DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 2, 2001
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Control problems in nonlinear systems

THE STRUCTURE MATRIX OF DYNAMICAL SYSTEM [1]

GEORGE OSIPENKO

St. Petersburg State Technical University, Russia
E-mail: math@math.hop.stu.neva.ru

SALIH AYTAR

Suleyman Demirel University, Turkey
E-mail: aytar@fef.sdu.edu.tr

SERGEY KOBYAKOV

St. Petersburg State Technical University, Russia
E-mail: ksergeu@mail.ru

1 Definitions and results.

Let us consider a discrete dynamical system governed by a homeomorphism $f : M \to M$, where M is a $C^\infty-$smooth compact manifold.

Definition 1 *[3] Let $\varepsilon > 0$ be given. An infinite in both direction sequence $\{x_k,\ k \in Z\}$ is named an ε-trajectory or pseudo-trajectory or pseudo-orbit of f if for any k the distance between the image $f(x_k)$ and x_{k+1} is less than ε:*

$$\rho(f(x_k), x_{k+1}) < \varepsilon,$$

[1] This research was supported by the Scientific and Technical Research Council of Turkey (TUBITAK-NATO-PC prog.) and Russian Program "Integracia".

where $\rho(x,y)$ is the distance between x and y on M and Z is the set of integers. A pseudo-trajectory $\{x_k\}$ is said to be ε-periodic if there is p such that $x_k = x_{k+p}$ for each $k \in Z$. In this case the number p is named a period of the pseudo-trajectory.

Definition 2 *[11] A point x is called chain recurrent if x is ε-periodic for each positive ε, i.e., there exists a periodic ε-trajectory passing through x. A chain recurrent set, denoted Q, is the set of all the chain recurrent points.*

It is known, that the chain recurrent set Q is invariant, closed, and contains periodic, homoclinic, nonwandering and other singular trajectories. It should be remarked that if a chain recurrent point is not periodic then there exists as small as one likes perturbation of f in C^0-topology for which this point is periodic [30],[39],[40]. One may say that a chain recurrent point may become periodic under a small C^0-perturbation of the map f.

A subset $\Omega \subset Q$ is called a component of the chain-recurrent set if each two points from Ω can be connected by a periodic ε-trajectory for any $\varepsilon > 0$. Denote by $\alpha(x)$ and $\omega(x)$ the α and $\omega-$limit sets of the trajectory through x, respectively. Let $\{Q_1, Q_2, Q_3, ...\}$ be chain recurrent set components of a dynamical system. A connection $Q_i \to Q_j$ is said to exist if there is a point x such that $\alpha(x) \subset Q_i$ and $\omega(x) \subset Q_j$. Let $g : M \to M$ be a continuous mapping and a distance $\rho(f,g) = \max_M \rho(f(x), g(x))$. Denote a support of the difference f and g by $\text{supp}(f-g) = \{x \in M : f(x) \neq g(x)\}$. The connections $\{Q_i \to Q_j\}$ are said to be stable if there exists $\varepsilon > 0$ such that any perturbation $g,\ \rho(f,g) < \varepsilon,\ \text{supp}(f-g) \subset M \backslash Q$, has the same connections $\{Q_i \to Q_j\}$.

Definition 3 *Let the matrix $S = (s_{ij})$ be such that $s_{ij} = 1$ if there is the connection $Q_i \to Q_j$, $s_{ii} = 1$ and $s_{ij} = 0$ in other case. The matrix S is named the structure matrix of dynamical system f.*

By the definition, the structure matrix is a topological invariant. A size on the structure matrix is $q \times q$, where q is a number of components. So the number q may be the infinity. The main result of paper is

Theorem 1 *If the dynamical system has a finite number of chain recurrent components with the stable connections then there exist a finite algorithm for construction of the structure matrix.*

