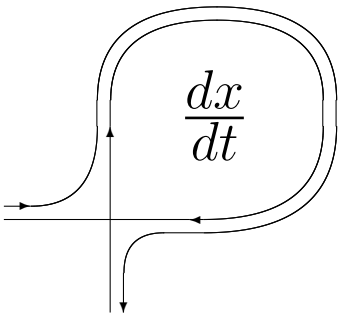

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Ordinary differential equations

AZBELEV'S W-TRANSFORM AND ITS APPLICATIONS IN MATHEMATICAL MODELING

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Abstract.

We propose a general method of reducing differential equations with a distributed delay function to finite or infinite systems of ordinary differential equations. The idea of the method goes back to the so-called "W-method" developed by N. V. Azbelev and his students (see e. g. [1] and references therein). As particular cases we obtain the famous "linear chain trick", on one hand, and the Krasovskii - Hale form of delay equations, on the other.

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In the paper, we explain also how this method can be used in mathematical modelling. We consider two examples: stability of delay differential equations and singular perturbations of delay differential equations.

1 Introduction

The W-method was primarily developed by N. V. Azbelev and his students to study boundary value problems and asymptotic properties of functional differential equations (see e. g. [1], [2] and references therein). The underlying idea is to represent a property of the solutions of the studied functional differential equation in terms of certain properties of an integral operator coming from another differential equation, which already has the required property. In case of asymptotic properties this idea is also combined with admissibility of the pairs of the appropriate functional spaces. The method is well-known to be efficient in both cases. The resulting integral transforms are sometimes called "the W - transforms".

In the two next sections we will give a short presentation of the method in its two (of the three known) forms.

2 Stability, admissibility and the W-transform

The first version of the W-transform is mainly used in the stability theory.

Suppose we are studying a linear functional (delay) equation which, in general, can be represented in the following form:

$$\begin{aligned} \dot{x}(t) &= (Vx)(t), \quad t > 0 \\ x(s) &= \varphi(s), \qquad s \leq 0, \end{aligned} \tag{1}$$

where V is a continuous Volterra operator.

If we want to apply Lyapunov's method we usually need to rewrite Eq. (1) in the Hale-Krasovskii form:

$$\dot{x}(t) = \tilde{V}x_t, \quad t > 0 \tag{2}$$

where $\tilde{V}$ is a certain functional on the space of the initial functions φ.

To apply the W-method we need another representation of Eq. (1) based on a certain splitting of the operator V, which moves the initial function φ to the right hand side of Eq. (1).

Setting

$$(V^+x)(t) := (Vx_+)(t), \quad t > 0$$
$$(V^-x)(s) := (Vx_-)(s), \quad s < 0,$$

where

$$x^+ := x\mathbf{1}_{[0,+\infty)}, \quad x^- := x\mathbf{1}_{(-\infty,0)} = \varphi\mathbf{1}_{(-\infty,0)},$$

we obtain the following form of Eq. (1):

$$\begin{aligned} \dot{x} &= V^+x + V^-\varphi, \quad t > 0 \\ x(0) &= x_0 := \varphi(0). \end{aligned} \tag{3}$$

Example 2.1

$$\dot{x} = \int_{-\infty}^{t} x(s)d_sR(t,s) = \int_{0}^{t} x(s)d_sR(t,s) + \int_{-\infty}^{0} \varphi(s)d_sR(t,s).$$

□

Denoting $f := V^-\varphi$ we obtain

$$\begin{aligned} \dot{x} &= V^+x + f, \qquad t > 0 \\ x(0) &= x_0 := \varphi(0). \end{aligned} \tag{4}$$

By this, the initial function φ becomes a part of the right hand side of Eq. (4).

Definition 1 . *We say that the pair of normed functional spaces (D, B) is admissible for Eq. $\dot{x} = V^+x + f$ if for any $f \in B$ and any $x_0 \in \mathbf{R}^n$ the solution x of (4) belongs to the space D and*

$$||x_f(\cdot, x_0)||_D \le \bar{c}(||x_0||_{\mathbf{R}^n} + ||f||_B).$$

Asymptotic properties of Eq. (1) can be now replaced by admissibility properties for Eq. (4), which can be studied by the W-transform. To do it, we introduce another, simpler equation that already has the required admissibility property:

$$\begin{aligned} \dot{x} &= Q^+x + g, \quad t > 0 \\ x(0) &= x_0. \end{aligned} \tag{5}$$

The representation formula for solutions of Eq. (5) gives

$$x(t) = U(t)x_0 + (Wg)(t), \tag{6}$$

where

$$(Wg)(t) = \int_0^t C(t,s)g(s)ds.$$

This operator defines essentially the W-transform which is to be applied to (4). This can be done in two manners.

