On the stability of nonlinear systems with the monotonic differentiable nonlinear characteristics

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Abstract.

Some new frequency criteria of stability of pulse systems with the monotonic differentiable static characteristics of pulse element are obtained.

1 Problem setting

Suppose, nonlinear operator $\mathcal{M}$, mapping a continuous signal $\sigma(t)$ on the modulator input into a signal $f(t)$ on its output, has the following properties.

a) For any $\sigma(t) \in C[0, +\infty)$ there exists a sequence $t_n \ (n = 0, 1, \ldots; \ t_0 = 0)$ such that

$$\delta_0 T \leq t_{n+1} - t_n \leq T \quad (0 < \delta_0 < 1, \ T > 0)$$

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and the function $f(t)$ is piecewise continuous and does not change its sign on the interval $[t_n, t_{n+1})$;
b) $t_n$ depends only on $\sigma(t)$ for $t \leq t_n$, $f(t)$ depends only on $\sigma(\tau)$ for $\tau \leq t$;
c) for any $n$ there exists $\tilde{t}_n \in [t_n, t_{n+1})$ such that the mean value of the $n$-th pulse

$$v_n = \frac{1}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} f(t) \, dt$$

is related with $\sigma(\tilde{t}_n)$ by formula

$$v_n = \varphi(\sigma(\tilde{t}_n)),$$

where $\varphi(\sigma)$ is a continuously differentiable function (static characteristics of pulse element) such that :

$$0 < \frac{\varphi(\sigma)}{\sigma} < \frac{1}{\sigma_*} \text{ for } \sigma \neq 0$$

$$0 \leq \frac{d\varphi}{d\sigma} \leq l.$$  

$$\frac{\varphi(\sigma)}{|\sigma|} \to 0 \text{ for } |\sigma| \to \infty.$$

Properties a), b), c) turn out to be ordinary for the most of modulators used in technology.

Consider a pulse system described by the following functional differential equation

$$\dot{x} = Ax + bf, \quad \sigma = c'x, \quad f = M\sigma,$$  

where $A$ is a constant Hurwitz $m \times m$-matrix, $b$ and $c$ are constant $m$-dimensional columns

The problem is to define the properties of the transfer functions $W(p) = c'(A - pI_m)^{-1}b$, which assure the asymptotics $x(t) \to 0$ as $t \to +\infty$ for any $x(0)$.

2 The formulation of result

Consider system (5) and suppose that the following conditions

$$\lim_{p \to \infty} pW(p) = \lim_{p \to \infty} p^2W(p) = 0.$$  

(6)
are satisfied.

**Theorem** Suppose that the transfer function $W(p)$ is nondegenerate, relations (1)-(4), (6) are valid, and there exist positive constants $\tau, \tau_1, \varepsilon_1, \varepsilon_2$ and $\kappa \geq 0$ such that for all $\omega \in [0, +\infty]$ the frequency condition holds

$$\alpha(\omega) \beta(\omega) - |\delta(\omega)|^2 > 0,$$

(7)

where

$$\alpha(\omega) = \tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3} - \varepsilon_3 \omega^2 |W(i\omega)|^2 + (\tau + \kappa \omega^2) Re W(i\omega) - \kappa \varepsilon_1 \omega^4 |W(i\omega)|^2,$$

$$\beta(\omega) = \tau_1 - \varepsilon_3 \omega^4 |W(i\omega)|^2 - \kappa \varepsilon_1 [\kappa_1 + i\omega^3 W(i\omega)]^2,$$

$$\delta(\omega) = \kappa \kappa_1 \frac{i\omega}{2} (\tau + \kappa \omega^2) W(i\omega) - \varepsilon_3 i\omega^3 |W(i\omega)|^2 - \kappa \varepsilon_1 [(\kappa_1 + i\omega^3 W(i\omega)) \omega^2 (W(i\omega))],$$

$$\nu = \tau(\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3}, \varepsilon_3 = \frac{T^2}{\pi^2} \left( \frac{\kappa l^2}{\varepsilon_1} + \frac{\tau}{\varepsilon_2} \right),$$

$$\kappa_1 = \lim_{p \to \infty} p^3 W(p).$$

Then solutions of system (5) have asymptotics $x(t) \to 0$ as $t \to +\infty$ for any $x(0)$.