Example 1 ***A perturbed pendulum system on cylindric phase space.*** *Let us consider a system of the form*

$$\begin{aligned} \varphi' &= y, \\ y' &= -\sin\varphi - \varepsilon y, \end{aligned}$$

where $\varphi \in S^1$ is an angle, $0 \le \varphi < 2\pi$, $y \in R$, the parameter $\varepsilon > 0$. So the phase space is the cylinder $P = S^1 \times R$. The system has two equilibriums $A(0,0)$ and $B(\pi,0)$. The first point is a focus, and the second is a hyperbolic point. Because $\varepsilon > 0$, the infinity set $S^1 \times \pm\infty$ can be considered as an unstable point C. The system do not have other chain recurrent points. So there are tree components of chain recurrent set $\{A, B, C\}$. The trajectories can start at C and finish at A or at B. The unstable separatrices of B finish at A. Thus the structure matrix if of the form

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

where $A = Q_1$, $B = Q_2$ and $C = Q_3$. Moreover these connections are stable.

2 Symbolic image [24]

Let $C = \{M(1), \cdots, M(n)\}$ be a finite covering of M by closed sets. The sets $M(i)$ are called cells or boxes of the covering.

Definition 4 *Let G be a directed graph having s vertices where each vertex i corresponds to the cell $M(i)$. The vertices i and j are connected by a directed edge $i \to j$ if and only if $M(j) \cap f(M(i)) \neq \emptyset$. The graph G is called a symbolic image of f with respect to the covering C.*

Denote by Ver the set of vertices of G. The graph G can be considered as a correspondence $G : Ver \to Ver$ between the vertices. Graph G is uniquely determined by its $n \times n$ matrix of transitions $\Pi = (\pi_{ij})$: $\pi_{ij} = 1$ if and only if there is the directed edge $i \to j$, otherwise $\pi_{ij} = 0$. Indeed we can use the transition matrix without the symbolic image which is a convenient geometrical tool only. Much of an effective information of a dynamical system may come from the investigation of a symbolic image. It is natural to consider the symbolic image as a finite discrete approximation of the mapping f.

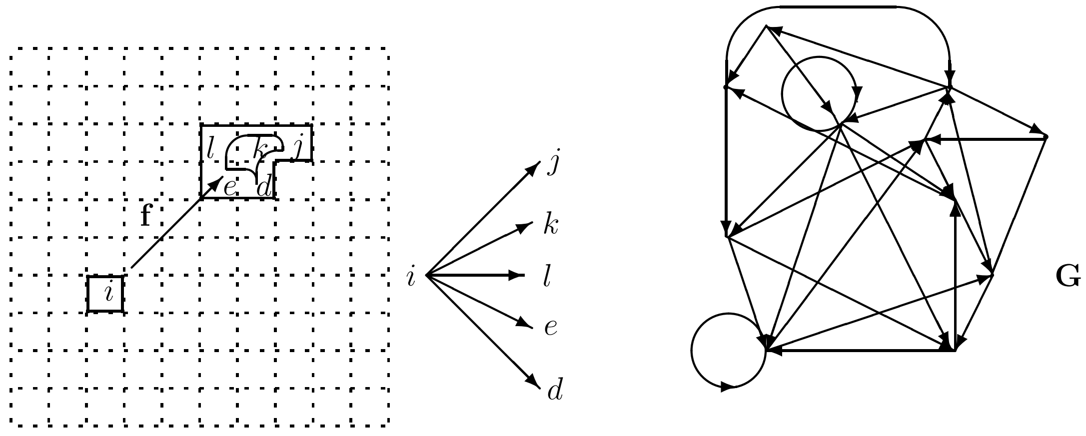


Figure 1: **Construction of a Symbolic Image.**

Definition 5 *An infinite in both direction sequence* $\{z_k\}$ *of vertices on the graph* G *is called an admissible path (or simply a path) if for each* k *the graph* G *contains the directed edge* $z_k \to z_{k+1}$. *A path* $\{z_k\}$ *is said to be p-periodic if* $z_k = z_{k+p}$ *for each* $k \in Z$.

