DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 1, 2002
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Applications to physics, electrotechnics, and electronics

INHOMOGENEOUS, NON-LINEAR AND ANISOTROPIC SYSTEMS WITH RANDOM MAGNETISATION MAIN DIRECTIONS

Part two: Minimisation of functional issue and FEM equations for 3D field produced by permanent magnets

BERE IOAN
Assoc. Prof., “Politehnica” University,
Department of Electrotechnics,
Timisoara, Romania
E-mail: ibere@et.utt.ro

Abstract

The paper presents the *minimization of functional issue*, associated to the finite elements method (FEM) and *establish on detail the equations* to solve the problem of the three-dimensional (3D) magnetic field, in inhomogeneous, non-linear and anisotropic domains with random main directions of magnetization. Numerical computation of the algebraic equations system obtained, it could be established the state quantities of the magnetic field or other quantities which are interested in.

Key words: permanent magnets, FEM, inhomogenity, nonlinearity, anisotropy.

4. The minimization of the energy functional

On FEM, the functional must be minimized to get the equations system which follow to solve the field problem. If the finite elements of 3D discretization mesh are small enough, the relation (33) from [1] become

$$F = \sum_{\lambda=1}^{m'} \left[A'_{xx} \left(\frac{\partial V_H}{\partial x} \right)^2 + A'_{yy} \left(\frac{\partial V_H}{\partial y} \right)^2 + A'_{zz} \left(\frac{\partial V_H}{\partial z} \right)^2 + \right.$$
$$\left. + A'_{xy} \left(\frac{\partial V_H}{\partial x} \right)\left(\frac{\partial V_H}{\partial y} \right) + A'_{yz} \left(\frac{\partial V_H}{\partial y} \right)\left(\frac{\partial V_H}{\partial z} \right) + A'_{zx} \left(\frac{\partial V_H}{\partial z} \right)\left(\frac{\partial V_H}{\partial x} \right) \right]_\lambda \cdot v_\lambda +$$
$$+ \sum_{\lambda=1}^{m''} \left[A''_{xx} \left(\frac{\partial V_H}{\partial x} \right)^2 + A''_{yy} \left(\frac{\partial V_H}{\partial y} \right)^2 + A''_{zz} \left(\frac{\partial V_H}{\partial z} \right)^2 + \right.$$
$$+ A''_{xy} \left(\frac{\partial V_H}{\partial x} \right)\left(\frac{\partial V_H}{\partial y} \right) + A''_{yz} \left(\frac{\partial V_H}{\partial y} \right)\left(\frac{\partial V_H}{\partial z} \right) + A''_{zx} \left(\frac{\partial V_H}{\partial z} \right)\left(\frac{\partial V_H}{\partial x} \right) -$$
$$\left. - K''_x \left(\frac{\partial V_H}{\partial x} \right) - K''_y \left(\frac{\partial V_H}{\partial y} \right) - K''_z \left(\frac{\partial V_H}{\partial z} \right) \right]_\lambda \cdot v_\lambda +$$
$$+ \sum_{\lambda=1}^{m'_N} (B_n)_\lambda (V_H)_\lambda (S'_N)_\lambda + \sum_{\lambda=1}^{m''_N} (B_n)_\lambda (V_H)_\lambda (S''_N)_\lambda , \quad (34)$$

where v_λ is the volume of the finite element λ.

The magnetic scalar potential $(V_H)_\lambda$ – unknown function in a current point $P(x, y, z)$ inside the finite element $\lambda = \overline{1, m}$ – could be written depending on the magnetic scalar potentials V_{Hi}, V_{Hj}, V_{Hk}, $V_{H\ell}$ of the i, j, k, ℓ nodes (fig.4) which determine an irregular tetrahedron (finite element). The expression of $(V_H)_\lambda$, for random λ, may be written [2] under the form

