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A Note on the Derivation of Governing Equations of Envelopes Created by Nearly Bi-chromatic Waves —The modified Schrödinger equation and the modified Schrödinger-Nohara equation—

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Abstract

In this paper, two concepts of directional waves: directional, nearly monochromatic waves and directional, nearly bi-chromatic waves are presented. Directional, nearly monochromatic waves are propagation-direction-based nearly monochromatic waves, whose energy is almost concentrated in a single propagation direction. Directional, nearly bi-chromatic waves are the ones whose energy is almost concentrated in two wave propagation directions and the approach of these directions is very close. We show that the modified Schrödinger equation, which is modified from the Schrödinger equation, governs the envelope created by nearly bi-chromatic waves and also the modified Schrödinger-Nohara equation modified from the Schrödinger-Nohara equation governs the envelope surface created by directional, nearly bi-chromatic waves.

1. INTRODUCTION

In the course of studying the physical characteristics of group waves (wave packets)[1][2] in the water, the Schrödinger equation has played an important role in the mathematical and physical analysis. The envelope, which is created by group waves, has been considered to model them. We can consider two types of model of group waves as nearly monochromatic waves and nearly bi-chromatic waves. In the former waves, the energy is almost concentrated in a single wavenumber (Definition 1). Whereas in the latter waves, the energy is almost concentrated in two wavenumbers and the approach of these wavenumbers is very close (Definition 3). The Schrödinger equation basically governs the envelope created by nearly monochromatic waves [3]. The past studies of modeling of group waves almost focused on nearly monochromatic waves [4][5]. This paper deals with nearly bi-chromatic waves in the model of group waves.

We present two concepts of directional waves: directional, nearly monochromatic waves and directional, nearly bi-chromatic waves. Directional, nearly monochromatic waves are propagation-direction-based nearly monochromatic waves, whose energy is almost concentrated in a single propagation direction (Definition 2). The Schrödinger-Nohara equation governs the envelope surface created by directional, nearly monochromatic waves [6][7]. Directional, nearly bi-chromatic waves are the ones whose energy is almost concentrated in two wave propagation directions and the approach of these directions is very close (Definition 4).

We show that the modified Schrödinger equation, which is modified from the Schrödinger equation, governs the envelope created by nearly bi-chromatic waves and also the modified Schrödinger-Nohara equation modified from the Schrödinger-Nohara equation governs the envelope surface created by directional, nearly bi-chromatic waves in this paper.

The following section presents preliminaries of the envelope equations of nearly monochromatic waves (the Schrödinger equation) and directional, nearly monochromatic waves (the Schrödinger-Nohara equation) as well. The third section defines nearly bi-chromatic waves and directional, nearly bi-chromatic waves and then presents main results.

2. Preliminaries

Definition 1 *Nearly monochromatic waves* $u(x,t)$ [3] *are defined by the waves whose energy is almost concentrated in one wavenumber as follows* [10]*:*

$$u(x,t) = \int_k S_1(k) e^{i\{kx-\omega(k)t\}} dk. \tag{1}$$

Here k*,* t*,* x*,* ω *and* S_1 *denote wavenumber, time, space, angular frequency of dispersive characteristics with respect to wavenumber and spectrum of nearly monochromatic waves, respectively. The profile of the spectrum of nearly monochromatic waves has a peak at* $k = k_0$ *and spreads* ϵ_k *(sufficiently small) around* k_0*.*

Remark 1 Nearly monochromatic waves create an envelope and we can see it as *swell* in ocean.

