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CONNECTION BETWEEN PAINLEVÉ ANALYSIS AND OPTIMAL SYSTEMS

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Abstract

In this article we discuss two methods of solving systems of partial differential equations. First we describe the Painlevé test. In a second step a similarity analysis is carried out to motivate optimal systems. We present procedures of solving partial differential equations, which connect both methods. The connection between the methods is examplified by the KdV equation. Special points are the applications of the theorem of Strampp and the calculation of nonclassical symmetries via the algorithm of Clarkson.

Introduction

A large number of physical phenomena is described by systems of partial or ordinary differential equations. During the last years it became more and more important to characterise biological, chemical and physical phenomenons with

nonlinear partial differential equations (PDEs). So the development of analytical methods of solving such PDEs becomes more significant. Today there exist three algorithms which are powerful and appropriate to solve such equations:
- the inverse scattering transform
- the calculation of similarity solutions by Lie's theory, and
- the calculation of solutions with the Painlevé Ansatz.

The aim of this paper is to connect the method of Lie [1] and the Painlevé Ansatz [2] to construct solutions with the aid of optimal systems. The first section contains the main facts about the Painlevé test. In section 2 we discuss the connection between the Painlevé tests developed by Weiss, Tabor, Carnevale (WTC-Test) [3] and by Ablowitz, Ramani, Segur [2]. We apply these tests to the Boussinesq equation to solve it. The third section connects the WTC-test and Lie's similarity analysis. We demonstrate the joint methods by applying it to the KdV equation. The fourth section introduces optimal systems. We discuss the preliminary method by Olver [1] for the KdV and nonlinear Schrödinger equation. The last section makes some concluding remarks.

1 Facts about Painlevé Property and the Painlevé Test

The generation of the Painlevé test is traced back to S. Kovalevskaya's conjecture in the years 1889, 1890 [4, 5]. She demonstrated a connection between the complete solvability of a finite dimensional Hamilton system and the analytical structure of the equations of motions on a complex surface. Her discussion was based on the problem that a rigid body moves around a fix point. She tried to find parameters in such a way that the solutions should not have movable branch points (essential singularities in the complex surface) depending on initial conditions. For these cases she solved the equations of motion explicitly [6]. Kovalevskaya was motivated by papers written by Painlevé (1888) [7] and Fuchs (1884) [8].

The main idea behind this investigation was the question if singularities appearing in the solutions are fixed or movable. Painlevé showed in 1888 that the singularities in the solution of a first-order differential equation

$$y\left(x, f, \frac{df}{dx}\right) = 0 \tag{1}$$

are poles and/or algebraic branch points [9]. In (1) y is an analytical polynom in f and $\frac{df}{dx}$. The power expansion found by Kovalevskaya is based on a method

introduced by Frobenius in the year 1873. After this Painlevé and Gambier [10] catalogued all ordinary differential equations of order one and two due to the fact that they allow poles as moveable singularities in the complex surface. Out of these attributes there emerged the Painlevé property. All solutions of the known ordinary differential equations except for six allow convergent series. These six exceptions are called the "Painlevé transcendents".
The interest in this fact was recovered by Ablowitz [2, 11]. He found that the Painlevé transcendents result from ordinary differential equations which are generated by the reduction of partial differential equations solvable via inverse scattering transformation .

Let us first look at the ordinary differential equations

$$y\left(x, f, \frac{df}{dx}, \frac{d^2f}{dx^2}, \cdots, \frac{d^nf}{dx^n}\right) = 0 \tag{2}$$

of order $n \in \mathbb{N}$ in the complex surface. Equation (2) depends on dependent variables f and the independent variables x. Equation (2) contains a meromorphic function f [12] which is one of the basics for the Painlevé property [6]. The practical calculation of the Painlevé property is bases on the following theorem by Steeb [6]:

Theorem 1.1 *An ordinary differential equation of order n allows the Painlevé property. Then the general solution of the differential equation has the generalised Laurent expansion at any point $z_1 \in \mathbb{C}$*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_1)^{j+m}, \qquad m \in \mathbb{R}, \tag{3}$$

furthermore there exist $n-1$ free constants.

This theorem allows to define a test to verify the Painlevé property [6]. However we note that condition (3) in Theorem 1.1 is not sufficient, because moveable essential singularities are not excluded.

In the following we describe how to transfer this Painlevé test to partial differential equations. We examine a system of partial differential equations of higher order with n independent and m dependent variables. We denote the independent variables by $x = (x_1, x_2, ..., x_n)$ and the dependent variables by $u = \left(u^1, u^2, ..., u^m\right)$. Let $F_1, ..., F_l$ be functions defined on the manifold $M{:}X \times$

$U \times U^{(r)} \to \mathrm{IR}$, $\left(x, u, u^{(r)}\right) \mapsto F_\nu \left(x, u, u^{(r)}\right)$, $\quad \nu = 1, ..., l$, and

$$u^{(r)} = \left(u^1_{x_1}, u^2_{x_2}, \cdots, u^1_{x_n}, u^2_{x_1}, \cdots, u^m_{x,n}, u^1_{x_1 x_1}, \cdots, u^m_{\underbrace{x_n \cdots x_n}_{r\ times}} \right). \tag{4}$$

are the derivatives in M. The system of differential equations under consideration now is

$$F\left(x, u, u^{(r)}\right) := \left(F_1\left(x, u, u^{(r)}\right), \cdots, F_l\left(x, u, u^{(r)}\right)\right) = 0. \tag{5}$$

We call a solution of (5) a function that satisfies $F\left(x, f(x), f^{(r)}(x)\right) \equiv 0$ if $f : X \to U, x \mapsto f(x)$ is given by (3).

We assume that all functions we are dealing with are analytic and therefore can be differentiated as often as necessary. In the case of partial differential equations we deal with functions with more than one variable. This extended manifold is known as an analytical hypersurface [13]. The Painlevé property on a analytic hypersurface is defined in [13]. To calculate solutions for $F_\nu\left(x, u, u^{(r)}\right)$, we need the definition giving by Weiss [3]

Definition 1.1 *A partial differential equation of order r passes the Painlevé test from Weiss-Tabor-Carnevale (WTC-Test), if the solution can be expressed in the Form*

$$u = \sum_{j=0}^{\infty} u_j \left(x_1, \cdots, x_n\right) \Phi^{j+\alpha} \tag{6}$$

where $\alpha \in \mathbb{Z}$, u_j is holomorph and k of these u_j are not determined.

We note:

1. Some authors don't distinguish between the Painlevé property and the Painlevé test.

   The ansatz from definition 1.1 is called weak Painlevé property. So we can define that a partial differential equation possesses the weak Painlevé property if equation (5) passes the Painlevé test.

2. Condition written in definition 1.1 is not sufficient because essential singularities are not pointed out (see for example [14]).

In the course of the analysis we must differentiate series. So we have to apply the theorem of Osgood [15]. This allows us to differentiate an infinite series of functions in n complex variables $z_1, ..., z_n$ which are holomorph[1] on a $2n$ dimensional surface T. You can conceive the product $u_j\Phi^{j+\alpha}$ in (6) as a function of $\tilde{u}_j$. Applying the series ansatz by Weierstrass to (6), we need the condition that $\tilde{u}_j$ is holomorph. A holomorphic function is the product of two holomorphic functions. This fact is the reason that u_j and Φ must have this property. Furthermore we take the elements of this series and differentiate them one by one.