$$(V_H)_\lambda = \frac{1}{6v_\lambda} \left[(c_{ii} + c_{ji}x + c_{ki}y + c_{\ell i}z) V_{Hi} + \right.$$
$$+ (c_{ij} + c_{jj}x + c_{kj}y + c_{\ell j}z) V_{Hj} + (c_{ik} + c_{jk}x + c_{kk}y + c_{\ell k}z) V_{Hk} +$$
$$\left. + (c_{i\ell} + c_{j\ell}x + c_{k\ell}y + c_{\ell\ell}z) V_{H\ell} \right]_\lambda \quad (35)$$

where: $v_\lambda = \pm \det (C)_\lambda / 6$; $v_\lambda > 0$; c_{rs} $(r, s = i, j, k, \ell)$ are the algebraic complements (cofactors) in the matrix $(C)_\lambda$ of the nodes coordinates which determine the tetrahedron λ:

$$(C)_\lambda = \begin{pmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_\ell & y_\ell & z_\ell \end{pmatrix}_\lambda . \tag{36}$$

From (35), for the terms of relation (34), results:

$$\left(\frac{\partial V_H}{\partial x}\right)_\lambda = \frac{1}{6v_\lambda}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)_\lambda , \tag{37}$$

$$\left(\frac{\partial V_H}{\partial y}\right)_\lambda = \frac{1}{6v_\lambda}\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)_\lambda , \tag{38}$$

$$\left(\frac{\partial V_H}{\partial z}\right)_\lambda = \frac{1}{6v_\lambda}\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)_\lambda . \tag{39}$$

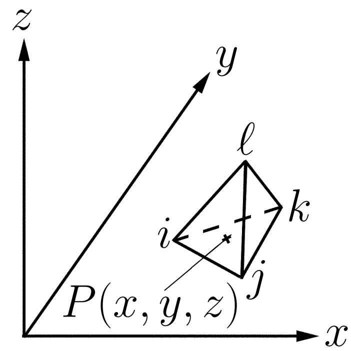

Fig.4. Finite element

The last two sums of expression (34) may be not detailed in a general treatment, because the boundary conditions for the possible concrete cases are different. In practice we can also have cases when the Neumann conditions are equal to zero what correspond to canceling of the last two sums. With the object of continuing the general analysis of the minimization process of functional, we will detail only the first two sums of relation (34), following to take into account, for concrete cases, the terms corresponding to the Neumann boundary conditions too. With these mentions, taking into account the relations (37, 38, 39), the functional (34) become

$$F = \sum_{\lambda=1}^{m'} \frac{1}{36v_\lambda}\Big[A'_{xx}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)^2+A'_{yy}(c_{ki}V_{Hi}+$$

$$+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell})^2+A'_{zz}(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell})^2+$$

$$+A'_{xy}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)+$$

$$+A'_{yz}\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)+$$

$$+A'_{zx}\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)]_{\lambda}+$$

$$+\sum_{\lambda=1}^{m''}\left\{\frac{1}{36v_{\lambda}}\left[A''_{xx}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)^2+A''_{yy}\left(c_{ki}V_{Hi}+\right.\right.\right.$$

$$\left.+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)^2+A''_{zz}\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)^2+$$

$$+A''_{xy}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)+$$

$$+A''_{yz}\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+c_{k\ell}V_{H\ell}\right)\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)+$$

$$+A''_{zx}\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)]_{\lambda}-$$

$$-\tfrac{1}{6}\left[K''_{x}\left(c_{ji}V_{Hi}+c_{jj}V_{Hj}+c_{jk}V_{Hk}+c_{j\ell}V_{H\ell}\right)+K''_{y}\left(c_{ki}V_{Hi}+c_{kj}V_{Hj}+c_{kk}V_{Hk}+\right.\right.$$

$$\left.\left.\left.+\,c_{k\ell}V_{H\ell}\right)+K''_{z}\left(c_{\ell i}V_{Hi}+c_{\ell j}V_{Hj}+c_{\ell k}V_{Hk}+c_{\ell\ell}V_{H\ell}\right)\right]_{\lambda}\right\}. \quad (40)$$