Theorem 1 *The envelope created by nearly monochromatic waves satisfies the following linear Schrödinger equation.*[3][6]

$$i\left(\frac{\partial A(x,t)}{\partial t} + \omega'(k_0)\frac{\partial A(x,t)}{\partial x}\right) + \frac{1}{2!}\omega''(k_0)\frac{\partial^2 A(x,t)}{\partial x^2} = 0 \tag{2}$$

Remark 2 Here $A(x,t)$ is the amplitude of nearly monochromatic waves of the function of space x and time t. $A(x,t)$ acts as the envelope of traveling waves. $\omega^{(n)}(k)$ means the n-th derivative of ω with respect to k.

proof We consider plane traveling waves $u(x,t)$ with dispersive characteristics of the form:

$$u(x,t) = A(x,t)e^{i\{k_0x-\omega(k_0)t\}} \tag{3}$$

as an approximation of the class of $u(x,t)$ in Equation (1). This is based on the assumption that most of the energy is concentrated in one wavenumber k_0

(nearly monochromatic waves) and the amplitude $A(x,t)$ is not constant but varies slowly in space and time. So, $A(x,t)$ is derived from Equations (1) and (3) as follows:

$$A(x,t) = \int_k S_1(k) e^{iP_1(x,k,t)} dk, \tag{4}$$

where

$$P_1(x,k,t) = (k-k_0)x - \{\omega(k) - \omega(k_0)\}t. \tag{5}$$

The time derivative of $A(x,t)$ is

$$\frac{\partial A(x,t)}{\partial t} = \int_k (-i)\left\{\omega(k) - \omega(k_0)\right\} S_1(k) e^{iP_1(x,k,t)} dk. \tag{6}$$

Moreover, the spatial derivative of $A(x,t)$ is obtained as follows:

$$\frac{\partial^n A(x,t)}{\partial x^n} = \int_k i^n (k-k_0)^n S_1(k) e^{iP_1(x,k,t)} dk, \quad n = 1, 2, 3, \cdots. \tag{7}$$

Equation (7) shows the spatial derivative of the envelope equals an envelope of the modified spectrum $S_{1,m}^{(n)}(k)$, i.e.,

$$\frac{\partial^n A(x,t)}{\partial x^n} = \int_k S_{1,m}^{(n)}(k) e^{iP_1(x,k,t)} dk, \tag{8}$$

where

$$S_{1,m}^{(n)}(k) = i^n (k-k_0)^n S_1(k). \tag{9}$$

On the other hand, the dispersion relation of $\omega(k)$ can be written as the following Taylor expansion based on the profile of the spectrum defined by Definition 1:

$$\omega(k) = \omega(k_0) + \omega'(k_0)(k-k_0) + \frac{1}{2!}\omega''(k_0)(k-k_0)^2 + \frac{1}{3!}\omega'''(k_0)(k-k_0)^3 + \cdots. \tag{10}$$

Substituting the relations of Equations (7) and (10) into Equation (6) leads to the following:

$$\frac{\partial A(x,t)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n i^{n-1} \frac{\omega^{(n)}(k_0)}{n!} \frac{\partial^n A(x,t)}{\partial x^n}. \tag{11}$$

Equation (11) represents the higher order governing equation that governs the amplitude of nearly monochromatic waves, namely, the equation that the envelope of nearly monochromatic waves satisfies. Neglecting the third and higher order of spatial derivatives in Equation (11), we obtain the linear Schrödinger equation of Theorem 1. *qed*

Theorem 2 *The envelope surface created by two-dimensional nearly monochromatic waves with a propagation direction* (θ_0) *satisfies the following Schrödinger equation.*[6][7]

$$i\left\{\frac{\partial A^f(x,y,t)}{\partial t}+\omega^{'}(k_0)\left(\cos\theta_0\frac{\partial A^f(x,y,t)}{\partial x}+\sin\theta_0\frac{\partial A^f(x,y,t)}{\partial y}\right)\right\}$$
$$+\frac{1}{2!}\omega^{''}(k_0)\left(\frac{\partial^2 A^f(x,y,t)}{\partial x^2}+\frac{\partial^2 A^f(x,y,t)}{\partial y^2}\right)=0 \qquad (12)$$