There exists another Painlevé test given by [16] who is distinguished from the WTC-test by assumptions and practicability:

Definition 1.2 *A partial differential equation (5) passes the Painlevé test from Ablowitz, Ramani, Segur (ARS-test), if the classical and the nonclassical symmetry reductions pass the Painlevé test, perhaps after transformations.*

This procedure is very difficult because you need all symmetry reductions of the original equation . The solution of this problem was the development of the WTC-test which can be easily applied to partial differential equations. However, it was shown [17] that if an equation passes the WTC-test it also pasess the ARS-test, but not vice versa. In the following part we describe how to apply the Painlevé test to partial differential equations. We look at a system of differential equations (5) with $x = (x_1, x_2, ..., x_n)$ and $u = \left(u^1, u^2, ..., u^m\right)$.

We assume that the number of equations coincide with the number of dependent variables (but it can happen that $n \neq m$). We make the following ansatz to expand the functions u^k; $k = 1, ..., m$ about an analytic manifold Φ:

$$u^k(x,t) = \Phi^{\alpha_k} \sum_{j_k=0}^{\infty} u^k_{j_k} \Phi^{j_k}(x,t); k = 1, ..., m; \quad \alpha_k \in \mathbb{Z} \tag{7}$$

$\alpha_k; \quad k = 1, ..., m$ form a m-tuple in short $\alpha := (\alpha_1, \alpha_2, ..., \alpha_m)$. The following steps discuss the necessary calculations to come to the statement if the discussed equation passes the Painlevé-test:

Step 1: To determine α, we have to look at the system of equations

$$F\left(x, u_{(0)}, u^{(r)}_{(0)}\right) = 0 \tag{8}$$

where

$$u_{(0)} = \left(u^1_0 \Phi^{\alpha_1}, \cdots, u^m_0 \Phi^{\alpha_m}\right). \tag{9}$$

[1]Osgood is using the word analytic but he means the complex differentiation relating to $z_1, ..., z_n$ at every point inside of this surface.

In each equation from (5) we determine for each nonvanishing term of the sum the exponent of the power of Φ. We get determining equations for the tuple α out of the fact that we look for the smallest exponents appearing in at least two terms of the sum of each equation. The complete set of determining equations for α depends on n, m and r. They are parameters in system (5).

Definition 1.3 *Terms leading to the smallest exponents of Φ for the ν-th equation in (5) are called the leading terms of the equation*

$$F_\nu\left(x, u, u^{(r)}\right) = 0; \quad \nu = 1, ..., m.$$

If one of the coefficients of α is not in $\mathbb{Z}$, the system doesn't possess the Painlevé property. If you cannot calculate all $\alpha_k, k = 1, ..., n$ meaning, that some can be chosen arbitrary, we continue with the Painlevé test under the assumption $\alpha \in \mathbb{Z}^m$ and restrict this arbitrary α in a later step.

Step 2: All known coefficients $\alpha_k \in \mathbb{Z}^m$ will be inserted in (5). We neglect all terms with the exception of the leading ones. If the equation can be reduced to the form

$$u_0^k u_0^l \frac{\partial^{|J|}\Phi}{\partial x_{j_1} \cdots \partial x_{j_{|J|}}} = 0,$$

the cases $u_0^k = 0$ and $u_0^l = 0$ are inconsistent with the ansatz, because the series starts with powers of Φ^{α_k}. The partial derivatives are written with a multi-index. In the cases that the assumption $\alpha \in \mathbb{Z}^m$ isn't valid or the last case has happened you can try to transform equation (5) into a form which passes step 1 and step 2 of the Painlevé test

Step 3: After doing these examinations we insert the generalised Laurent-series (6) into equation (5) and take into account the values of α_k. We assume that these series are convergent or absolutely convergent without making examinations about this question. We now factor out Φ^{σ_ν} in the ν-th equation where σ_ν is the leading power of this equation. The gained result is

$$\sum_{j=0}^{\infty} G_j^\nu\left(u_j, u_{j-1}^{(r)}, \cdots, u_0^{(r)}, \Phi^{(r)}\right)\Phi^j = 0. \tag{10}$$

Now we make a comparison of coefficients in powers of Φ leading to

$$\bigwedge_{j \in \mathbb{N}_0} G_j^\nu\left(u_j, \cdots, \Phi^{(r)}\right) = 0.$$

These equations lead to a recursion equation for arbitrary but fixed j,

$$P^{\nu}\left(j, u_0, \Phi^{(r)}\right) u_{j,\nu}^{T} = \tilde{G}^{\nu}\left(u_{j-1}, ..., u_0, \Phi^{(r)}\right),$$

where $P^{\nu}\left(j, u_0, \Phi^{(r)}\right) := \left(P_1^{\nu}\left(j, u_0, \Phi^{(r)}\right), ..., P_m^{\nu}\left(j, u_0, \Phi^{(r)}\right)\right)$. T denotes a transposition and $P_k^{\nu}, k = 1, ..., m$ are the coefficients in front of u^k. All ν equations can be put together to a matrix equation

$$M\underline{u} = \underline{V}\left(u_{j-1}, u_{j-2}, \cdots, \Phi, \Phi_{x_1}, \cdots\right).$$

To solve this matrix equation in a definite manner it is necessary that *det M* doesn't vanish. So we define

Definition 1.4 *The spots $a_j,\ j = 1, ..., r$, where the determinant of M is vanishing are called resonance spots. At these spots the functions $u_{a_j}^k$ are arbitrary.*

This arbitrary function is necessary for solving problems including initial- and boundary condition problems. We must adapt the solutions to these problems.

The gain of this procedure is:

1. One resonance spot is $a_1 = -1$. This corresponds with the arbitrariness of Φ.

2. If u_0 is arbitrary we find the resonance spot $a_2 = 0$.

3. If all resonance spots are $a \in \mathbb{Z}$ and have the rows of the vector $\underline{V}$ the same factor as the belonged rows of the matrix M, so the system posses the Painlevé property. Further more we find that all coefficients u^k with the index $j = a_i$ are arbitrary functions.

4. Some resonance spots may be not an element of $\mathbb{Z}$. For this case we find integrable as well as non-integrable equations. An example for this is the Harry-Dym equation. The case of rational resonance spots is called weak Painlevé property.

5. If all resonance spots are in $\mathbb{Z}$ but the rows of the vector on the right hand aren't linear dependent like described in 3) then we distinguish between two cases:

(a) We can generate the linear dependence if one introduces logarithm Psi-series [2] or:

(b) We get conditions of compatibility out of the assumption that the matrix equation is solvable. The solutions had to perform these additional equations. In the last two cases the Painlevé test is violated.

To check whether the resonance spots are valid or not you form the matrix equation for the special j and examine the linear dependence of the rows. It must be accomplish

$$Rang\ M = Rang\ M_{erw}. \tag{11}$$

For constructing solutions we make a cut of the several series u^k at the power Φ^0 and put this terms in system (5). A comparison of coefficients of powers of Φ leads to an overdetermined system of equations which we have to solve. There are several varieties for solving:

1. You can simplify the system by inserting an equation into other equations of the system and making a very strong assumption. In this case a solution will be found.

2. We can add and subtract terms to remove interfering terms. You have to pay attention to the fact that the Painlevé property respectively the passing of the Painlevé test wouldn't be disturbed. This means, that the resonance spots can be displaced and the condition (11) can loose its validity because of adding and subtracting terms. A procedure to find special terms which don't change the Painlevé property is given in a paper by Newell. [21]

1.1 The Boussinesq Equation and Painlevé tests

In this subsection we will examine the Boussinesq equation. This equation introduced by Boussinesq in 1871 describes the expansion of long waves in shallow water. However, this type of equation is applied to a wide range of physical problems containing one-dimensional nonlinear lattice wave, vibrations in a nonlinear chains, and sound waves of ions in a plasma.