The expression (40) could be written more concentrated, in the following way

$$F=\sum_{\lambda=1}^{m'}F'_{\lambda}+\sum_{\lambda=1}^{m''}\left(F''_{\lambda}-F''_{p\lambda}\right), \quad (41)$$

where: F'_{λ} is the general term of the first sum;
F''_{λ} is the general term of the second sum;
$F''_{p\lambda}$ is the second general term of the second sum.
To minimize the functional we have to cancel its derivatives by magnetic scalar potential of the n nodes from the discretization mesh. Using the concentrated form (41), results

$$\sum_{\lambda=1}^{m'}\frac{\partial F'_{\lambda}}{\partial V_{Hi}}+\sum_{\lambda=1}^{m''}\frac{\partial F''_{\lambda}}{\partial V_{Hi}}-\sum_{\lambda=1}^{m''}\frac{\partial F''_{p\lambda}}{\partial V_{Hi}}=0; \quad i=\overline{1,n}. \quad (42)$$

The system (42) contains n algebraic equations, where the unknowns are the magnetic scalar potentials V_{Hi} $(i=\overline{1,n})$ the 3D discretization mesh of the studied domain. In particular cases, where the studied domain has not permanent magnetization zones (permanent magnets), the terms of the last sum are canceled $\left(K''_x = K''_y = K''_z = 0\right)$.

5. The establish of FEM equations

For every node of the discretization mesh, it must be known the coordinates (x, y, z), the adjacent finite elements and the neighboring nodes. In order to establish the rules for writing the equations system, two local numbering are introduced: N_{vi} is the number of the neighbor nodes to the node i; M_{ai} is the number of adjacent elements to the node i. The concrete values for N_{vi} and M_{ai} depends on the position of every node $i = \overline{1, n}$ in the discretization mesh. For instance, for any node i from inside of the tetrahedral mesh, $N_{vi} = 12$ and $M_{ai} = 20$ [3]. The magnetic scalar potential V_{Hi} of the node i appears only in elementary functionless F'_λ, F''_λ and $F''_{p\lambda}$ which belong to the tetrahedral adjacent to the node i. Thus, the equation from the system (42) concerning to any node i from inside of the discretization mesh, which has $M_{ai} \equiv a$ adjacent elements, will be

$$\frac{\partial}{\partial V_{Hi}}\left(\sum_{\lambda=1}^{a} F'_\lambda\right) + \frac{\partial}{\partial V_{Hi}}\left(\sum_{\lambda=1}^{a} F''_\lambda\right) - \frac{\partial}{\partial V_{Hi}}\left(\sum_{\lambda=1}^{a} F''_{p\lambda}\right) = 0, \tag{43}$$

where are made the notations

$$\begin{cases} CVj &= c_{ji}V_{Hi} + c_{jj}V_{Hj} + c_{jk}V_{Hk} + c_{j\ell}V_{H\ell} \\ CVk &= c_{ki}V_{Hi} + c_{kj}V_{Hj} + c_{kk}V_{Hk} + c_{k\ell}V_{H\ell} \\ CV\ell &= c_{\ell i}V_{Hi} + c_{\ell j}V_{Hj} + c_{\ell k}V_{Hk} + c_{\ell\ell}V_{H\ell} \end{cases}, \tag{44}$$

with which the expressions of elementary functionless are:

$$\begin{cases} F'_1 = \frac{1}{36v_1}\Big[(A'_{xx})_1(CVj)_1^2 + \left(A'_{yy}\right)_1(CVk)_1^2 + (A'_{zz})_1(CV\ell)_1^2 + \\ \qquad + \left(A'_{xy}\right)_1(CVj)_1(CVk)_1 + \left(A'_{yz}\right)_1(CVk)_1(CV\ell)_1 + \\ \qquad + (A'_{zx})_1(CV\ell)_1(CVj)_1\Big], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F'_a = \frac{1}{36v_a}\Big[(A'_{xx})_a(CVj)_a^2 + \left(A'_{yy}\right)_a(CVk)_a^2 + (A'_{zz})_a(CV\ell)_a^2 + \\ \qquad + \left(A'_{xy}\right)_a(CVj)_a(CVk)_a + \left(A'_{yz}\right)_a(CVk)_a(CV\ell)_a + \\ \qquad + (A'_{zx})_a(CV\ell)_a(CVj)_a\Big], \end{cases} \tag{45}$$