There is a natural connection between the admissible paths on the symbolic image G and the ε -trajectories of the homeomorphism f. It can be said that an admissible path is the trace of an ε-trajectory and vice versa. However, there are some relationships between the parameters of the symbolic image and the number ε for which the connections take place [26].

Definition 6 *A vertex of the symbolic image is called recurrent if there is a periodic path passing through it. The set of recurrent vertices is denoted by RV. A pair of recurrent vertices* i, j *are called equivalent if there is a periodic path through* i *and* j.

The recurrent vertices $\{i\}$ are uniquely defined by the nonzero diagonal elements $\pi_{ii} \neq 0$ of the powers of the transitions matrix Π^m, $m \leq n$, where n is the number of the covering cells. By Definition 6, the set of recurrent vertices RV decomposes into several classes $\{H_k\}$ of equivalent recurrent vertices.

Denote by $P(d)$ the union of the cells $M(i)$ for which the vertex i is recurrent, i.e.,

$$P(d) = \{\cup M(i) : i \text{ is recurrent}\}, \tag{2.1}$$

where d is the largest diameter of the cells $M(i)$. As before we consider the dependence of P on the largest diameter d. Denote by $T(d)$ the union of the cells $M(k)$ for which the vertex k is non recurrent, i.e.,

$$T(d) = \{\cup M(k) : \ k \textit{ is non recurrent}\}.$$

The following theorem describes the properties of the sets $P(d)$ and $T(d)$.

Theorem 2 *[26] (i) The set $P(d)$ is a closed neighborhood of the chain recurrent set., i.e., $Q \subset P(d)$.*

(ii) For any neighborhood V of Q there exists $d > 0$ such that $P(d) \subset V$, i.e., if the largest diameter d is small enough then this neighborhood is sufficiently small.

(iii) The chain recurrent set Q coincides with the intersection of the sets $P(d)$ for all positive d:

$$Q = \bigcap_{d>0} P(d). \tag{2.2}$$

(iv) The points of T are not chain recurrent, i.e., $T \cap Q = \emptyset$.

Let us introduce a quasi-order relation between the vertices of the symbolic image. We consider $i \prec j$ if and only if there exists an admissible path of the form

$$i = i_0, i_1, i_2, ..., i_m = j.$$

Hence, a vertex i is recurrent if and only if $i \prec i$, and a pair of recurrent vertices $i,\ j$ are equivalent if and only if $i \prec j \prec i$.

Proposition 1 *[1] The vertices of a symbolic image G can be renumbered such that*

- *the equivalent recurrent vertices are numbered with consecutive integers,*
- *the new numbers $i,\ j$ of other vertices are chosen such that $i < j$ if $i \prec j \not\prec i$.*

In other words, the transition matrix is of the form

$$\Pi = \begin{pmatrix} (\Pi_1) & \cdots & \cdots & \cdots & \cdots \\ & \ddots & & & \\ 0 & & (\Pi_k) & \cdots & \cdots \\ & \ddots & & \ddots & \\ 0 & & 0 & & (\Pi_s) \end{pmatrix} \tag{2.3}$$

where the elements under the diagonal blocks are zeros, each diagonal block Π_k corresponds to either a class of equivalent recurrent vertices H_k or a nonrecurrent vertex. In the last case Π_k coincides with a single zero. The renumbering described in Proposition 1 is not uniquely defined.

3 Algorithms.

In order to find the transition matrix of the form (3) we have to renumbering vertices on symbolic image. For this we give an algorithm which finds classes of recurrent equivalent vertices and establishes a partial order between these classes and nonrecurrent vertices.

Our aim is to construct new graph by identifying equivalent vertices of the initial directed graph G.

The algorithm consist of four main parts:

1. Continuation,
2. Comparison,
3. Identification,
4. Forbidden.

Let us consider a path $\{i_0, i_1, i_2, ..., i_k\}$ of length k.