Remark 3 *Here* $A^f(x,y,t)$ *is the amplitude of two-dimensional nearly monochromatic waves of the function of two dimensional spaces* x, y *and time* t. $A^f(x,y,t)$ *acts as the envelope surface of traveling waves. Equation* (*12*) *has the robustness about the propagation direction; namely, the small variation of the propagation direction makes no change of the original equation* [8].

proof Simple expansion from Theorem 1 [6]. *qed*

Definition 2 *Directional, nearly monochromatic waves* $u(x,y,t)$ *are defined by the waves whose energy is almost concentrated in one propagation direction of waves as follows* [6]*:*

$$u(x,y,t)=\int_\theta G_1(\theta)e^{i\{k_0x\cos\theta+k_0y\sin\theta-\omega(k_0)t\}}d\theta, \qquad (13)$$

where $G_1(\theta)$ *denotes the directional spectrum of waves considered here. The profile of the directional spectrum of waves has a peak at* $\theta=\theta_0$ *and spreads* ϵ_θ *(sufficiently small) around* θ_0.

Remark 4 Directional, nearly monochromatic waves also create an envelope surface.

Theorem 3 *The envelope surface created by directional, nearly monochromatic waves satisfies the following Schrödinger-Nohara equation.* [6][7]

$$\cos\theta_0\frac{\partial A^\theta(x,y)}{\partial x}+\sin\theta_0\frac{\partial A^\theta(x,y)}{\partial y}=\frac{i}{2k_0}\left(\frac{\partial^2 A^\theta(x,y)}{\partial x^2}+\frac{\partial^2 A^\theta(x,y)}{\partial y^2}\right) \qquad (14)$$

Remark 5 *Here* $A^{\theta}(x,y)$ *is the amplitude of directional, nearly monochromatic waves of the function of two dimensional spaces* x *and* y*.* $A^{\theta}(x,y)$ *acts as the time-invariant envelope surface of traveling waves.* θ_0 *denotes an almost concentrated wave propagation direction. The Schrödinger-Nohara equation shows the steady state of Equation (12). Therefore, the envelope surface created by directional, nearly monochromatic waves equals the steady state of the envelope surface created by two-dimensional wavenumber-based nearly monochromatic waves.*

proof Directional, nearly monochromatic waves have the fixed wavenumber k_0 but spread over small propagating directions ϵ_{θ} around the direction θ_0, so most of the wave energy is concentrated in one propagating direction θ_0. We then can assume that two-dimensional plane traveling waves $u(x,y,t)$ have the form as:

$$u(x,y,t) = A^{\theta}(x,y,t)e^{i\{k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t\}}. \tag{15}$$

HereC$A^{\theta}(x,y,t)$ indicates the amplitude of nearly monochromatic waves in terms of the directionality. $A^{\theta}(x,y,t)$ is immediately obtained from Equations (13) and (15) as follows:

$$\begin{aligned} A^{\theta}(x,y,t) &= \int_{\theta} G_1(\theta)e^{iP_1^{\theta}(x,y,\theta)}d\theta \\ &= A^{\theta}(x,y), \end{aligned} \tag{16}$$

$$P_1^{\theta}(x,y,\theta) = k_0 x(\cos\theta - \cos\theta_0) + k_0 y(\sin\theta - \sin\theta_0). \tag{17}$$

We can find that $A^{\theta}(x,y,t)$ becomes time-invariant, i.e., $A^{\theta}(x,y)$ from Equations (16) and (17).