This theorem extends result, obtained in [1] under condition $\kappa_1 = 0$, to case $\kappa_1 \neq 0$.

### 3 The proof of theorem

We introduce, following [2], the functions $v(t) = v_n$ for $t_n \leq t < t_{n+1}$, $u(t) = \int_0^t [f(t) - v(t)] dt$, and $y = x - bu$ and transform system (5) to the form

$$\dot{y} = Ay + bv + Abu.$$  

(8)

The objective of such a transformation is in finding the system such that the function $f$ is excluded, the function $v$ is a ”frozen” function $\varphi(\sigma(t))$, and the function $u$ is small in a certain sense.

By (6) we have

$$c' b = c' Ab = 0.$$  

(9)

Consider now the Lyapunov function [3]

$$V = y^* Hy - \kappa c' Ay \varphi(\sigma),$$  

(10)
where \( H \in \mathbb{R}^{k \times k} \) is a constant positively definite matrix, which will be given below. Differentiating (10) by using system (8) and taking into account the following equality

\[
\dot{\sigma} = c' Ay,
\]

which is resulted from (9), we obtain

\[
\dot{V} = W_1 - \kappa c' A^2 y \varphi - \kappa (\dot{\sigma})^2 \frac{d \varphi}{d \sigma},
\]

where \( W_1 = 2y' H (Ay + bv + Abu) \). Applying the S-procedure with the coefficients \( \tau \) and \( \tau_1 \), we transform the above relation into

\[
\dot{V} = W_1 - \kappa c' A^2 y \varphi + W_2 + \tau (\sigma - \sigma_*) v + \tau_1 \left( \frac{T^2}{3} v^2 - u^2 \right),
\]

where \( \sigma(t) = \sigma(\tilde{t}_n) \) for \( t_n \leq t < t_{n+1} \),

\[
W_2 = -\kappa \sigma^2 \frac{d \varphi}{d \sigma} - \tau (\sigma - \sigma_*) v - \tau_1 \left( \frac{T^2}{3} v^2 - u^2 \right).
\]

Using (2), (3) and the property, stated in [2],

\[
\int_{t_n}^{t_{n+1}} u^2 dt \leq \frac{(t_{n+1} - t_n)^2}{3} \int_{t_n}^{t_{n+1}} v^2 dt
\]

the following estimate holds

\[
\int_{t_n}^{t_{n+1}} W_2 dt \leq 0.
\]

Having performed the changes of variables in (12), namely, \( \varphi = v + (\varphi - v) \), \( \sigma = c'y + (\sigma - \sigma) \), we obtain

\[
\dot{V} = W_1 + W_2 - \kappa c' A^2 y v + \tau (c'y - \sigma_*) v + W_3 + \tau_1 \left( \frac{T^2}{3} v^2 - u^2 \right),
\]

where \( W_3 = \kappa c' A^2 y (v - \varphi) + \tau (\sigma - \sigma_*) v \). By (3), estimate \( W_3 \) takes the form

\[
W_3 \leq \kappa [\varepsilon_1 (c'A^2 y)^2 + \frac{l^2 (\sigma - \sigma)^2}{4 \varepsilon_1}] + \tau [\varepsilon_2 v^2 + (\sigma - \sigma)^2]
\]

According to the inequality of Virtinger [2] and property (1), the following estimate