$$\begin{cases} F_1'' = \frac{1}{36 v_1}\Big[(A_{xx}'')_1 (CVj)_1^2 + \big(A_{yy}''\big)_1 (CVk)_1^2 + (A_{zz}'')_1 (CV\ell)_1^2 + \\ \qquad + \big(A_{xy}''\big)_1 (CVj)_1 (CVk)_1 + \big(A_{yz}''\big)_1 (CVk)_1 (CV\ell)_1 + \\ \qquad\qquad + (A_{zx}'')_1 (CV\ell)_1 (CVj)_1\Big], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_a'' = \frac{1}{36 v_a}\Big[(A_{xx}'')_a (CVj)_a^2 + \big(A_{yy}''\big)_a (CVk)_a^2 + (A_{zz}'')_a (CV\ell)_a^2 + \\ \qquad + \big(A_{xy}''\big)_a (CVj)_a (CVk)_a + \big(A_{yz}''\big)_a (CVk)_a (CV\ell)_a + \\ \qquad\qquad + (A_{zx}'')_a (CV\ell)_a (CVj)_a\Big], \end{cases} \tag{46}$$

$$\begin{cases} F_{p1}'' = \frac{1}{6}\Big[(K_x'')_1 (CVj)_1 + \big(K_y''\big)_1 (CVk)_1 + (K_z'')_1 (CV\ell)_1\Big], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{pa}'' = \frac{1}{6}\Big[(K_x'')_a (CVj)_a + \big(K_y''\big)_a (CVk)_a + (K_z'')_a (CV\ell)_a\Big]. \end{cases} \tag{47}$$

In the relations (45, 46, 47) the subscripts $(1, 2, 3, \ldots, a)$ written after the parenthesis show that the quantities $(CVj, CVk, CV\ell)$, $(V_{Hi}, V_{Hj}, V_{Hk}, V_{H\ell})$, c_{rs} $(r = j, k, \ell;\ s = i, j, k, \ell)$, $\left(A_{xx}', A_{yy}', A_{zz}', A_{xy}', A_{yz}', A_{zx}'\right)$, $\left(A_{xx}'', A_{yy}'', A_{zz}'', A_{xy}'', A_{yz}'', A_{zx}''\right)$, correspond to the finite element with the same index, and $(v_1, v_2, \ldots, v_a)$ are the adjacent tetrahedral volumes of the node i for what the relation is written. The algebraic complements c_{rs} of the matrix $(C)_\lambda$ – written with the nodes coordinates i, j, k, ℓ which determine the finite element λ – are constant quantities for an established discretization mesh. The elementary functionals F_λ'' and $F_{p\lambda}''$ are different by zero only for the finite elements from the zone with permanent magnetization, where the functionals F_λ' are null. If the element λ is in the zone without permanent magnet, $F_\lambda' \neq 0$ and $F_\lambda'' = F_{p\lambda}'' = 0$.