[2] series of the form

$$u^k = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} u^k_{j,n}\Phi^{j-\alpha_k}\left(\Phi^{a_i}\ln\Phi\right)^n$$

(vgl. Clarkson [18], Tabor [19], Levine [20])

Here we examine the Boussinesq equation in the form

$$u_{tt} + au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} = 0 \tag{12}$$

$$\Rightarrow \quad u_{tt} + au_{xx} + 2bu_x^2 + 2buu_{xx} + cu_{xxxx} = 0 \tag{13}$$

and make for the WTC-Test the ansatz.

$$u = \sum_{j=0}^{\infty} u_j \Phi^{j+\alpha}. \tag{14}$$

First, as mentioned in step 1, we introduce the terms of order zero in the equation to determine u_0 and α. We get the solution

$$\alpha = -2; \qquad u_0 = -6\frac{c}{b}\left(\frac{\partial \Phi}{\partial x}\right)^2.$$

The last three terms of equation (13) are the leading ones. We put the series and their derivatives into the differential equation and make an index - transformation. Since the derived equation is valid for all j we get the following recursion equation:

$$u_{j-2}(j-4)(j-5)\left(\frac{\partial \Phi}{\partial t}\right)^2 + 2(j-5)\frac{\partial u_{j-3}}{\partial t}\frac{\partial \Phi}{\partial t} + u_{j-3}(j-5)\frac{\partial^2 \Phi}{\partial t^2} +$$

$$\frac{\partial^2 u_{j-4}}{\partial t^2} + au_{j-2}(j-4)(j-5)\left(\frac{\partial \Phi}{\partial x}\right)^2 + 2a\frac{\partial u_{j-3}}{\partial x}(j-5)\frac{\partial \Phi}{\partial x} +$$

$$au_{j-3}(j-5)\frac{\partial^2 \Phi}{\partial x^2} + a\frac{\partial^2 u_{j-4}}{\partial x^2} + 2b\sum_{k=0}^{j}\frac{\partial u_k}{\partial x}\frac{\partial u_{j-k-2}}{\partial x} +$$

$$2b\sum_{k=0}^{j}\frac{\partial u_k}{\partial u}_{\,j-k-1}(j-k-3)\frac{\partial \Phi}{\partial x} + 2b\sum_{k=0}^{j}u_k(k-2)\frac{\partial u_{j-k-1}}{\partial x}\frac{\partial \Phi}{\partial x} +$$

$$2b\sum_{K=0}^{j}u_k(k-2)u_{j-k}(j-k-2)\left(\frac{\partial \Phi}{\partial x}\right)^2 +$$

$$2b\sum_{k=0}^{j}u_k\frac{\partial^2 u_{j-k-2}}{\partial x^2} + 4b\sum_{k=0}^{j}u_k\frac{\partial u_{j-k-1}}{\partial x}(j-k-3)\frac{\partial \Phi}{\partial x} +$$

$$2b\sum_{k=0}^{j}u_k u_{j-k}(j-k-2)(j-k-3)\left(\frac{\partial \Phi}{\partial x}\right)^2 +$$

$$
\begin{aligned}
&2b\sum_{k=0}^{j} u_k u_{j-k-1}(j-k-3)\frac{\partial^2\Phi}{\partial x^2} + \\
&c(j-5)(j-4)(j-3)(j-2)u_j\left(\frac{\partial\Phi}{\partial x}\right)^4 + \\
&4c(j-5)(j-4)(j-3)\frac{\partial u_{j-1}}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)^3 + \\
&6c(j-5)(j-4)(j-3)u_{j-1}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \\
&12c(j-5)(j-4)\frac{\partial\Phi}{\partial x}\frac{\partial u_{j-2}}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \\
&3c(j-5)(j-4)u_{j-2}\left(\frac{\partial^2\Phi}{\partial x^2}\right)^2 + 6c(j-5)(j-4)\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2 u_{j-2}}{\partial x^2} + \\
&6c(j-5)\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 u_{j-3}}{\partial x^2} + 3c(j-5)(j-4)u_{j-2}\frac{\partial\Phi}{\partial x}\frac{\partial^3\Phi}{\partial x^3} + \\
&4c(j-5)\frac{\partial u_{j-3}}{\partial x}\frac{\partial^3\Phi}{\partial x^3} + 4c(j-5)\frac{\partial\Phi}{\partial x}\frac{\partial^3 u_{j-3}}{\partial x^3} + \\
&c(j-5)u_{j-3}\frac{\partial^4\Phi}{\partial x^4} + c\frac{\partial^4 u_{j-4}}{\partial x^4} = 0
\end{aligned}
\tag{15}
$$

We solve this equation for the leading terms and calculate the resonance spots. One finds $j = -1, 4, 5, 6$.

Now we insert numbers of $\mathbb{N}$ for j in (15) and get the u_j's:

$$
\begin{aligned}
j = 0 \ &: \ u_0 = -6\frac{c}{b}\left(\frac{\partial\Phi}{\partial x}\right)^2, \\
j = 1 \ &: \ u_1 = 6\frac{c}{b}\frac{\partial^2\Phi}{\partial x^2}, \\
j = 2 \ &: \ u_2 = -\frac{\left(\frac{\partial\Phi}{\partial t}\right)^2 + a\left(\frac{\partial\Phi}{\partial x}\right)^2 - 3c\left(\frac{\partial^2\Phi}{\partial x^2}\right)^2 + 4c\frac{\partial\Phi}{\partial x}\frac{\partial^3\Phi}{\partial x^3}}{2b\left(\frac{\partial\Phi}{\partial x}\right)^2}, \\
j = 3 \ &: \ u_3 = -\frac{-\frac{\partial^2\Phi}{\partial x^2}\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial t}\right)^2\frac{\partial^2\Phi}{\partial x^2} - 3c\left(\frac{\partial^2\Phi}{\partial x^2}\right)^3}{2b\left(\frac{\partial\Phi}{\partial x}\right)^4} + \\
&\qquad \frac{4c\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^3\Phi}{\partial x^3} - c\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^4\Phi}{\partial x^4}}{2b\left(\frac{\partial\Phi}{\partial x}\right)^4}.
\end{aligned}
$$

For $j = 4, j = 5$ and $j = 6$ the equation (15) is fulfilled identically. So the Boussinesq equation passes the WTC-Test.

In the following part of this section we deal with the symmetry analysis. Applying Lie's theory by using the functions of MathLie [22], we get the infinitesimal symmetries of equation (13):

$$\xi^1 = c_1 + \frac{c_3}{2}x, \quad \xi^2 = c_2 + c_3 t, \quad \eta^1 = -\frac{c_3\,(a+2bu)}{2b},$$

which are connected with the generators

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad \text{and} \quad v_3 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{a+2bu}{2b}\frac{\partial}{\partial u}$$

The commutator table is

	v_1	v_2	v_3
v_1	0	0	$-\frac{1}{2}v_1$
v_2	0	0	$-v_2$
v_3	$\frac{1}{2}v_1$	v_2	0

In the next step we reduce the Boussinesq equation with these generators:

1. v_1: The corresponding similarity variable is $\lambda = t, u = F(\lambda)$, and the reduced equation is
$$F''(\lambda) = 0.$$
We find the solution $u = a\lambda + b$. This function hasn't any singularities. It follows that this equation posses the Painlevé property for ordinary differential equations.