The partial derivatives of $A^{\theta}(x,y)$ with respect to x and y are as follows:

$$\frac{\partial A^{\theta}(x,y)}{\partial x} = -ik_0\cos\theta_0 A^{\theta}(x,y) + ik_0\int_{\theta}\cos\theta G_1(\theta)e^{iP_1^{\theta}(x,y,\theta)}d\theta, \tag{18}$$

$$\frac{\partial A^{\theta}(x,y)}{\partial y} = -ik_0\sin\theta_0 A^{\theta}(x,y) + ik_0\int_{\theta}\sin\theta G_1(\theta)e^{iP_1^{\theta}(x,y,\theta)}d\theta. \tag{19}$$

In general, we obtain the following relations.

$$\int_\theta \cos^n \theta G_1(\theta) e^{iP_1^\theta(x,y,\theta)} d\theta =$$

$$\begin{pmatrix} \cos^n \theta_0 & {}_nC_1 \cos^{n-1}\theta_0(-\frac{i}{k_0}) & {}_nC_2 \cos^{n-2}\theta_0(-\frac{i}{k_0})^2 & \cdots & (-\frac{i}{k_0})^n \end{pmatrix} \tag{20}$$

$$\times \begin{pmatrix} A^\theta(x,y) & \frac{\partial A^\theta(x,y)}{\partial x} & \frac{\partial^2 A^\theta(x,y)}{\partial x^2} & \cdots & \frac{\partial^n A^\theta(x,y)}{\partial x^n} \end{pmatrix}^T$$

$$\int_\theta \sin^n \theta G_1(\theta) e^{iP_1^\theta(x,y,\theta)} d\theta =$$

$$\begin{pmatrix} \sin^n \theta_0 & {}_nC_1 \sin^{n-1}\theta_0(-\frac{i}{k_0}) & {}_nC_2 \sin^{n-2}\theta_0(-\frac{i}{k_0})^2 & \cdots & -(\frac{i}{k_0})^n \end{pmatrix} \tag{22}$$

$$\times \begin{pmatrix} A^\theta(x,y) & \frac{\partial A^\theta(x,y)}{\partial y} & \frac{\partial^2 A^\theta(x,y)}{\partial y^2} & \cdots & \frac{\partial^n A^\theta(x,y)}{\partial y^n} \end{pmatrix}^T$$

where the superscript of T means the transpose of vectors. The Schrödinger-Nohara equation is obtained by $n = 2$ in Equations (20) and (21). *qed*

3. Main Results

Definition 3 *Nearly bi-chromatic waves $u(x,t)$ are defined by the waves whose energy is almost concentrated in two wavenumbers and the approach of these wavenumbers is very close as follows:*

$$u(x,t) = \int_k S_2(k) e^{i\{kx - \omega(k)t\}} dk. \tag{24}$$

Here S_2 denotes the spectrum of nearly bi-chromatic waves. The profile of the spectrum of nearly bi-chromatic waves has two peaks at $k = k_0$ and $k = k_1$ and spreads ϵ_k (sufficiently small) around k_0 and k_1. Moreover, k_0 and k_1 are very closely approached with each other as follows:

$$k_1 = k_0 + \Delta_k \quad (\Delta_k / k_0 \ll 1). \tag{25}$$

Remark 6 Nearly bi-chromatic waves create an envelope and we can see it as *beat* in ocean.

Theorem 4 *The envelope created by nearly bi-chromatic waves satisfies the following modified Schrödinger equation.*

$$i\left\{\frac{\partial A(x,t)}{\partial t}+\left(\omega^{'}(k_0)+\frac{\Delta_k e^{ig(x,t)}}{1+e^{ig(x,t)}}\omega^{''}(k_0)\right)\frac{\partial A(x,t)}{\partial x}\right\}$$
$$+\ \frac{1}{2!}\left(\omega^{''}(k_0)+\frac{\Delta_k e^{ig(x,t)}}{1+e^{ig(x,t)}}\omega^{'''}(k_0)\right)\frac{\partial^2 A(x,t)}{\partial x^2}=0, \qquad (26)$$

where

$$g(x,t)=\Delta_k\left(x-\frac{\omega(k_0)}{2k_0}t\right)=\Delta_k(x-\omega^{'}(k_0)t).^1 \qquad (27)$$

Remark 7 The modified Schrödinger equation becomes the Schrödinger equation of Theorem 1 when $\Delta_k=0$; namely, $\Delta_k=0$ means that nearly bi-chromatic waves change to nearly monochromatic waves.