Derive the relations (45, 46, 47), the general forms of these three kind of terms from functional becomes:

$$\frac{\partial F_\lambda'}{\partial V_{Hi}} = \frac{1}{18 v_\lambda}\Big\{\left[A_{xx}' c_{ji}^2 + A_{yy}' c_{ki}^2 + A_{zz}' c_{\ell i}^2 + A_{xy}' c_{ji} c_{ki} + A_{yz}' c_{ki} c_{\ell i} + A_{zx}' c_{\ell i} c_{ji}\right] V_{Hi} +$$

$$+ \left[A_{xx}' c_{ji} c_{jj} + A_{yy}' c_{ki} c_{kj} + A_{zz}' c_{\ell i} c_{\ell j} + \tfrac{1}{2} A_{xy}' \left(c_{ji} c_{kj} + c_{ki} c_{jj}\right) + \right.$$

$$\left. + \tfrac{1}{2} A_{yz}' \left(c_{ki} c_{\ell j} + c_{\ell i} c_{kj}\right) + \tfrac{1}{2} A_{zx}' \left(c_{\ell i} c_{jj} + c_{ji} c_{\ell j}\right)\right] V_{Hj} +$$

$$+ \left[A_{xx}' c_{ji} c_{jk} + A_{yy}' c_{ki} c_{kk} + A_{zz}' c_{\ell i} c_{\ell k} + \tfrac{1}{2} A_{xy}' \left(c_{ji} c_{kk} + c_{ki} c_{jk}\right) + \right.$$

$$+\tfrac{1}{2}A'_{yz}\left(c_{ki}c_{\ell k}+c_{\ell i}c_{kk}\right)+\tfrac{1}{2}A'_{zx}\left(c_{\ell i}c_{jk}+c_{ji}c_{\ell k}\right)\big]\,V_{Hk}+$$

$$+\ \big[A'_{xx}c_{ji}c_{j\ell}+A'_{yy}c_{ki}c_{k\ell}+A'_{zz}c_{\ell i}c_{\ell\ell}+\tfrac{1}{2}A'_{xy}\left(c_{ji}c_{k\ell}+c_{ki}c_{j\ell}\right)+$$

$$+\tfrac{1}{2}A'_{yz}\left(c_{ki}c_{\ell\ell}+c_{\ell i}c_{k\ell}\right)+\tfrac{1}{2}A'_{zx}\left(c_{\ell i}c_{j\ell}+c_{ji}c_{\ell\ell}\right)\big]\,V_{H\ell}\Big\}_{\lambda}, \quad (48)$$

$$\frac{\partial F''_{\lambda}}{\partial V_{Hi}}=\frac{1}{18v_{\lambda}}\Big\{\big[A''_{xx}c_{ji}^{2}+A''_{yy}c_{ki}^{2}+A''_{zz}c_{\ell i}^{2}+A''_{xy}c_{ji}c_{ki}+A''_{yz}c_{ki}c_{\ell i}+A''_{zx}c_{\ell i}c_{ji}\big]\,V_{Hi}+$$

$$+\big[A''_{xx}c_{ji}c_{jj}+A''_{yy}c_{ki}c_{kj}+A''_{zz}c_{\ell i}c_{\ell j}+\tfrac{1}{2}A''_{xy}\left(c_{ji}c_{kj}+c_{ki}c_{jj}\right)+$$

$$+\ \tfrac{1}{2}A''_{yz}\left(c_{ki}c_{\ell j}+c_{\ell i}c_{kj}\right)+\tfrac{1}{2}A''_{zx}\left(c_{\ell i}c_{jj}+c_{ji}c_{\ell j}\right)\big]\,V_{Hj}+$$

$$+\big[A''_{xx}c_{ji}c_{jk}+A''_{yy}c_{ki}c_{kk}+A''_{zz}c_{\ell i}c_{\ell k}+\tfrac{1}{2}A''_{xy}\left(c_{ji}c_{kk}+c_{ki}c_{jk}\right)+$$

$$+\tfrac{1}{2}A''_{yz}\left(c_{ki}c_{\ell k}+c_{\ell i}c_{kk}\right)+\tfrac{1}{2}A''_{zx}\left(c_{\ell i}c_{jk}+c_{ji}c_{\ell k}\right)\big]\,V_{Hk}+$$

$$+\ \big[A''_{xx}c_{ji}c_{j\ell}+A''_{yy}c_{ki}c_{k\ell}+A''_{zz}c_{\ell i}c_{\ell\ell}+\tfrac{1}{2}A''_{xy}\left(c_{ji}c_{k\ell}+c_{ki}c_{j\ell}\right)+$$