2. v_2: The similarity solution for this generator is $\lambda = x, u = F(\lambda)$. We find the following reduced equation:
$$2b\,(F'(\lambda))^2 + aF''(\lambda) + 2bF(\lambda)F''(\lambda) + cF^{(iv)}(\lambda) = 0.$$

3. $v_1 + v_2$: We get the following similarity solution for this generator:
$$\lambda = \frac{k_1 t - k_2 x}{k_1}; \quad u = F(\lambda).$$
The reduction is
$$2bk_1^2k_2^2\,(F'(\lambda))^2 + k_1^4F''(\lambda) + ak_1^2k_2^2F''(\lambda) + 2bk_1^2k_2^2F(\lambda)F''(\lambda) + ck_2^4F^{(iv)}(\lambda) = 0.$$
After two integrations we find
$$bk_1^2k_2^2\,(F(\lambda))^2 + k_1^4F(\lambda) + ak_1^2k_2^2F(\lambda) + ck_2^4F''(\lambda) = A\lambda + B.$$

4. v_3: The similarity representation is

$$\lambda = \frac{x^2}{t}, \quad F = x^2\left(a + 2bu\right).$$

This leads to the reduction

$$\frac{10}{\lambda^4}\left(F(\lambda)\right)^2 + \frac{120c}{\lambda^4}F(\lambda) - \frac{14}{\lambda^2}F(\lambda)F'(\lambda) + \frac{4}{\lambda^2}F(\lambda)F''(\lambda) -$$
$$\frac{120c}{\lambda^3}F'(\lambda) + \frac{2}{\lambda}F'(\lambda) + \frac{4}{\lambda^2}\left(F''(\lambda)\right) + \frac{60c}{\lambda^2}F''(\lambda) +$$
$$F''(\lambda) - \frac{16c}{\lambda}F'''(\lambda) + 16cF^{(iv)}(\lambda) = 0$$

On the other hand, we can examine (13) by means of the non-classical symmetry reduction. The main topics about this procedure are collected in [23, 24]. Following Clarkson [23] we make the following ansatz to calculate the symmetries:

$$u(x,t) = \alpha(x,t) + \beta(x,t)w\left(z(x,t)\right)$$

where α, β and $z(x,t)$ are unknown functions. This ansatz is inserted into the differential equation (13) and order by w, powers of w and derivatives of w. We find:

$$c\beta\left(\frac{\partial z}{\partial x}\right)^4\frac{d^4w}{dz^4} + \left(4c\frac{\partial\beta}{\partial x}\left(\frac{\partial z}{\partial x}\right)^3 + 6c\beta\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial x^2}\right)\frac{d^3w}{dz^3} +$$
$$\left(\beta\left(\frac{\partial z}{\partial t}\right)^2 + a\beta\left(\frac{\partial z}{\partial x}\right)^2 + 2b\alpha\beta\left(\frac{\partial z}{\partial x}\right)^2 + 6c\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2\beta}{\partial x^2} +\right.$$
$$\left.12c\frac{\partial\beta}{\partial x}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x^2} + 3c\beta\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + 4c\beta\frac{\partial z}{\partial x}\frac{\partial^3 z}{\partial x^3}\right)\frac{d^2z}{dz^2} +$$
$$\left(2\frac{\partial\beta}{\partial t}\frac{\partial z}{\partial t} + \beta\frac{\partial^2 z}{\partial t^2} + 2a\frac{\partial\beta}{\partial x}\frac{\partial z}{\partial x} + a\beta\frac{\partial^2 z}{\partial x^2} + 4b\beta\frac{\partial\alpha}{\partial x}\frac{\partial z}{\partial x} + 2b\alpha\beta\frac{\partial^2 z}{\partial x^2} +\right.$$
$$\left.6c\frac{\partial^2\beta}{\partial x^2}\frac{\partial^2 z}{\partial x^2} + 4c\frac{\partial z}{\partial x}\frac{\partial^3\beta}{\partial x^3} + 4c\frac{\partial\beta}{\partial x}\frac{\partial^3 z}{\partial x^3} + c\beta\frac{\partial^4 z}{\partial x^4}\right)\frac{dw}{dz} +$$

$$
\begin{aligned}
&\left(8b\beta\frac{\partial\beta}{\partial x}\frac{\partial z}{\partial x}+2b\beta^2\frac{\partial^2 z}{\partial x^2}\right)w\frac{dw}{dz}+\\
&\left(\frac{\partial^2\beta}{\partial t^2}+a\frac{\partial^2\beta}{\partial x^2}+4b\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}+2b\beta\frac{\partial^2\alpha}{\partial x^2}+2b\alpha\frac{\partial^2\alpha_2}{\partial x^2}+c\frac{\partial^4\beta}{\partial x^4}\right)w+\\
&2b\beta^2\left(\frac{\partial z}{\partial x}\right)^2 w\frac{d^2w}{dz^2}+2b\beta^2\left(\frac{\partial z}{\partial x}\right)^2\left(\frac{dw}{dz}\right)^2+\\
&\left(2b\left(\frac{\partial\beta}{\partial x}\right)^2+2b\beta\frac{\partial^2\alpha_2}{\partial x^2}\right)w^2+\\
&\frac{\partial^2\alpha_1}{\partial t^2}+a\frac{\partial^2\alpha}{\partial x^2}+2b\alpha\frac{\partial^2\alpha}{\partial x^2}+c\frac{\partial^4\alpha}{\partial x^4}+2b\left(\frac{\partial\alpha}{\partial x}\right)^2=0.
\end{aligned}
\tag{16}
$$

We now have to derive an ordinary differential equation for $w(z)$ from (16). Therefore the factors in front of several derivatives and powers of $w(z)$ have to be only functions of z. So we get conditions for $\alpha(x,t),\beta(x,t)$ and $z(x,t)$. So each solution of (16) will lead to a similarity solution. We refer the reader to the cited paper of Clarkson [23] and references included therein to get more information. During the following calculations we have the following arbitrariness:

1. The factor in front of the highest derivative has to be normalized. So it follows that they are of the type
$$\beta\Gamma(z)\frac{\partial^4 z}{\partial x^4}$$
with the arbitrary function $\Gamma(z)$ remaining to be determined.

2. We name the unknown functions of z with a Greek letter so that we can denote the result after transformations like derivation, integration, exponentiation, scaling etc. with the same letter. (So we denote for example the derivative $\Gamma'(z)$ with $\Gamma(z)$)

3. Without loosing any generality we make the following statements about $\alpha(x,t),\beta(x,t),z(x,t)$and $w(z(x,t))$:

   (a) Is $\alpha(x,t)$ given by
   $$\alpha=\alpha(x,t)+\beta(x,t)\Omega(z),$$
   we can set $\Omega\equiv 0$ after the transformation
   $$w(z)\rightarrow w(z)-\Omega(z).$$

(b) If $\alpha_2(x,t)$ has the form

$$\beta = \beta_0(x,t)\Omega(z),$$

we can set $\Omega(z) \equiv 1$ via the transformation $w(z) \to \frac{w(z)}{\Omega(z)}$.

(c) We determine $z(x,t)$ with the equation

$$\Omega(z) = z_0(x,t)$$

where $\Omega(z)$ is invertible. We put $\Omega(z) = z$ via the substitution $z \to \Omega^{-1}(z)$.