proof The following equation can be assumed by the definition of nearly bi-chromatic waves.

$$u(x,t)=A(x,t)\left\{e^{i(k_0x-\omega(k_0)t)}+e^{i(k_1x-\omega(k_1)t)}\right\} \qquad (28)$$

This is based on the assumption that most of the energy is concentrated in two wavenumbers k_0 and k_1, which are very closely approached with each other as described Definition 3. $\omega(k_1)$ is also written by

$$\omega(k_1)=\omega(k_0)+\Delta_\omega, \qquad (29)$$

where

$$\Delta_\omega=\omega^{'}(k_0)\Delta_k.^2 \qquad (30)$$

$A(x,t)$ is derived from Equations (22) and (26) using Equations (23) and (27) as follows:

$$A(x,t)=\frac{1}{1+e^{ig(x,t)}}\int_k S_2(k)e^{iP_2(x,k,t)}dk. \qquad (31)$$

where $P_2(x,k,t)=P_1(x,k,t)$.

The time derivative of $A(x,t)$ is

$$\frac{\partial A(x,t)}{\partial t} = \omega^{'}(k_0)\frac{i\Delta_k e^{ig(x,t)}}{(1+e^{ig(x,t)})^2}\int_k S_2(k)e^{iP_2(x,k,t)}dk$$
$$+\ \frac{1}{1+e^{ig(x,t)}}\int_k(-i)(\omega(k)-\omega(k_0))S_2(k)e^{iP_2(x,k,t)}dk. \qquad (32)$$

[1] The dispersion relation of $\omega(k)$ is presented by $\omega(k)=\sqrt{gk\tanh(hk)}$ in water waves [9]. Here g and h denote acceleration due to gravity and uniform water depth, respectively. If $hk\gg 1$ (deep water), then $\omega(k)=\sqrt{gk}$. So, $\omega^{'}(k_0)=\frac{1}{2}\sqrt{\frac{g}{k_0}}=\frac{\omega(k_0)}{2k_0}$.

[2] $\omega(k_1)=\sqrt{gk_1}=\sqrt{g(k_0+\Delta_k)}=\sqrt{gk_0}\ \sqrt{1+\frac{\Delta_k}{k_0}}\cong\omega(k_0)(1+\frac{\Delta_k}{2k_0})$. So, $\Delta_\omega=\frac{\omega(k_0)}{2k_0}\Delta_k=\omega^{'}(k_0)\Delta_k$.

Moreover, the spatial derivative of $A(x,t)$ is obtained by neglecting the second and higher order of Δ_k as follows:

$$\begin{aligned}\frac{\partial^n A(x,t)}{\partial x^n} &\cong \frac{-in\Delta_k e^{ig(x,t)}}{(1+e^{ig(x,t)})^2}\int_k i^{n-1}(k-k_0)^{n-1}S_2(k)e^{iP_2(x,k,t)}dk \\ &+ \frac{1}{1+e^{ig(x,t)}}\int_k i^n(k-k_0)^n S_2(k)e^{iP_2(x,k,t)}dk,\ n=1,2,3,\cdots .\end{aligned} \tag{33}$$

Substituting the relations of Equations (10) and (31) into Equation (30) leads to the following:

$$\begin{aligned}\frac{\partial A(x,t)}{\partial t} &= \sum_{n=1}^{\infty}(-1)^n i^{n-1}\frac{\omega^{(n)}(k_0)}{n!}\frac{\partial^n A(x,t)}{\partial x^n} \\ &+ \frac{\Delta_k e^{ig(x,t)}}{1+e^{ig(x,t)}}\sum_{n=1}^{\infty}(-i)^{n+1}\frac{\omega^{(n+1)}(k_0)}{n!}\frac{\partial^n A(x,t)}{\partial x^n}\end{aligned} \tag{34}$$

Equation (32) represents the higher order governing equation, which governs the amplitude of nearly bi-chromatic waves; namely, the equation that the envelope of nearly bi-chromatic waves satisfies. Neglecting the third and higher order of spatial derivatives in Equation (32), we obtain the modified Schrödinger equation of Theorem 4. *qed*

Remark 8 Equation (32) becomes to be identical with Equation (11) when $\Delta_k = 0$ (namely, when nearly bi-chromatic waves change to nearly monochromatic waves).