The left W-transform.

$$\begin{aligned} & \dot{x} = Q^+x + V^+x - Q^+x + f \\ \Rightarrow \quad & x = Ux_0 + W(V^+ - Q^+)x + Wf, \\ \text{or} \quad & (I - \Theta)x = Ux_0 + Wf, \end{aligned}$$

where $\Theta := W(V^+ - Q^+)$.

The right W-transform.

$$\begin{aligned} & Q^+x + g = \dot{x} = V^+x + f \\ \Rightarrow \quad & g = (V^+ - Q^+)(Ux_0 + Wg) + f, \\ \text{or} \quad & (I - \tilde{\Theta})g = f + (V^+ - Q^+)Ux_0, \end{aligned}$$

where $\tilde{\Theta} := (V^+ - Q^+)W$.

Proposition 1 . *Assume that $I - \Theta$ is invertible in D (or $I - \tilde{\Theta}$ is invertible in B). Then (D, B)-admissibility for Eq. (5) implies (D, B)-admissibility for Eq. (4).*

Proof. Consider the simpler case of the left W-transform. The result is then readily seen from the formula $(I - \Theta)x = Ux_0 + Wf$. Indeed, if the pair (D, B) is admissible for the reference equation (5), then, according to the representation (6), $Ux_0 + Wf \in D$. Invertibility of the operator $(I - \Theta)$ shows that any solution x of Eq. (4) belongs to the space D, too.

A similar argument can be used in the case of the right W-transform. □

3 Reference equations in a different state space

In [6] a different approach to applying the W-transform to functional differential equations was suggested. The purpose was to justify a general method of reducing differential equations with a distributed delay function to finite or infinite systems of *ordinary differential equations*. This mens that the W-transformed equation will not be an integral equation any more, but a system

of ordinary differential equations. The price we should pay for this is that the reference equations should be defined in a different state space.

Let us shortly describe this approach.

We are assumed given a nonlinear n-dimensional delay differential equation

$$x'(t) = f(t, (\Re x)(t)), \qquad t \in I_+, \tag{7}$$

$$x(\tau) = \varphi(\tau), \qquad \tau \in I_-, \tag{8}$$

with a delay operator $\Re$ to be specified below. Here I_+ is either $(0, T]$, or $(0, \infty)$, while I_- is either $[-H, 0]$, or $(-\infty, 0]$, where T and H are both positive numbers. The interval I_-, where the "initial function" φ is defined, is determined by the delay operator $\Re$: if e.g. Equation (7) includes only expressions like $x(t - h_i),\ i = 1, ..., m$, then $H = \max h_i$.

The state variable x is an n-vector, that is $x(t) \in \mathbf{R}^n$ for any t. Any solution $x(t)$ of (7) is supposed to be absolutely continuous for positive t.

We will also write

$$I := I_- \cup I_+. \tag{9}$$

In order to describe assumptions on the delay operator $\Re$ we, as in the previous section, have to split it into two parts:

$$\Re = \Re_- + \Re_+, \tag{10}$$

where

$$(\Re_+ x)(t) = \int_0^t d_s\, R(t, s)x(s), \qquad t \in I_+, \tag{11}$$

and

$$(\Re_- \varphi)(t) = \int_{I_-} d_s\, R(t, s)\varphi(s), \qquad t \in I_+. \tag{12}$$

The measurable $\nu \times n$-matrix $R(t, s),\ t \in I_+, s \in I$ is assumed to have two properties.

R1. The function

$$\varphi_R(t) := \int_{I_-} d_s\, R(t, s)\varphi(s) \tag{13}$$

is locally Lebesgue-integrable on the interval I_+.

R2. For any finite $s, A \in I_+$ the entries $r_{ij}(\cdot, s)\,(1 \le i \le \nu, 1 \le j \le n)$ of the matrix $R(\cdot, s)$ as well as the functions $\mathrm{Var}_{s\in[0,A]} r_{ij}(\cdot, s)$ are Lebesgue-integrable on I_+.