"Continuation" constructs a continuation of path $w = \{i_0, i_1, i_2, ..., i_k\}$ on an edge as follows. If there is an edge $i_k \to j$, $i_k \neq j$ we put $i_{k+1} = j$ and construct of a new path of the form $w^* = \{i_0, i_1, i_2, ..., i_k, i_{k+1}\}$.

"Comparison" compares the last vertex j with the vertices of path $w = \{i_0, i_1, i_2, ..., i_k\}$, i.e., we check the equality $j = i_m$, $m = k-1, k-2, ..., 1, 0$.

"Identification": if there exists $j = i_s$, than we have the circle $\{i_s, i_{s+1}, ..., i_k, j\}$, i.e., these vertices are equivalent recurrent vertices. In this case we identify the vertices $\{i_s, i_{s+1}, ..., i_k, j\}$ and replace their by new vertex j^*.

"Forbidden" : if we can not construct the continuation, i.e., there is not an edge $i_k \to j$, $i_k \neq j$, we form a forbidden vertex $k_l = i_k$ and put k_l in the set of forbidden vertices N. Then we consider the path $\{i_0, i_1, i_2, ..., i_{k-1}\}$. The "forbidden" gives a partial order between the forbidden vertices. The order is determined by the index l, which is increased by each step of the forbidden.

By repeating the "continuation", "comparison", "identification" and "forbidden" we come to a graph with recurrent vertices only of the form $k \to k \in N$.

Moreover a final recurrent vertex k_l corresponds to the class of equivalent recurrent vertices of the initial graph. In this case the transition matrix of constructed graph is of the form

$$\Pi^* = \begin{pmatrix} (\pi^*_{11}) & \cdots & \cdots & \cdots & \cdots \\ & \ddots & & & \\ 0 & & (\pi^*_{ll}) & \cdots & \cdots \\ & \ddots & & \ddots & \\ 0 & & 0 & & (\pi^*_{ss}) \end{pmatrix}, \qquad (3.4)$$

where $(\pi^*_{ll}), (1 \leq l \leq s)$ is either 1 or 0, 1 corresponding to a recurrent vertex and 0 nonrecurrent vertex.The initial transition matrix takes the form (4) if we make the renumbering according to the constructed partial order.

Now we give the algorithm. We will denote the forbidden vertices by N, at first $N = \phi$. Let s be number of vertices from N. Let $Ver = \{i\}$ be a set of vertices of symbolic image. Consider a path $w = \{i_0, i_1, i_2, ..., i_k\}$, let k be a index of last vertex in w, suppose these vertices are different and i_0 is named initial vertex in graph. At first $w = \phi$. Let m be a index of vertex in w for "Comparison". Let x be a index of vertex in w for "Identification".

Algorithm 1

1. Initiation

$N = \phi; \quad w = \phi; \quad k = 0; \quad s = 0$

2. Continuation

If $\exists i \notin N$ then add i in w

Else go to 7

3. Continuation

If $\exists i_k \to j, j \notin N, j \neq i_k$ then add j in w; $k = k + 1$

Else go to 6

4. Comparison

For m since $k - 1$ down to 0

If $i_m = i_k$ then go to 5

Go to 3

5. Identification

Add new vertex j^* in Ver

For x since k down to $m+1$

For all $i_x \to i^*$ add new edge $j^* \to i^*$; delete edge $i_x \to i^*$

For all $i^* \to i_x$ add new edge $i^* \to j^*$; delete edge $i^* \to i_x$

Delete vertex i_x from Ver

Delete vertex i_x from w

Delete vertex i_m from w

Add j^* in w; $k = m$

Go to 3

6. Forbidden

Add i_k in N; $s = s + 1$

Delete i_k from w; $k = k - 1$

If $w = \phi$ then go to 2

Else go to 3

7. End

From the previous explanation it follows

Proposition 2 *If the directed graph G has class of equivalent recurrent vertices $K = \{i_0, i_1, i_2, ..., i_k\}$ then the algorithm identifies K to one vertex.*

Algorithm 2

Suppose we have a directed graph G which has a classes of equivalent recurrent vertices consisting of one vertex $\{j_k\}$, i.e., there exists a closed path only of the form $j_k \to j_k$. Let $\{i_0, i_1, i_2, ..., i_s\}$ be a set of nonrecurrent vertices.