$$+\tfrac{1}{2}A''_{yz}\left(c_{ki}c_{\ell\ell}+c_{\ell i}c_{k\ell}\right)+\tfrac{1}{2}A''_{zx}\left(c_{\ell i}c_{j\ell}+c_{ji}c_{\ell\ell}\right)\big]\,V_{H\ell}\Big\}_{\lambda}, \quad (49)$$

$$\frac{\partial F''_{p\lambda}}{\partial V_{Hi}}=\tfrac{1}{6}\left(K''_{x}c_{ji}+K''_{y}c_{ki}+K''_{z}c_{\ell i}\right)_{\lambda}, \quad (50)$$

where $\lambda=1,2,3,\,\dots\,,a$.

We notice that for specify equation i of the system (42), the term having the form (50) represents the free terms of the equation, because it doesn't contain the magnetic scalar potentials of the nodes (the unknowns of the problem). After the introduction of the terms (48, 49, 50) in relation (43), results the general form of an equation from the system (42), written for any node i from inside of the discretization mesh

$$C_{i}V_{Hi}+\sum_{t=1}^{N_{vi}}C_{it}V_{Ht}=TL_{i};\quad i=\overline{1,n}, \quad (51)$$

where we note that:

$$C_{i}\ =\ \sum_{t=1}^{a}\frac{1}{18v_{t}}\left(A_{xx}c_{ji}^{2}+A_{yy}c_{ki}^{2}+A_{zz}c_{\ell i}^{2}+A_{xy}c_{ji}c_{ki}+A_{yz}c_{ki}c_{\ell i}+A_{zx}c_{\ell i}c_{ji}\right)_{t}, \quad (52)$$

$$C_{it} = \sum_{t=1}^{w} \frac{1}{18v_t} \left[A_{xx}c_{ji}c_{jj} + A_{yy}c_{ki}c_{kj} + A_{zz}c_{\ell i}c_{\ell j} + \tfrac{1}{2}A_{xy}\left(c_{ji}c_{kj} + c_{ki}c_{jj}\right) + \right.$$
$$\left. + \tfrac{1}{2}A_{yz}\left(c_{ki}c_{\ell j} + c_{\ell i}c_{kj}\right) + \tfrac{1}{2}A_{zx}\left(c_{\ell i}c_{jj} + c_{ji}c_{\ell j}\right)\right]_t, \quad (53)$$

$$TL_i = \frac{1}{6}\sum_{t=1}^{a}\left(K_x''c_{ji} + K_y''c_{ki} + K_z''c_{\ell i}\right)_t. \quad (54)$$

In relations (52, 53) – obtained by writing more concentrated the relations (48, 49) – the terms $(A_{xx})_t$, $(A_{yy})_t$, $(A_{zz})_t$, $(A_{xy})_t$, $(A_{yz})_t$, $(A_{zx})_t$ contain the components of the tensors of the ferromagnetic materials permeability (μ_u, μ_v, μ_w), of the air – gap ($\mu_u = \mu_v = \mu_w = \mu_0$), or of the permanent magnet (μ_{pu}, μ_{pv}, μ_{pw}) (s.[1]), as the finite element t – adjacent to the node i for which is written the relation (51) – is situated in the ferromagnetic material zone, in air – gap or in permanent magnet. The current index t from the expressions (52, 53, 54) refers to the finite element $\lambda = t$, which could particularized for all the nodes $i = \overline{1,n}$ of the discretization mesh, identifying the elements $(1 \div M_{ai})$ from around of each node. The quantity w from the relation (53) represent the number of finite element (of the M_{ai} adjacent to the node i) to which any neighbor node (of the N_{vi} neighbor to the node i) is adjacent. For the analyzed case (i – an inner node) $w = 5$ [3]. The coefficients c_{rs} from the relations (52, 53, 54) are obtained by identifying the nodes j, k and ℓ of each finite element around the common node i for what the relation (51) is written. Therefore, for each node i with unknown magnetic scalar potential an equation having the form (51) could be written. In relation with the concrete position of each node i, the number of the neighbor nodes N_{vi}, respectively of the adjacent elements M_{ai} of the node i, gets particularly values. For the nodes on the boundary surface with non-null Neumann conditions of the studied domain, in equation (43) we introduce supplementary particular terms correspond to these conditions (the last two sums from energetic functional (34)).