If Property **R2** is fulfilled, then the operator $\Re_+$ will be bounded as a linear operator from the space $D^1([0,A];\mathbf{R}^n)$ (of all absolutely continuous functions from $[0,A]$ to $\mathbf{R}^n$) to the Lebesgue space $L^1([0,A];\mathbf{R}^\nu)$ (see [1] for the details). We will denote the norm of this operator by R_A. It can easily be checked (see again [1]) that

$$R_T \le R_A \ (T \le A) \ \text{ and } \ \lim_{A\to+0} R_A = 0. \tag{14}$$

The function $f(\cdot,\cdot) : [0,T] \times \mathbf{R}^\nu \to \mathbf{R}^n$ will in the sequel be assumed to have three properties.

f1. The function $f(\cdot,u)$ is measurable for any $u \in \mathbf{R}^\nu$.

f2. The function $f(\cdot,0)$ is bounded on I_+, i. e. $|f(t,0)| \le C$, for some constant C.

f3. f is Lipschitz: there exists a constant L such that

$$|f(t,u) - f(t,v)| \le L|u-v| \tag{15}$$

for all $t \in I_+$.

Note that Conditions **f2 - f3** imply that $|f(t,u)| \le L|u| + C$ for any $t \in I_+$ and $u \in \mathbf{R}^\nu$.

In what follows we assume that the initial functions φ are taken from a fixed Banach space $\mathcal{S}$, called in the sequel "the space of initial functions".

In contrast to the previous section, the reference equation

$$(\mathcal{L}z)(t) = y(t), \qquad t \in I_+. \tag{16}$$

will now be defined in a given **separable Banach space** B, so that $z(t) \in B$ for any $t \in I_+$. We assume $\mathcal{L}$ to be a linear operator generating **an ordinary differential equation** in the space B. This situation admits a rigorous description if one uses the notion of a local operator. The definition below is a slight modification of that used in [9] (see also [10]).

Definition 2 *An operator $\mathcal{L} : \mathcal{D} \to L^1_{loc}(I_+, B)$, defined on a subset $\mathcal{D}$ of the space $C_{loc}(I_+, B)$ is called* **local** *if for any $x, y \in \mathcal{D}$, $t_0 \in I_+$, $\varepsilon > 0$ satisfying $x(t) = y(t)$ for $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap I_+$, one has $(\mathcal{L}x)(t) = (\mathcal{L}y)(t)$ for almost all $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap I_+$.*

The definition gives a precise meaning to the intuitive expression "independence from the past" and covers e. g. differentiation and multiplications by

linear operators. The most typical example of a linear equation with a local operator is therefore an ordinary differential equation of the form

$$z'(t) = A(t)z(t) + y(t), \qquad t \in I_+, \tag{17}$$

where $A(t) : B \to B,\ t \in I_+$ is a family of bounded linear operators. In this section we will mostly deal with this equation, but the usage of more general reference equations can also be of interest (see e. g. Section 6 for further details).

The reference equation 16 is supplied with the usual initial condition

$$z(0) = z_0 \in B. \tag{18}$$

In the sequel we shall assume that for any admissible function y and any $z_0 \in B$ the reference equation (16) has the unique solution

$$z(t) := W^t(y, z_0) \tag{19}$$

We do not describe here a functional space of all possible y, nor specify we a nature of dependence of z on y and z_0, as it is not important for the general scheme.

Remark 3.1 *The nature of the W-transform just defined is somewhat different from what we had in Section 2. Indeed, the space state of the reference equation (16) does not coincide with that of Equation (7).*

Now we explain how the reference equation (16) can be related to the delay equation (7). The idea is to use (16) for a representation of the delay operator $\Re$.

Given an operator $\Re$ in Equation (7) and an initial condition (8), we assume that there exist

W1. A reference equation (16), supplied with an initial condition (18), which generates a bounded operator $W^t(y, z_0)$ for any $t \in I_+$;

W2. A family of linear vector-functionals $l^t : B \to \mathbf{R}^\nu$, $t \in I_+$;

W3. Linear bounded maps $p : \mathbf{R}^n \to B$ and $q : \mathcal{S} \to B$;

such that

$$(\Re x)(t) = l^t W^t(px, q\varphi), \qquad t \in I_+, \tag{20}$$

for any $x \in C(I_+, \mathbf{R}^n)$ and any φ satisfying **R1**.