Definition 4 *Directional, nearly bi-chromatic waves $u(x,y,t)$ are defined by the waves whose energy is almost concentrated in two wave propagation directions and the approach of these directions is very close as follows:*

$$u(x,y,t) = \int_\theta G_2(\theta)e^{i\{k_0 x\cos\theta + k_0 y\sin\theta - \omega(k_0)t\}}d\theta. \tag{35}$$

Here the directional spectrum G_2 has two peaks at $\theta=\theta_0$ and $\theta=\theta_1$ and spreads ϵ_θ (sufficiently small) around θ_0 and θ_1. Moreover, θ_0 and θ_1 are very closely approached with each other as follows:

$$\theta_1 = \theta_0 + \Delta_\theta \quad (\Delta_\theta/\theta_0 \ll 1). \tag{36}$$

Theorem 5 *The envelope created by directional, nearly bi-chromatic waves satisfies the following modified Schrödinger-Nohara equation.*

$$\begin{pmatrix} 1 & \frac{-\Delta_\theta e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}} \end{pmatrix} \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & -\cos\theta_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A^\theta(x,y)}{\partial x} \\ \frac{\partial A^\theta(x,y)}{\partial y} \end{pmatrix}$$

$$= \frac{i}{2k_0}\left(\frac{\partial^2 A^\theta(x,y)}{\partial x^2} + \frac{\partial^2 A^\theta(x,y)}{\partial y^2}\right), \tag{37}$$

where

$$h(x,y) = k_0 x \sin\theta_0 - k_0 y \cos\theta_0. \tag{38}$$

Remark 9 The modified Schrödinger-Nohara equation becomes to be identical with the Schrödinger-Nohara equation of Theorem 3 when $\Delta_\theta = 0$ (namely, when directional, nearly bi-chromatic waves change to directional, nearly monochromatic waves).

proof The following equation can be assumed by the definition of directional, nearly bi-chromatic waves.

$$u(x,y,t) = A^\theta(x,y,t)\left\{e^{i(k_0 x\cos\theta_0 + k_0 y \sin\theta_0 - \omega(k_0)t)} + e^{i(k_0 x\cos\theta_1 + k_0 y\sin\theta_1 - \omega(k_0)t)}\right\} \tag{39}$$

HereC$A^\theta(x,y,t)$ indicates the amplitude of nearly bi-chromatic waves in terms of the directionality. This is based on the assumption that most of the energy is concentrated in two wave propagation directions θ_0 and θ_1 as described Definition 4. Then $A^\theta(x,y,t)$ is immediately obtained from Equations (33) and (37) using Equation (34) as follows:

$$\begin{aligned} A^\theta(x,y,t) &= \frac{1}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta G_2(\theta) e^{iP_2^\theta(x,y,\theta)} d\theta \\ &= A^\theta(x,y) \end{aligned} \tag{40}$$

where $P_2^\theta(x,y,\theta) = P_1^\theta(x,y,\theta)$. $A^\theta(x,y,t)$ becomes to be a time-invariant form of $A^\theta(x,y)$ same as Equation (16). Here we obtain the first order partial derivatives of $A^\theta(x,y)$ with respect to x and y as follows:

$$\begin{aligned} \frac{\partial A^\theta(x,y)}{\partial x} &= -\left(ik_0\cos\theta_0 - \frac{i\Delta_\theta k_0 \sin\theta_0 e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}}\right) A^\theta(x,y) \\ &+ \frac{ik_0}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \cos\theta G_2(\theta) e^{iP_2^\theta(x,y,\theta)} d\theta, \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{\partial A^\theta(x,y)}{\partial y} &= -\left(ik_0\sin\theta_0 + \frac{i\Delta_\theta k_0 \cos\theta_0 e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}}\right) A^\theta(x,y) \\ &+ \frac{ik_0}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \sin\theta G_2(\theta) e^{iP_2^\theta(x,y,\theta)} d\theta. \end{aligned} \tag{42}$$