If we look at equation (16) we find that the coefficients of $w\frac{d^2w}{dz^2}$ and $\left(\frac{dw}{dz}\right)^2$ are equal. So we make the ansatz

$$c\beta z_x^4\Gamma(z) = 2b\beta^2 z_x^2.$$

It follows

$$\beta = \frac{c}{2b} z_x^2\Gamma(z).$$

Now we are looking at the coefficient of w'''. For this the equation

$$c\beta z_x^4\Gamma(z) = 4c\beta_x z_x^3 + 6cz_x^2 z_{xx}$$

is valid where $\Gamma(z)$ is to be determined. We put in β and β_x and divide by z_x^5. After rescaling of $\Gamma(z)$ we get

$$z_x\Gamma(z) - \frac{z_{xx}}{z_x} = 0.$$

This equation will be integrated over x. During this calculation we partially integrate the first term and apply remark 2. After exponentiating the result under condition 2) we get $\Gamma(z) = x\Theta(t)+\Sigma(t)$, using 3), we have $z = x\theta(t)+\sigma(t)$. and $\beta = \frac{c}{2b}\theta^2(t)$. The next step considers coefficients of w''. The corresponding equation is

$$\begin{aligned} c\beta z_x^4\Gamma(z) = \quad & \beta(z_t)^2 + a\beta z_{xx} + 2b\alpha\beta z_x^2 + 6c\beta_{xx}(z_x)^2 + 12c\alpha_x z_x z_{xx} + \\ & 3c\beta(z_{xx})^2 + 4\beta c z_x z_{xxx}. \end{aligned}$$

After inserting all results, we solve this equation with resprect to α and use remark 1). It follows

$$\alpha = -\frac{1}{2b\theta^2}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right)^2.$$

Putting this into the differential equation (16) we get

$$\theta^6\left(w^{iv}+ww''+(w')^2\right)+\frac{c}{2b}\theta^2\left(x\frac{d^2\theta}{dt^2}+\frac{d^2\sigma}{dt^2}\right)w'+\frac{c}{b}\theta\frac{d^2\theta}{dt^2}w+$$
$$\alpha_{tt}+a\alpha_{xx}+2b\alpha\alpha_{xx}+c\alpha_{xxxx}+2b(\alpha_x)^2=0.$$

Taking the coefficients of the derivatives and powers of w and determine them as functions of z, we end up with

$$\theta^6\gamma_1(z) = \frac{c}{2b}\theta^2\left(x\frac{d^2\theta}{dt^2}+\frac{d^2\sigma}{dt^2}\right) \tag{17}$$
$$\theta^6\gamma_2(z) = \frac{c}{b}\theta\frac{d^2\theta}{dt^2} \tag{18}$$
$$\begin{aligned}\theta^6\gamma_3(z) = &-\frac{a}{b\theta^2}\left(\frac{\partial\theta}{\partial t}\right)^2+\frac{1}{b\theta^3}\left(\frac{d\sigma}{dt}+x\frac{d\theta}{dt}\right)^2\frac{d^2\theta}{dt^2}+\\ &\frac{4}{b\theta^3}\frac{d\theta}{dt}\left(\frac{d\sigma}{dt}+x\frac{d\theta}{dt}\right)\left(\frac{d^2\sigma}{dt^2}+x\frac{d^2\theta}{dt^2}\right)-\\ &\frac{1}{b\theta^2}\left(\frac{d^2\sigma}{dt^2}+x\frac{d^2\theta}{dt^2}\right)^2-\frac{1}{b\theta^2}\left(\frac{d\sigma}{dt}+x\frac{d\theta}{dt}\right)\left(\frac{d^3\sigma}{dt^3}+x\frac{d^3\theta}{dt^3}\right)\end{aligned} \tag{19}$$

Since x in (17) is linear, we make the following ansatz for γ_1: $\gamma_1(z)=Az+B$. From the comparison of coefficients it follows

$$\frac{d^2\theta}{dt^2} = \frac{2b}{c}\theta^5A \tag{20}$$
$$\frac{d^2\sigma}{dt^2} = \frac{2b}{c}\left(A\sigma+B\right)\theta^4. \tag{21}$$

We put this into equation (18) and get $\gamma_2(z)=2A$. Finally let us examine (20). We insert α and its derivatives and compare the coefficients. For $\gamma_3(z)$ me make the ansatz $\gamma_3(z)=\tilde{\alpha}x^2+\tilde{\beta}x+\tilde{\gamma}$ With (20) and (21) we find

$$\begin{aligned}\tilde{\alpha} &= -\frac{4b}{c^2}\theta^8A^2,\\ \tilde{\beta} &= -\frac{8b}{c^2}\theta^7A^2\sigma-\frac{8b}{c^2}AB\theta^7,\\ \tilde{\gamma} &= -\frac{a}{b\theta^2}-\frac{4b}{c^2}\theta^6A^2\sigma^2-\frac{8b}{c^2}\theta^6AB\sigma-\frac{4b}{c^2}\theta^6B^2.\end{aligned}$$

The final result is

$$u(x,t) = \frac{c}{2b}\theta^2 w(z) - \frac{1}{2b\theta^2}\left(x\frac{\partial\theta}{\partial t} + \frac{\partial\sigma}{\partial t}\right)^2, \tag{22}$$

$$z(x,t) = x\theta(t) + \sigma(t); \quad \frac{d^2\theta}{dt^2} = \frac{2b}{c}\theta^5 A; \quad \frac{d^2\sigma}{dt^2} = \frac{2b}{c}(A\sigma + B)\theta^4, \tag{23}$$

$$w^{(iv)} + ww'' + (w')^2 + (Az+B)w' + 2Aw = \frac{4b}{c^2}(Az+B)^2 + \frac{a}{b\theta^8}\left(\frac{d\theta}{dt}\right)^2 \tag{24}$$

We can show [23] that the general form of all differential equations of the form

$$w^{(iv)} + ww'' + (w')^2 + f(z)w' + g(z)w = h(z)$$

where $f(z), g(z), h(z)$ are analytic functions which possess the Painlevé property is of the type

$$w^{(iv)} + ww'' + (w')^2 + (Az+B)w' + 2Aw = 2(Az+B)^2 \tag{25}$$

For equation (25) it is possible to distinguish several cases for the constants A and B. We put in (24) $a = 0$.

For example if we choose $A = 0$ and $B = 0$, it follows from (20) and (21) that

$$\theta(t) = a_1 t + a_0; \quad \sigma(t) = b_1 t + b_0$$

and

$$\begin{aligned} u(x,t) &= \frac{c}{2b}(a_1 t + a_0)^2 w(z) - \frac{1}{2b}\left(\frac{a_1 x + b1}{a_1 t + a_0}\right)^2, \\ z &= x(a_1 t + a_0) + b_1 t + b_0, \\ w'' + \frac{1}{2}w^2 &= c_1 z + c_0. \end{aligned}$$

The last differential equation is equivalent to the first Painlevé transcendent. With this procedure, we can determine similarity reductions which possess the Painlevé property. It follows that the Boussinesq equation passes the ARS-Test.

We observe that it is very difficult to show if a partial differential equation passes the ARS-Test because we have to find all symmetry reductions.