\[
\int_{t_n}^{t_{n+1}} (\sigma - \sigma)^2 dt \leq \frac{4T^2}{\pi^2} \int_{t_n}^{t_{n+1}} \dot{\sigma}^2 dt
\]
is valid. Therefore by (15) and (11)
\[
\int_{t_n}^{t_{n+1}} W_3 dt \leq \tau \varepsilon_2 \int_{t_n}^{t_{n+1}} v^2 dt + \varepsilon_3 \int_{t_n}^{t_{n+1}} (c' Ay)^2 dt + \kappa \varepsilon_1 \int_{t_n}^{t_{n+1}} (c' A^2 y) dt.
\]
By (13), (14), (16) we have
\[
\int_{t_n}^{t_{n+1}} \dot{V} dt \leq \int_{t_n}^{t_{n+1}} (W_1 - G) dt,
\]
where G is a quadratic form with the real coefficients
\[
G(y, v, u) = \left[ \tau (\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3} \right] v^2 + \\
+ \tau_1 u^2 - \varepsilon_3 (c' Ay)^2 + (\kappa c' A^2 y - \tau c' y) v - \tau_1 |u|^2 - \kappa \varepsilon_1 |c' A^2 y|^2.
\]
Extending it to the Hermitean one, we obtain
\[
G(y, v, u) = \left[ \tau (\sigma_* - \varepsilon_2) - \frac{\tau_1 T^2}{3} \right] |v|^2 + \tau_1 |u|^2 - \varepsilon_3 |c' Ay|^2 + \\
+ Re[(\kappa c' A^2 y - \tau c' y)\overline{v}], -\tau_1 |u|^2 - \kappa \varepsilon_1 |c' A^2 y|^2,
\]
where v, u are complex numbers, \(\overline{v}\) is a complex number, associated with v, y \(\in \mathbb{C}^k\). Having performed the Laplace transformation with the zero initial conditions in (8) and saving a notation of variables, we arrive at a formula
\[
y = -(A - p I_m)^{-1} bv - (A - p I_m)^{-1} Abu.
\]
Using the representation \(A = (A - p I_m) + p I_m\) and properties (9), we find
\[
c' y = - W(p) v - p W(p) u, \\
c' Ay = - p W(p) v - p^2 W(p) u, \\
c' A^2 y = \kappa_1 u - p^3 W(p) v - p^2 W(p) u.
\]
Substituting these expressions into (18) and putting \(p = i \omega\), we obtain
\[
G \mid_{p=i\omega} = \alpha(\omega) |v|^2 + \beta(\omega) |u|^2 + 2 Re[\delta(\omega) u\overline{v}].
\]
If the Hermitean matrix
\[
\begin{pmatrix}
\alpha & \delta \\
\overline{\delta} & \beta
\end{pmatrix}
\]
is positively definite, then by the frequency theorem of V.A. Yakubovich for the nondegenerate case [4] there exist \( \mu > 0 \) and a positively definite matrix \( H \) such that the term under integral sign in (17) may be estimated as follows

\[
W_1 - G < -\mu(u^2 + v^2 + |y|^2).
\]

The Silvester criterion implies that for the positive definiteness of this matrix it is necessary and sufficient that for all \( \omega \in [0, +\infty] \) the inequalities

\[
\alpha(\omega) > 0, \quad \alpha(\omega)\beta(\omega) - |\delta(\omega)|^2 > 0
\]

were satisfied. The second inequality coincides with frequency condition (7) and therefore is also satisfied. The first inequality follows directly from the second one. Really, as \( \omega \to \infty \) \( \beta \to \tau_1 \), \( \delta \to 0 \), and, consequently, \( \alpha(\infty) > 0 \) and \( \alpha(\omega) \) cannot be zero for any \( \omega \).

Thus estimate (19) is proved.

Relations (17) and (19) resulted in the inequality

\[
V \mid_{t=t_n} + \mu \int_0^{t_n} (v^2 + u^2 + |y|^2)dt \leq V \mid_{t=0}.
\]

By (4) and by the positive definiteness of the matrix \( H \) \( V \to +\infty \) as \( |y| \to \infty \). Therefore (20) implies that \( u, v \in L_2[0, +\infty) \). Since the matrix \( A \) is the Hurwitzian one, it follows from (8) that \( y(t) \to 0 \) as \( t \to +\infty \). From that \( v \in L_2[0, +\infty) \) and from (1) the asymptotics \( v(t) \to 0 \) as \( t \to +\infty \) follows. In [2] it is stated that \( |u| \leq T|v| \) and therefore \( u \to 0 \) as \( t \to +\infty \). Finally, from the relation \( x = y - bu \) it follows that \( x \to 0 \) as \( t \to +\infty \). The proof of theorem is completed.

References


