$$C_3 : \begin{cases} \mu_1 = h_1(\mu_2, \dots, \mu_n) \\ \mu_1 = h_2(\mu_2, \dots, \mu_n) \end{cases}$$

is an $(n-2)$*-dimensional, analytic, cusp bifurcation surface of multiplicity-three limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$ *and*

$$C_2^{\pm} : \mu_1 = g^{\pm}(\mu_2, \dots, \mu_n)$$

are two $(n-1)$*-dimensional, analytic, fold bifurcation surfaces of multiplicity-two limit cycles of* (9) *which intersect in a cusp along* C_3.

Theorem 4. *Suppose that* $n \ge 4$, *that for* $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$ *system* (9) *has a multiplicity-four limit cycle* L_0, *that* $d_{\mu_1}(0, \boldsymbol{\mu}_0) \ne 0$, $d_{r\mu_1}(0, \boldsymbol{\mu}_0) \ne 0$, $d_{rr\mu_1}(0, \boldsymbol{\mu}_0) \ne 0$, *and that for* $j = 2, \dots, n$,

$$\frac{\partial(d, d_r)}{\partial(\mu_1, \mu_j)}(0, \boldsymbol{\mu}_0) \ne 0, \quad \frac{\partial(d, d_{rr})}{\partial(\mu_1, \mu_j)}(0, \boldsymbol{\mu}_0) \ne 0, \quad \frac{\partial(d_r, d_{rr})}{\partial(\mu_1, \mu_j)}(0, \boldsymbol{\mu}_0) \ne 0.$$

Then given $\epsilon > 0$, *there is a* $\delta > 0$ *and constants* $\sigma_{jk} = \pm 1$ *for* $j = 2, \dots, n$, $k = 1, 2$, *and there exist unique functions* $g_i(\mu_2, \dots, \mu_n)$, $h_k^{\pm}(\mu_2, \dots, \mu_n)$ *and* $F_i(\mu_2, \dots, \mu_n)$, *with* $g_i(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = h_k^{\pm}(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = F_i(\mu_2^{(0)}, \dots, \mu_n^{(0)}) = \mu_1^{(0)}$, *for* $i = 0, 1, 2$ *and* $k = 1, 2$, *where* F_i *is defined and analytic for* $i = 0, 1, 2$, *and* $|\mu_j - \mu_j^{(0)}| < \delta$, $j = 2, \dots, n$, $h_k^{\pm}$ *are defined and continuous for* $0 \le \sigma_{jk}(\mu_j - \mu_j^{(0)}) < \delta$ *and analytic for* $0 < \sigma_{jk}(\mu_j - \mu_j^{(0)}) < \delta$, $j = 2, \dots, n$, $k = 1, 2$, *and for* $i = 0, 1, 2$, g_i *is defined and analytic in the cuspidal region between the surfaces* $\mu_1 = h_2^{\pm}(\mu_2, \dots, \mu_n)$, *which intersect in a cusp, and* g_i *is continuous in the closure of that region, such that*

$$C_4 : \begin{cases} \mu_1 = F_0(\mu_2, \dots, \mu_n) \\ \mu_1 = F_1(\mu_2, \dots, \mu_n) \\ \mu_1 = F_2(\mu_2, \dots, \mu_n) \end{cases}$$

is an $(n-3)$*-dimensional, analytic, swallow-tail bifurcation surface of multiplicity-four limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$ *which is the intersection of two* $(n-2)$*-dimensional, analytic, cusp bifurcation surfaces of multiplicity-three limit cycles of* (9),

$$C_3^{\pm} : \begin{cases} \mu_1 = h_1^{\pm}(\mu_2, \dots, \mu_n) \\ \mu_1 = h_2^{\pm}(\mu_2, \dots, \mu_n) \end{cases}$$

which intersect in a cusp along C_4; *furthermore,* $C_3^+ = C_2^{(0)} \bigcap C_2^{(1)}$ *and* $C_3^- = C_2^{(0)} \bigcap C_2^{(2)}$ *where for* $i = 0, 1, 2$,

$$C_2^i : \mu_1 = g_i(\mu_2, \dots , \mu_n)$$

are $(n-1)$*-dimensional, analytic, fold bifurcation surfaces of multiplicity-two limit cycles of* (9) *which intersect in cusps along* $C_3^{\pm}$ *and in an* $(n-2)$*-dimensional, analytic surface* $C_2^{(1)} \bigcap C_2^{(2)}$ *on which* (9) *has two multiplicity-two limit cycles.*