2 The Theorem of Strampp and Symmetry analysis

First of all we will examine how a Laurent-series (6) of the Painlevé test changes under a similarity reduction. We look at an equation of the form

$$u_t = K(u, u_x, \cdots, u_{rx}) \tag{26}$$

where K is a polynomial in u and the spatial derivatives up to order r, $x \in \mathrm{IR}^2$. The independent variables are $x_1 = x$ and $x_2 = t$. Furthermore, we assume that equation (26) possesses the Painlevé property and the series (6). The generator of u in (26) is assumed to allow the ansatz

$$v = \xi^x(x,t)\frac{\partial}{\partial x} + \xi^t(x,t)\frac{\partial}{\partial t} + \eta^u(x,t,u)\frac{\partial}{\partial u}$$

resulting into specific infinitesimals. Knowing the infinitesimals, we apply Fuchs's procedure [25] to reduce equation (26) to

$$\eta^u(x,c,v) - \xi^x(s,c)\frac{d}{d\lambda}v = \xi^t(s,c)K\left(v, v', ..., v^{(n)}\right). \tag{27}$$

It can be shown that the related Laurent-series are of the form [26]

$$u(x,t) = \left(x - \left\{\frac{s_0 - h(t)}{g(t)}\right\}\right)^{\alpha} \sum_{j=0}^{\infty} u_j \left(x - \left\{\frac{s_0 - h(t)}{g(t)}\right\}\right)^j, \tag{28}$$

where $u_j = c_j\ G(t)\ g(t)^{j-a}$ for $j \neq a$ and $u_a = c_a G(t) + H(t)$. If we compare (28) with (6) we find, that (28) does not allow other resonance spots than we found in (11).

For example, we consider the Burgers equation [26]

$$u_t + uu_x + u_{xx} = 0. \tag{29}$$

The Laurent-series is given by

$$u = \Phi^{-1} \sum_{j=0}^{\infty} u_j \Phi^j.$$

The generators for (29) are

$$v_1 = \gamma\frac{\partial}{\partial x} + \frac{\partial}{\partial t}; \quad \gamma = const, \tag{30}$$

$$v_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \tag{31}$$

$$v_3 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - tu)\frac{\partial}{\partial u}, \tag{32}$$

$$v_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{33}$$

From generator (30) it follows that $u(x,t) = v(\lambda)$; $\lambda = x - \gamma t$. We find

$$v'' + (v + \gamma)v' = 0. \tag{34}$$

With (31), we get $u(x,t) = \frac{1}{\sqrt{t}}v(\lambda), \quad \lambda = \frac{x}{\sqrt{t}}$ and

$$v'' + vv' - \frac{1}{2}\lambda v' = 0. \tag{35}$$

If we consider (32), we calculate $u(x,t) = \lambda + \frac{1}{t}\left(v - \lambda\right), \lambda = \frac{x}{t}$. The corresponding reduced differential equation is

$$v'' + vv' - (\lambda v)' + \lambda = 0. \tag{36}$$

Generator (33) leads to $u(x,t) = v + \frac{x}{\lambda}, \quad \lambda = t$ and

$$\lambda v' + v = 0. \tag{37}$$

If we inspect equations (34), (35) and (36) we find that they allow the following Laurent expansion over a movable pole:

$$v(\lambda) = (\lambda - \lambda_0)^{-1} \sum_{j=0}^{\infty} c_j (\lambda - \lambda_0)^j \tag{38}$$

where c_2 is an arbitrary constant. Using this equation for (34), (35) and (36), we get the following series for similarity solutions.
In case of (34):

$$u(x,t) = (x - \gamma t - \lambda_0)^{-1} \sum_{j=0}^{\infty} c_j \left(x - \gamma t - \lambda_0\right)^j,$$

for relation (35):

$$u(x,t) = \left(x - \lambda_0\sqrt{t}\right)^{-1} \sum_{j=0}^{\infty} c_j t^{-\frac{j}{2}} \left(x - \lambda_0\sqrt{t}\right)^j,$$

and for the reduction (36):

$$\begin{aligned} u(x,t) &= c_0 \left(x - \lambda_0 t\right)^{-1} + \lambda_0 + \left(c_1 - \lambda_0\right) t^{-1} + \left(c_2 t^{-2} + t - 1\right) \left(x - \lambda_0 t\right) \\ &\quad + \sum_{j=3}^{\infty} c_j t^{-j} \left(x - \lambda_0 t\right)^{j-1}. \end{aligned}$$

Another possibility to construct solutions when making a similarity analysis is the following procedure:
Let $x_1 = x$ and $x_2 = t$ be the independent variables. Let us use a linearized form of equation (26) for the solution u, given by

$$v_t = K'(u)v = \frac{\partial}{\partial \varepsilon} K\left(u + \varepsilon v\right)\Big|_{\varepsilon=0}. \tag{39}$$

Each solution v of this equation is a symmetry or an infinitesimal transformation of u, meaning that (26) is invariant under $u + \varepsilon v$ which is motivated by the following theorem by Strampp [27]:

Theorem 2.1 *Let*

$$u_t = K\left(u, u^{(r)}\right), \tag{40}$$

be a given partial differential equation where K is a polynomial in u and in the spatial derivatives up to order r. Furthermore, we know the expansion of u in the form

$$u = \sum_{j=0}^{\infty} u_j \Phi^{j+\alpha}. \tag{41}$$

Equation (40) passes the Painlevé test.

If we stop the series at a spot $a > 0$ then u_{a-1} is an infinitesimal transformation of u_a.

The prove of the theorem can be found in [27].

Let us apply this theorem to the KdV equation to calculate solutions connecting the WTC-test and infinitesimal transformations. The KdV-equation ist

$$u_t + uu_x + u_{xxx} = 0 \tag{42}$$

The system of the broken series reads

$$\Phi^{-5} \ : \ -2u_0^2\frac{\partial\Phi}{\partial x} - 24u_0\left(\frac{\partial\Phi}{\partial x}\right)^3 = 0, \tag{43}$$

$$\begin{aligned}\Phi^{-4} \ : \ & -3u_1u_0\frac{\partial\Phi}{\partial x} - 6u_1\left(\frac{\partial\Phi}{\partial x}\right)^3 + 18\frac{\partial u_0}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)^2 + \\ & 18u_0\frac{\partial^2\Phi}{\partial x^2}\frac{\partial\Phi}{\partial x} + u_0\frac{\partial u_0}{\partial x} = 0,\end{aligned} \tag{44}$$

$$\begin{aligned}\Phi^{-3} \ : \ & -2u_2u_0\frac{\partial\Phi}{\partial x} - u_1^2\frac{\partial\Phi}{\partial x} + u_1\frac{\partial u_0}{\partial x} + 6u_1\frac{\partial^2\Phi}{\partial x^2}\frac{\partial\Phi}{\partial x} + u_0\frac{\partial u_1}{\partial x} - 2u_0\frac{\partial^3\Phi}{\partial x^3} - \\ & 2u_0\frac{\partial\Phi}{\partial t} + 6\frac{\partial u_1}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)^2 - 6\frac{\partial^2 u_0}{\partial x^2} - 6\frac{\partial u_0}{\partial x}\frac{\partial^2\Phi}{\partial x^2} = 0,\end{aligned} \tag{45}$$

$$\begin{aligned}
\Phi^{-2} &: -u_2 u_1 \frac{\partial \Phi}{\partial x} + u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} - u_1 \frac{\partial^3 \Phi}{\partial x^3} - u_1 \frac{\partial \Phi}{\partial t} + u_0 \frac{\partial u_2}{\partial x} - \\
&\quad 3 \frac{\partial^2 u_1}{\partial x^2} \frac{\partial \Phi}{\partial x} - 3 \frac{\partial u_1}{\partial x} \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^3 u_0}{\partial x^3} + \frac{\partial u_0}{\partial t} = 0, \qquad (46) \\
\Phi^{-1} &: u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + \frac{\partial^3 u_1}{\partial x^3} \frac{\partial u_1}{\partial t} = 0, \qquad (47) \\
\Phi^{0} &: u_2 \frac{\partial u_2}{\partial x} + \frac{\partial^3 u_2}{\partial x^3} + \frac{\partial u_2}{\partial t} = 0. \qquad (48)
\end{aligned}$$