Our aim is to construct new graph NG with the same recurrent vertices $\{j_k\}$ and without nonrecurrent vertices. An edge $j_k \to j_l$ exists on NG if and only if there is a path $j_k \to ... \to j_l$ on the initial graph G.

Now we give the algorithm. The main idea is to replace nonrecurrent vertex i^* by a collection of edges $j \to l$ if the path $j \to i^* \to l$ exists. Let k is an index of nonrecurrent vertices.

1. $k = 0$

2. For all j^* such that $j^* \to i_k$

For all l^* such that $i_k \to l^*$

Add new edge $j^* \to l^*$

Delete edge $i_k \to l^*$

Delete edge $j^* \to i_k$

Delete vertex i_k

3. If $k < s$ then $k := k + 1$; go to 1

4. End

From the previous explanation it follows

Proposition 3 *The graphs G and NG have the same recurrent vertices and new graph has an edge $j_k \to j_l$ if and only if there is a path $j_k \to ... \to j_l$ on the initial graph G.*

By applying the algorithm 1 and 2 to the symbolic image the structure matrix of dynamical system is constructed.

4 Examples.

Now we can present some results obtained by program realization of our algorithms. We indicate the structure matrix of two test systems and show appropriate phase portraits of these systems, where each component of chain recurrent set is separated from others. The links between components correspond to non-diagonal "1" of structure matrix. The indexes of components are defined on the proper picture.

Test system 1. The Van-der-Pol equation

$$\begin{cases} x' &= y \\ y' &= y(1 - x^2) - x \end{cases}$$

The appropriate structure matrix is

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

see Picture 1.