Theorem 5. *Given* $m \geq 2$. *Suppose that* $n \geq m$, *that for* $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$ *polynomial system* (9) *has a multiplicity-*m *limit cycle* L_0, *that*

$$\frac{\partial d}{\partial \mu_1}(0, \boldsymbol{\mu}_0) \neq 0, \quad \frac{\partial d_r}{\partial \mu_1}(0, \boldsymbol{\mu}_0) \neq 0, \quad \dots , \quad \frac{\partial d_r^{(m-2)}}{\partial \mu_1}(0, \boldsymbol{\mu}_0) \neq 0,$$

and that

$$\frac{\partial (d_r^{(i)}, d_r^{(j)})}{\partial (\mu_1, \mu_k)}(0, \boldsymbol{\mu}_0) \neq 0$$

for $i, j = 0, \dots , m-2$ *with* $i \neq j$ *and* $k = 2, \dots , n$. *Then given* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for* $\|\boldsymbol{\mu} - \boldsymbol{\mu}_0\| < \delta$, *system* (9) *has*

(1) *a unique* $(n-m+1)$*-dimensional analytic surface* C_m *of multiplicity-*m *limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$;

(2) *two* $(n-m+2)$*-dimensional analytic surfaces* C_{m-1} *of multiplicity-*$(m-1)$ *limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$ *which intersect in a cusp along* C_m;

...

(j) *exactly* j, $(n-m+j)$*-dimensional analytic surfaces* C_{m-j+1} *of multiplicity-*$(m-j+1)$ *limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$ *which intersect pairwise in cusps along the bifurcation surfaces* C_{m-j+2};

...

$(m-1)$ *exactly* $(m-1)$, $(n-1)$*-dimensional analytic fold bifurcation surfaces* C_2 *of multiplicity-two limit cycles of* (9) *through the point* $\boldsymbol{\mu}_0$ *which intersect pairwise in a cusp along the* $(n-2)$*-dimensional cusp bifurcation surfaces* C_3.

Remark. As in [13], it can be shown that the set of polynomial vector fields $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$ satisfying the hypotheses of Theorem 5 is an open, dense subset of the set of all polynomial vector fields having a multiplicity-m limit cycle L_0 at a point $\boldsymbol{\mu} = \boldsymbol{\mu}_0 \in \boldsymbol{R}^n$; i.e., the codimension $(m-1)$ bifurcation at L_0, described in Theorem 5, is generic.

4 Global bifurcations

In this section we use the results from Sections 2 and 3 to develop a global approach to the study of limit cycle bifurcations. For instance, in [8] by means of Abelian integrals a complete study of quadratic three-parameter unfoldings of some integrable system was carried out and for small perturbations of the system a versal bifurcation diagram and global phase portraits including the precise number and configuration of the limit cycles were obtained. We use more general Wintner–Perko termination principle [13] describing a global behaviour of multiple limit cycles to connect all local bifurcations of limit cycles and to develop a global bifurcation theory of polynomial systems (9) on the whole parameter space. For (9) this principle can be formulated in the following way:

Theorem 6 (Termination Principle). *Any one-parameter family of multiplicity-m limit cycles of a polynomial system* (9) *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* (9) *which is either open or cyclic. If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of* (9)*, which is typically a fine focus of multiplicity m, or on a* (*compound*) *separatrix cycle of* (9)*, which is also typically of multiplicity m.*

In order to obtain a one-parameter family of multiplicity-m limit cycles of (9), we can use the results on establishing the local existence of the corresponding bifurcation surfaces which were formulated in [13] for the case when $n = m$. To show that such a one-parameter family of multiplicity-m limit cycles can be uniquely continued through any bifurcation and to prove the termination principle, L. M. Perko used arcs and paths of (multiplicity-m) limit cycles which were originally introduced by J. Mallet-Paret and J. A. Yorke in their work [15]. After defining arcs and paths of multiplicity-m limit cycles of (9), he applied Puiseux series as in [12] to show how the Poincaré map or displacement function for (9) can be used to define a local analytic path of multiplicity-m limit cycles, how this path can be uniquely continued through any bifurcation and how it can be extended to a unique maximal one-parameter family of multiplicity-m limit cycles of (9) which is either open or cyclic and which satisfies the Termination Principle. This principle implies that the boundary of any global multiple limit cycle bifurcation surface typically (generically) consists of Hopf bifurcation surfaces of the same multiplicity and/or homoclinic (or heteroclinic) loop bifurcation surfaces also of the same multiplicity. However, there are examples [13] which show that in non-generic cases, a one-parameter family of multiplicity-m

limit cycles can terminate at a singular point (focus) or on a separatrix cycle of multiplicity $k > m$; it can also terminate at a center or at a generate singular point in a Bogdanov–Takens (or cusp) bifurcation.