The generators are:

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad v_4 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u}.$$

Let us make a linear combination of all the generators as an ansatz

$$v = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial t} + \gamma \left(t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right) + \delta \left(x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u} \right)$$

and transform it to the form

$$v = \left[(\gamma - 2\delta u) + \frac{\partial u}{\partial x} (\alpha + \gamma t - \delta x) - \frac{\partial u}{\partial t} (\beta + 3\delta t) \right] \frac{\partial}{\partial u}, \qquad (49)$$

using

$$v = \left[\eta^\alpha - \sum_i \frac{\partial u^\alpha}{\partial x^i} \xi^i \right] \frac{\partial}{\partial u_\alpha}.$$

(see [28] p. 261 theorem 5.2.3-1). After setting $u = u_2$ in 49 we apply this equation to u_2. Therefore we can represent u_1 with the help of Strampp's theorem as

$$u_1 = (\gamma - 2\delta u_2) - \frac{\partial u_2}{\partial x} (\alpha + \gamma t + \delta x) - \frac{\partial u_2}{\partial t} (\beta + 3\delta t) \qquad (50)$$

If we insert the solution $u_2 = 0$ into (50), it follows that $u_1 = \gamma$ and with (44) we find $\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{12}\gamma$. An integration of the last relation leads to $\frac{\partial \Phi}{\partial x} = \frac{1}{12}\gamma x + g(t)$. If we integrate once more, we get $\Phi = \frac{1}{24}\gamma x^2 + g(t)x + h(t)$. Using this equation we find:

$$u_0 = -12 \left[\frac{1}{12} \gamma x + g(t) \right]^2 .$$

Inserting this result into system (41) - (47) we find that $\gamma = 0,\ g(t) = c_1,\ h(t) = c_2,\ c_1, c_2 = const$. Thus the final result is

$$\begin{aligned}
u_1 &= 0, \qquad (51) \\
u_2 &= 0, \qquad (52) \\
u_0 &= -12 c_1^2, \qquad (53) \\
\Phi &= c_1 x + c_2. \qquad (54)
\end{aligned}$$

The solution in original coordinates for u reads

$$u = \frac{-12c_1^2}{(c_1x + c_2)^2} = \frac{-12}{(x + \tilde{c_1})^2}.$$

If we check the conditions from [13], we get

$$grad\ \Phi = c_1 \neq 0$$

for $c_1 \neq 0$ at every arbitrary spot x_0. Furthermore we see that the solution u is meromorph. Now we can use this solution as a new solution u_2 and apply the transformaation [1] $u^3 = f(x - \varepsilon t) + \varepsilon$ to gain another solution

$$u_2 = \frac{-12}{(x + c_1t + c_2)^2} - c_1$$

We can repeat the last two steps again and again to create solutions.

3 Definition of Optimal Systems

We regard a system of differential equations (5) for which we have calculated the r-parametric maximal symmetry group. For every s-parametric subgroup H one is able to find a family of similarity solutions. The assumption is that $s \geq min\{r, n'\}, s, r, n' \in \mathbb{N}$. n' is the number of independent variables of F and r is the order of derivatives. Since in many cases there exists an infinite number of such subgroups it is impossible to calculate all similarity solutions relative to s-parametric subgroups. In this set there are similarity solutions which result from other similarity solutions of the same set applying a transformation of the symmetry group. It would be profitable to have a minimal list of similarity solutions such that with these elements one can get all other similarity solutions via transformation. Such a minimal list is called an optimal system and their elements are essentially different similarity solutions. The related inner-authomorphism can be defined by conjugation [29] and leads to a comparison of two elements g_1, g_2 of the Lie-group G.

Since two equivalence classes are either identical or disjunct, the set of all s-parametric subgroups is split into two disjunct equivalence classes.

We define an optimal system following Olver [1]. This definition reduces the task to classify subgroups of the maximal symmetry group S. Furthermore, we treat the problem to classify the subalgebras contained in the subgroups of the maximal symmetry group S. This is possible because there is a connection via the adjoint representation of the Lie group and the adjoint representation of the Lie algebra:

$$ad(v)(w) := \frac{d}{d\varepsilon} Ad\left(\exp\left(\varepsilon \underline{v}\right)\right)(w) = [\underline{w}, \underline{v}]$$

with

$$\begin{aligned} Ad\left(\exp\left(\varepsilon \underline{v}\right)\right) &= e^{\varepsilon\ ad(\underline{v})}\left(\underline{w}_0\right) = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\left(ad\left(\underline{v}\right)\right)^j\left(\underline{w}_0\right) \\ &= \underline{w}_0 + \varepsilon\left[\underline{w}_0, \underline{v}\right] + \frac{\varepsilon^2}{2}\left[\left[\underline{w}_0, \underline{v}\right], \underline{v}\right] + \cdots \end{aligned} \tag{55}$$

It can be also shown [1]:

H_s and $\tilde{H}_s$ are two connected s-parametric subgroups of the Lie group G with the corresponding s-dimensional Lie algebras $\mathcal{H}_s$ and $\tilde{\mathcal{H}}_s$, which are subalgebras of the Lie algebra $\mathcal{G}$ of G. H_s and $\tilde{H}_s$ are conjugated subgroups of G, $H_s \overset{G}{\sim} \tilde{H}_s$, if and only if there exists an inner automorphism $Ad(g) \in Int(\mathcal{G})(g \in G)$ for the related subalgebras

$$\mathcal{H}_s = Ad\left(g\right)\left(\tilde{\mathcal{H}}_s\right).$$

This directly leads to the definition of conjugated subalgebras and optimal systems [1].

To apply the calculation of optimal systems let us consider the KdV equation. Contrary to the presentation of Olver [1] we don't use invariants in our calculation.

The Lie algebra's basis corresponding to the KdV equation is given by

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial t}, \quad v_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad v_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{56}$$

The commutator table reads

	v_1	v_2	v_3	v_4
v_1	0	0	0	$-v_1$
v_2	0	0	$-v_1$	$-3v_2$
v_3	0	v_1	0	$2v_3$
v_4	v_1	$3v_2$	$-2v_3$	0

First we determine the adjoint representation Ad of the symmetry group by (55) and get

$$Ad\,(e^{\varepsilon_1 v_1}) \;=\; \begin{pmatrix} 1 & 0 & 0 & -\varepsilon_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Ad\,(e^{\varepsilon_2 v_2}) = \begin{pmatrix} 1 & 0 & -\varepsilon 2 & 0 \\ 0 & 1 & 0 & -2\varepsilon_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Ad\,(e^{\varepsilon_3 v_3}) \;=\; \begin{pmatrix} 1 & \varepsilon_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\varepsilon_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Ad\,(e^{\varepsilon_4 v_4}) = \begin{pmatrix} e^{\varepsilon_4} & 0 & 0 & 0 \\ 0 & e^{3\varepsilon_4} & 0 & 0 \\ 0 & 0 & e^{-2\varepsilon_4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjoint representation of an arbitrary element g of the group is gained from the product of the matrices above.