Moreover, the second order derivatives of $A^\theta(x,y)$ are obtained through the neglection of the second and higher order of Δ_θ as follows:

$$\begin{aligned}\frac{\partial^2 A^\theta(x,y)}{\partial x^2} &= -\left(k_0^2\cos^2\theta_0 - \frac{\Delta_\theta k_0^2 \sin 2\theta_0 e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}}\right)A^\theta(x,y)\\ &+ \left(\frac{2k_0^2\cos\theta_0}{1+e^{-i\Delta_\theta h(x,y)}} - \frac{2\Delta_\theta k_0^2\sin\theta_0 e^{-i\Delta_\theta h(x,y)}}{(1+e^{-i\Delta_\theta h(x,y)})^2}\right)\\ &\quad\times\int_\theta \cos\theta G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta \qquad (43)\\ &- \frac{k_0^2}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \cos^2\theta G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 A^\theta(x,y)}{\partial y^2} &= -\left(k_0^2\sin^2\theta_0 + \frac{\Delta_\theta k_0^2 \sin 2\theta_0 e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}}\right)A^\theta(x,y)\\ &+ \left(\frac{2k_0^2\sin\theta_0}{1+e^{-i\Delta_\theta h(x,y)}} + \frac{2\Delta_\theta k_0^2\cos\theta_0 e^{-i\Delta_\theta h(x,y)}}{(1+e^{-i\Delta_\theta h(x,y)})^2}\right)\\ &\quad\times\int_\theta \sin\theta G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta \qquad (44)\\ &- \frac{k_0^2}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \sin^2\theta G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta.\end{aligned}$$

From Equations (39) and (40) the following relations are obtained.

$$\begin{aligned}\cos\theta_0\frac{\partial A^\theta(x,y)}{\partial x} &+ \sin\theta_0\frac{\partial A^\theta(x,y)}{\partial y} =\\ -ik_0A^\theta(x,y) &+ \frac{ik_0}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \cos(\theta-\theta_0)G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta \qquad (45)\end{aligned}$$

$$\begin{aligned}\sin\theta_0\frac{\partial A^\theta(x,y)}{\partial x} &- \cos\theta_0\frac{\partial A^\theta(x,y)}{\partial y} =\\ \frac{i\Delta_\theta k_0 e^{-i\Delta_\theta h(x,y)}}{1+e^{-i\Delta_\theta h(x,y)}}A^\theta(x,y) &- \frac{ik_0}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \sin(\theta-\theta_0)G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta \qquad (46)\end{aligned}$$

We also obtain the following relation from Equations (41) and (42).

$$\begin{aligned}\frac{\partial^2 A^\theta(x,y)}{\partial x^2} &+ \frac{\partial^2 A^\theta(x,y)}{\partial y^2} =\\ -2k_0^2A^\theta(x,y) &+ \frac{2k_0^2}{1+e^{-i\Delta_\theta h(x,y)}}\int_\theta \cos(\theta-\theta_0)G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta \qquad (47)\\ &+ \frac{2\Delta_\theta k_0^2 e^{-i\Delta_\theta h(x,y)}}{(1+e^{-i\Delta_\theta h(x,y)})^2}\int_\theta \sin(\theta-\theta_0)G_2(\theta)e^{iP_2^\theta(x,y,\theta)}d\theta\end{aligned}$$

The modified Schrödinger-Nohara equation is obtained from Equations (43), (44) and (45). *qed*

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