Test system 2

$$\begin{cases} x' &= 2x\cos x - 5y \\ y' &= 2x \end{cases}$$

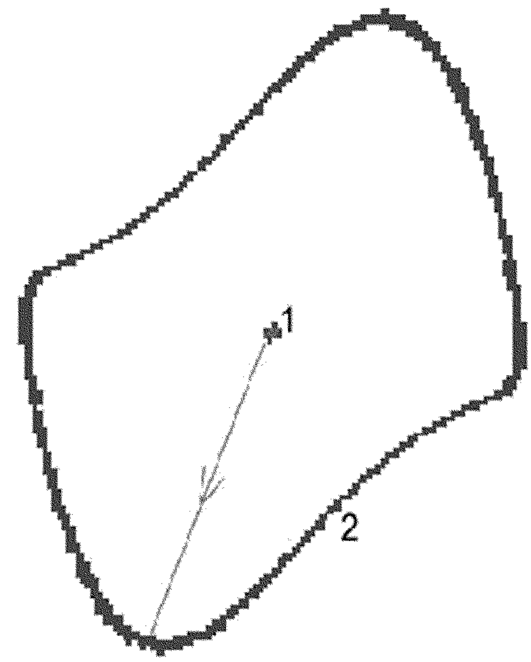
The appropriate structure matrix is

$$S = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

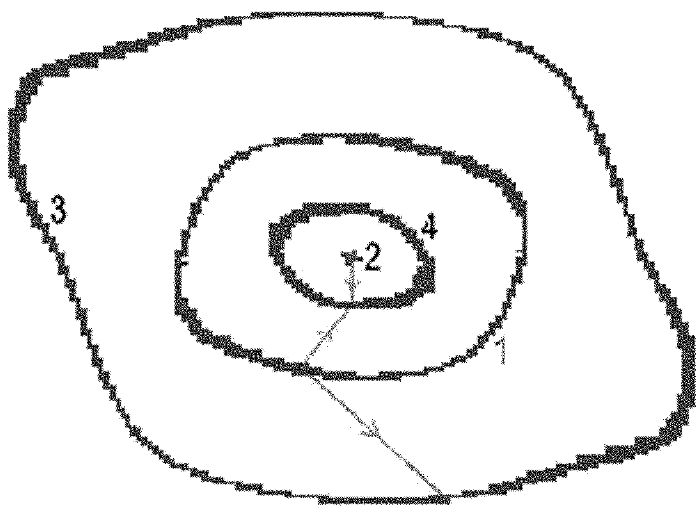
see Picture 2.

Acknowledgement

Authors would like to express their thanks to TUBITAK-NATO.



Picture 1. Test system 1. Van-der-Pol equation



Picture 2. Test system 2

References

[1] V.M.Alekseev, *Symbolic Dynamics, 11th Mathematical School*, Kiev, 1976 (in Russian).

[2] A.A.Andronov and L.S.Pontryagin, Rough Systems, *Doklady Academy Nauk SSSR,* v.14, no.5, 247-250 (1937) (in Russian).

[3] D.V. Anosov, *Geodesic flow on closed Riemannian manifold of negative curvature*, Trudy Math. Steclov Institute, v. 90, (1967) (in Russian).

[4] L.Auslender and R.MacKenzie, *Introduction to Differentiable Manifolds*, N.Y., 1963.

[5] N.P.Bhatia, G.P.Szego, *Stability theory of dynamical systems,* New-York, Springer, 1970.

[6] R.Bowen, Symbolic Dynamics, *Ann. Math. Soc. Providence, R.I.*, vol.8, (1982).

[7] I.U.Bronshtein, Theorem on structural stability of smooth extension of cascade, in *Algebraic invariants of dynamical systems, Mat. Issledovaniya,* v. 67, Kishinev, Shtinisa, 12-29 (1980) (in Russian).

[8] I.U.Bronshtein, *Nonautonomous dynamical system*, Kishinev, Shtinisa, 1984 (in Russian).

[9] B.F.Bylov, R.E.Vinograd, D.M.Grobman, V.V.Nemytskii, *Theory of Lyapunov exponents*, Moscow, Nauka, 1966 (in Russian).

[10] F.Colonius and W.Kliemann, The Morse spectrum of linear flows on vector bundles, *Trans. Amer. Math. Soc.*, 348, 4355-4388 (1996).

[11] C.Conley, Isolated Invariant set and the Morse Index, *CBMS Regional Conference Series, v.38, Amer.Math.Soc.*, Providence, (1978).

[12] B.Coomes, H.Kocak, K.Palmer, Computation of long period orbits in chaotic dynamical systems, *Aust. Math. Soc. Gaz.*, v.24, no.5, 183-190 (1997).

[13] M.Dellnitz, O.Junge, An adaptive subdivision technique for the approximation of attractors and invariant measures, *Comput. Visual. Sci.* 1, 63 - 68, (1998).

[14] M.Dellnitz, A.Hohmann, The computation of unstable manifolds using subdivision and continuation Broer, H. W.; Gils, S. A. van; Hoveijn, I.; Takens, F. (eds.): *Nonlinear Dynamical Systems and Chaos,* PNLDE 19, Birkhäuser, 449-459, 1996.

[15] P.Hartman, *Ordinary Differential Equations*, N.Y., 1964.

[16] M.Hirsch and S.Smale, *Differential Equations, Dynamical Systems and Linear Algebra,* N.Y. 1970.

[17] C.S.Hsu, *Cell-to-Cell Mapping.* Springer-Verlag, N.Y., 1987.

[18] F.Hunt, Unique ergodicity and the approximation of attractors and their invariant measures using Ulam s method, *Nonlinearity*, v. 11, no. 2, 307–317 (1998).