$$Ad_g = \begin{pmatrix} e^{\varepsilon_4} & \varepsilon_3 e^{3\varepsilon_4} & -2\varepsilon_2 e^{-2\varepsilon_4} & -\varepsilon_1 - 2\varepsilon_2\varepsilon_3 \\ 0 & e^{3\varepsilon_4} & 0 & -3\varepsilon_2 \\ 0 & 0 & 2e^{-2\varepsilon_4} & 2\varepsilon_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the following calculations let us make the ansatz

$$\frac{1}{a} Ad_g \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}, \tag{57}$$

$$\frac{1}{a} \begin{pmatrix} e^{\varepsilon_4}\alpha_1 + \varepsilon_3 e^{2\varepsilon_4}\alpha_2 - 2\varepsilon_2 e^{-2\varepsilon_4}\alpha_3 - \varepsilon_1\alpha_4 - 2\varepsilon_1 2\varepsilon_3\alpha_4 \\ e^{3\varepsilon_4}\alpha_2 - 3\varepsilon_2\alpha_4 \\ 2e^{2\varepsilon_4}\alpha_3 + 2\varepsilon_3\alpha_4 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

We are trying to simplify the right hand side by determining ε_i. We have to distinguish several cases referring to α_4

1. $\alpha_4 \neq 0$:

   We begin with the third component and set it to zero and solve it for ε_3.

The result is

$$\varepsilon_3 = -\frac{e^{-2\varepsilon_4}\alpha_3}{\alpha_4}$$

It follows $\beta_3 = 0$. After having set the second component to zero the solution for ε_2 is

$$\varepsilon_2 = \frac{e^{3\varepsilon_4}\alpha_2}{3\alpha_4}$$

and $\beta_2 = 0$. Now we set the first component zero and solve it for ε_1. We get

$$\varepsilon_1 = \frac{e^{\varepsilon_4}\alpha_1 + \varepsilon_3 e^{3\varepsilon_4}\alpha_2 - 2\varepsilon_2 e^{-2\varepsilon_4}\alpha_3 - 2\varepsilon_2\varepsilon_3\alpha_4}{\alpha_4}$$

and $\beta_1 = 0$. For the vector $\underline{\beta}$ in equation (57), we find

$$\underline{\beta} = (0, 0, 0, 1)$$

with $\frac{\alpha_4}{a} = 1$.

2. $\alpha_4 = 0$:
   The appropriate equation is

$$\frac{1}{a}\begin{pmatrix} e^{\varepsilon_4}\alpha_1 + \varepsilon_3 e^{3\varepsilon_4}\alpha_2 - 2\varepsilon_2 e^{-2\varepsilon_4}\alpha_3 \\ e^{3\varepsilon_4}\alpha_1 \\ 2e^{-2\varepsilon_4}\alpha_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

   In this case we have to distinguish some subcases too:

   (a) $\alpha_2 \neq 0, \alpha_3 \neq 0$
   You take the second component and put

$$e^{3\varepsilon_4}\alpha_2 = \begin{cases} +1 & \text{for} \quad \alpha_2 > 0 \\ -1 & \text{for} \quad \alpha_2 < 0 \end{cases}$$

   This ansatz is necessary because we have to take into account logarithms of this equation to get a solution for ε_4. This operation is defined only for positive values. We find

$$e^{3\varepsilon_4} = \frac{1}{|\alpha_2|}$$

   and $\beta_2 = \pm 1$.

From the same reasons, we choose the same ansatz for the third component and get $\beta_3 = \pm 1$. Then the first component will be set zero yields

$$\varepsilon_3 = \frac{-e^{\varepsilon_4}\alpha_1 + 2\varepsilon_2 e^{-2\varepsilon_4}\alpha_3}{\alpha_2 e^{3\varepsilon_4}}$$

It follows $\beta_1 = 0$. We obtain two different linear independent results:

$$\begin{aligned}\underline{\beta} &= (0,1,1,0) \quad \text{for} \quad \alpha_2 > 0; \alpha_3 > 0\\ \underline{\beta} &= (0,1,-1,0) \quad \text{for} \quad \alpha_2 > 0; \alpha_3 < 0.\end{aligned}$$

All other possibilities of combination are linear dependent.

(b) $\alpha_2 = 0; \alpha_3 \neq 0$:
The equation which corresponds to this case is

$$\frac{1}{a}\begin{pmatrix} e^{\varepsilon_4}\alpha_1 - 2\varepsilon_2 e^{-2\varepsilon_4}\alpha_3 \\ 0 \\ 2e^{-2\varepsilon_4}\alpha_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

From the third component it follows that

$$2e^{-2\varepsilon_4}\alpha_3 = \pm 1$$

and $\beta_3 = 1$. Putting the first component to zero this leads to

$$\varepsilon_2 = \frac{e^{\varepsilon_4}\alpha_1}{2e^{-2\varepsilon_4}\alpha_3},$$

and $\beta_1 = 0$. We get

$$\underline{\beta} = (0,0,1,0).$$

(c) $\alpha_2 = 0; \alpha_3 = 0; \alpha_1 \neq 0$
The equation for this case can be written as

$$\frac{1}{a}\begin{pmatrix} e^{\varepsilon_4}\alpha_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

We put the first component to zero and get

$$e^{\varepsilon_4} = \frac{1}{\alpha_1}$$

and $\beta_1 = 1$ and come to

$$\underline{\beta} = (1, 0, 0, 0) .$$

(d) $\alpha_3 = 0; \alpha_2 \neq 0; \alpha_1 \neq 0$:
The appropriate equation is

$$\frac{1}{a} \begin{pmatrix} e^{\varepsilon_4}\alpha_1 + \varepsilon_3 e^{3\varepsilon_4}\alpha_2 \\ e^{3\varepsilon_4}\alpha_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} .$$

If we set the second component equal to one, we get $\beta_2 = 1$. If we require the vanishing of the first component, we obtain

$$\varepsilon_3 = -\frac{e^{\varepsilon_4}\alpha_1}{e^{3\varepsilon_4}\alpha_2}$$

and the result for $\underline{\beta}$ is

$$\underline{\beta} = (0, 1, 0, 0) .$$

All other possible cases and settings lead to linear dependent vectors $\underline{\beta}$

As a result we find the following optimal system $\Theta_1^{\mathcal{G}}$:

$$\mathcal{H} = \{v_4\} , \mathcal{H} = \{v_2 + v_3\} , \mathcal{H} = \{v_2 - v_3\} , \mathcal{H} = \{v_3\} , \mathcal{H} = \{v_2\} , \mathcal{H} = \{v_1\} .$$

Knowing this system, we can derive solutions of the KdV-equation via reductions, and the methods discussed in section 3.

Similar calculations were carried out by us for the non-linear Schrödinger equation. The results will be published elsewhere.

4 Conclusions

These calculations demonstrated that there are two different possibilities to calculate solutions for non-linear partial differential equations. With the help of the Painlevé test, we gain solutions by Laurent- series. With the similarity analysis, we can determine the Lie algebra related to the equation and the classification of the Lie algebra allows to derive self-similar solutions. However, we demonstrated that both solution procedures are connected. With the Painlevé ansatz, we determined a system of partial differential equations resulting from the broken series. The similarity analyse leads directly to the generators of the equation. If we make an ansatz of the linearcombination of these generators we get recursively solutions. Putting this into the system of partial differential equations we can calculate all unknown functions.

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