Besides, the Termination Principle is too general to be applied directly to such specific problem as Hilbert's 16th Problem. For example, we do not know precisely what parameters of system (9) really control the multiple limit cycles, we have no complete information about the boundary of the global bifurcation surface of multiple limit cycles, we do not know how to separate the case when the maximal one-parameter family of multiple limit cycles is cyclic, etc. Therefore it makes sense to consider, first, the quadratic case of (9). By means of canonical systems (7), (8) we can outline the proof of the following conjecture:

Conjecture 1. *There exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, a quadratic system cannot have neither a multiplicity-four limit cycle nor four limit cycles around a singular point* (*focus*) *and the maximum multiplicity or the maximum number of limit cycles surrounding a focus is equal to three.*

Proof. Let us give a sketch of proof of this conjecture. The proof is carried out by contradiction. We suppose that system (7) containing three field-rotation parameters α, β, λ has four limit cycles around the origin; then we get into some three-dimensional domain of these parameters being restricted by some conditions on the rest two parameters a, c corresponding to the definite case of singular points in the phase plane [25]. The three-parameter domain of four limit cycles is bounded by three fold bifurcation surfaces forming a swallow-tail bifurcation surface of multiplicity-four limit cycles [13]. It can be shown that the corresponding maximal one-parameter family of multiplicity-four limit cycles cannot be cyclic and terminates either at the origin or on some separatrix cycle surrounding the origin, because its termination is indicated either by vanishing the divergence and the first focus quantity at the origin or by vanishing the divergence (or the equivalent value) at the saddle (or saddle points) lying on the separatrix cycle [23, 25]. Since we know absolutely precisely at least the cyclicity of the singular point (Bautin's result) which is equal to three, we have got a contradiction with the Termination Principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate [13]. This contradiction concludes the proof.

Since we know the concrete properties of all field-rotation parameters in (7), (8) and, besides, we are able to control simultaneously bifurcations of limit cycles around different singular points, we can formulate also

Conjecture 2. *The maximum number of limit cycles in a quadratic system is equal to four and the only possible their distribution is* (3 : 1).

5 Conclusion and applications

In a similar way cubic and more general polynomial systems can be considered. Thus, generalizing the obtained results and using the Termination Principle, we develop a global bifurcation theory of planar polynomial dynamical systems.

Perko's termination principle is a consequence of Wintner's principle of natural termination which was stated for higher-dimensional dynamical systems and was applied for studying one-parameter families of periodic orbits of the restricted three-body problem [21]. By means of Puiseux series, it was shown that in the analytic case any one-parameter family of periodic orbits can be uniquely continued through any bifurcation except a period-doubling bifurcation. Besides, there exist higher-dimensional systems where the periods in a one-parameter family can become unbounded in strange ways: for example, the periodic orbits may belong to a strange invariant set (strange attractor) generated at a bifurcation value for which there is a homoclinic tangency of the stable and unstable manifolds of the Poincaré map. Such bifurcations can occur even in three-dimensional quadratic systems of Lorenz type. It would be interesting to construct a three-dimensional system with a strange attractor

$$\dot{x} = P(x, y, \varepsilon z), \quad \dot{y} = Q(x, y, \varepsilon z), \quad \varepsilon\dot{z} = R(x, y, z, \varepsilon) \tag{10}$$

on the base, for example, of a planar quadratic system with two unstable foci and an invariant straight line.

Applying the obtained results, we can also carry out the global qualitative analysis of two-dimensional polynomial dynamical systems simulating complicated generation-recombination processes in semiconductors. Basing on these processes, new types of transistors are worked out in micro- and nanoelectronics, and the qualitative analysis of the mathematical models helps to obtain more optimal characteristics for the transistors. We consider also possibilities of application of the global bifurcation theory to the study of generalized (polynomial and nonpolynomial) higher-dimensional Lotka–Volterra systems describing the dynamics in complex ecological models.

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