6. Conclusions

If we write the equation (51) for all the nodes $i = \overline{1,n}$ with unknowns magnetic scalar potentials of the 3D discretization mesh in which the field problem is analyzed, a compatible system of algebraic equations is obtained. The values of the magnetic scalar potentials are gotten by numerical computation of the system. After that, we can get (s.eq. 37, 38, 39) the components of the magnetic field intensity for each finite element $\lambda = \overline{1, m}$:

$$(H_x)_\lambda = -\left(\frac{\partial V_H}{\partial x}\right)_\lambda = -\frac{1}{6v_\lambda}\left(c_{ji}V_{Hj} + c_{jj}V_{Hj} + c_{jk}V_{Hk} + c_{j\ell}V_{H\ell}\right)_\lambda, \quad (55)$$

$$(H_y)_\lambda = -\left(\frac{\partial V_H}{\partial y}\right)_\lambda = -\frac{1}{6v_\lambda}\left(c_{ki}V_{Hi} + c_{kj}V_{Hj} + c_{kk}V_{Hk} + c_{k\ell}V_{H\ell}\right)_\lambda, \quad (56)$$

$$(H_z)_\lambda = -\left(\frac{\partial V_H}{\partial z}\right)_\lambda = -\frac{1}{6v_\lambda}\left(c_{\ell i}V_{Hi} + c_{\ell j}V_{Hj} + c_{\ell k}V_{Hk} + c_{\ell\ell}V_{H\ell}\right)_\lambda. \quad (57)$$

The components of the magnetic flux density $\bar{\boldsymbol{B}}_\lambda$ are determined from $\bar{\boldsymbol{H}}_\lambda$ and the tensors of magnetic permeability, in the following way: in air-gap $\bar{\boldsymbol{B}}_\lambda = \mu_0 \bar{\boldsymbol{H}}_\lambda$; in ferromagnetic materials $\bar{\boldsymbol{B}}_\lambda = \bar{\bar{\mu}}_\lambda \bar{\boldsymbol{H}}_\lambda$; in permanent magnets $\bar{\boldsymbol{B}}_\lambda = \bar{\bar{\mu}}_{p\lambda} \bar{\boldsymbol{H}}_\lambda$.

In conclusion, by numerical computation of equation system (51), we get $\bar{\boldsymbol{H}}_\lambda$ and $\bar{\boldsymbol{B}}_\lambda$ ($\lambda = \overline{1, m}$) in each finite element of the studied domain which is 3D, inhomogeneous, nonlinear and anisotrop. After we know $\bar{\boldsymbol{H}}_\lambda$ and $\bar{\boldsymbol{B}}_\lambda$, we can determine other quantities which we are interested in, such as: magnetic voltage, magnetic flux through certain surfaces, forces (torque) applied on coils or electric conductors placed within the air-gap and so on.

References

[1] I.Bere, *Inhomogeneous, non-linear and anisotropic systems with random magnetisation main directions. Part one: The useful form of functional 3D for computation the field created by permanent magnets*, Electronic journal “Differential equations and control processes”, N 4, 2001, pp(26-35). (http://www.neva.ru/journal)

[2] I.Bere, *Relative permeability of nonlinear and anisotrop permanent magnets,* Proceedings of the scientific communications “A. Vlaicu” University, Arad, 1996, pp(9-14).

[3] K.H.Bachmann a.a., *Kleine Enzyklopadie der Mathematik*, Ed. Tehnica, Bucuresti, 1980, pp(239-240).