[19] A.N.Kolmogorov, S.W.Fomin. *An elements of the theory of functions and the functional analysis.* Science, Moscow, 1968 (in Russian).

[20] R.Mane, Characterizations of AS diffeomorphisms, *Lect. Notes Math.*, v. 597, 389-394 (1977).

[21] R.Mane, A proof of the C^1 stability conjecture, Publ. Math., Inst. Hautes Etud. Sci. 66, 161-210 (1988).

[22] Z.Nitecki, *Differentiable Dynamics.* London, 1971.

[23] Z.Nitecki and M.Shub, Filtrations, decompositions, and explosions, *Amer.J.of Math.*, v.97, no.4, 1029-1047 (1975).

[24] G.S.Osipenko, On a symbolic image of dynamical system, in *Boundary value problems*, Interuniv. Collect. sci. Works, Perm 101-105, (1983), (in Russian).

[25] G.S.Osipenko, Verification of the transversality condition by the symbolic-dynamical methods, *Differential Equations*, v.26, no.9, 1126-1132 (1990); translated from *Differentsialnye Uravneniya*, v.26, N9, 1528-1536, (1990).

[26] G.S.Osipenko, The periodic points and symbolic dynamics, in *Seminar on Dynamical Systems. Euler International Mathematical Institute, St.Petersburg, Russia, October and November, 1991*, Birkhauser Verlag, Basel, Prog. Nonlinear Differ. Equ. Appl. 12, 261-267 (1994).

[27] G.Osipenko, Morse Spectrum of Dynamical Systems and Symbolic Dynamics, *Proceedings of 15th IMACS World Congress,* v.1, 25-30 (1997).

[28] G.Osipenko and S.Campbell, Applied Symbolic Dynamics: Attractors and Filtrations, *Discrete and Continuous Dynamical Systems,* v.5, no.1&2, 43-60 (1999).

[29] G. Osipenko Spectrum of a Dynamical System and Applied Symbolic Dynamics, *Journal of Mathematical Analysis and Applications*, v. 252, no. 2, 587-616 (2000).

[30] S.Yu.Pilyugin, *The space of Dynamical Systems with C^0-Topology*, Springer-Verlag, Lec. Notes in Math., 1571, 1994.

[31] V.A.Pliss, *Integral Sets of Periodic System of Differential Equations*, Moscow, 1977 (in Russian)

[32] J.Robin, A structural stability theorem, *Annals of Math.*, v.94, no.3, 447-493 (1971).

[33] R.Robinson, Structural stability of C^1-diffeomorphism, *J. Diff. Equat.*, v.22, no.1, 28-73 (1976).

[34] R.Sacker and G.Sell, Existence of dichotomies and invariant splitting for linear differential systems *I-III*, *J. Diff. Equat.* v.15, no.3, 429-458 (1974), v.22, no.2, 476-522 (1976).

[35] R.Sacker and G.Sell, A spectral theory for liner differential systems, *J. Diff. Equat.,* v. 27, no. 3, 320-358 (1978).

[36] D.Salamon and E.Zehnder, Flows on vector bundles and hyperbolic sets, *Trans. Amer. Math. Soc.*, v. 306, no. 2, 623-649 (1988).

[37] J.Selgrade, Isolated invariant sets for flows on vector bundles*, Trans. Amer. Math. Soc.*, v. 203, 359-390 (1975).

[38] G.Sell, Nonautonomous differential equations and topological dynamics, *Trans. Amer. Math. Soc.*, 127, 241-283 (1967).

[39] A.N.Sharkovsky, Structure theory of differentiable dynamical systems and weak nonwandering points, *Abh. Akad. Wiss. DDR. Abt. Math Naturwiss. Techn.*, v.4, 193-200 (1977).

[40] M.Shub, Stabilite globale de systems denamiques, *Asterisque*, v. 56, 1-21 (1978).