We claim that if the representation (20) holds true, one can apply the W-method to reduce the delay equation (7) to a system of ordinary differential equations (in a Banach space).

Theorem 1 *Under assumptions* **W1 - W3** *Equation (7), supplied with the initial condition (8), is equivalent to the following system of ordinary differential equations in the space* $B \times \mathbf{R}^n$*:*

$$\begin{aligned} (\mathcal{L}z)(t) &= px(t) \\ & \qquad\qquad (t \in I_+) \\ x'(t) &= f(t, l^t z(t)) \end{aligned} \tag{21}$$

with the initial conditions

$$\begin{aligned} z(0) &= q\varphi \\ \\ x(0) &= \varphi(0). \end{aligned} \tag{22}$$

System (21) - (22) is related to Equation (7) via

$$z(t) = W^t(px, z_0), \qquad z_0 = q\varphi.$$

Proof. If $x(t)$ satisfies (21) for $t \in I_+$ and equals $\varphi(t)$ for $t \in I_-$, then by (20) and (19) we have

$$x'(t) = f(t, l^t z(t)) = f(t, l^t W^t(px, z_0)) = f(t, (\Re x)(t)).$$

If $x(t)$ satisfies (7) and (8), then, putting $z(t) = W^t(px, q\varphi)$ we observe that $z(t)$ solves the reference equation (16) with $y = px$ and the initial condition $z(0) = q\varphi$, while $x(t)$ solves the second equation of System (21) with the initial condition $x(0) = \varphi(0)$. □

Below we consider some examples illustrating this general framework.

4 An example: the linear chain trick

This is a well-known algorithm (see e.g. [5]) to reduce specific delay differential equations of the form (7) to finite dimensional systems of ordinary differential equations. The idea of the method is to represent the delay operator $\Re$ as a finite linear combination of "elementary" delay operators which admit a simple description. We present here a slightly modified version of the "trick" which is adjusted to our objectives.

In this section we demonstrate how the linear chain trick fits into the general scheme of the W-method described in the previous section.

Assume that the kernel $R(t,s)$ of the integral operator $\Re$ (see (11)) can be represented as follows:

$$d_s\, R(t,s) = G(t-s)ds, \tag{23}$$

where

$$G(u) = \sum_{i=1}^{m} c_i G_\alpha^i(u) \tag{24}$$

and

$$G_\alpha^i(u) = \frac{\alpha^i u^{i-1}}{(i-1)!} e^{-\alpha u} \tag{25}$$

The coefficients c_i are supposed to be $\nu \times n$-matrices and $\alpha > 0$.

In practice these assumptions mean that we exclude delay terms like $x(t-h)$ as the corresponding kernel in the integral representation (11) is discontinuous. The linear chain trick is only applicable to some classes of smooth delay functions and differential equations with distributed delays.

We also assume given a set I_- being either $(-\infty, 0]$, or $[-H, 0]$, where H is a positive number. The case of a finite H describes bounded time lags, otherwise one obtains equations with an unbounded time lag.

Clearly, the basis functions G_α^i have the following properties:

$$\begin{aligned} &1.\ G_\alpha^i(\infty) = 0, \\ &2.\ G_\alpha^i(0) = 0, \quad i \geq 2. \\ &3.\ G_\alpha^1(0) = \alpha \end{aligned} \tag{26}$$

It is also straightforward that

$$\begin{aligned} &\tfrac{d}{du} G_\alpha^i(u) = \alpha G_\alpha^{i-1}(u) - \alpha G_\alpha^i(u) \quad (i \geq 2) \\ &\tfrac{d}{du} G_\alpha^i(u) = -\alpha G_\alpha^i(u) \qquad\qquad (i = 1). \end{aligned} \tag{27}$$

We proceed with defining the reference equation for the linear chain trick. It will be an equation in the finite-dimensional Banach space $B = \mathbf{R}^{mn}$ given by

$$z'(t) = Az(t) + y(t), \tag{28}$$

where

$$A = \begin{pmatrix} -\alpha E_n & 0 & 0 & \dots & 0 \\ \alpha E_n & -\alpha E_n & 0 & \dots & 0 \\ 0 & \alpha E_n & -\alpha E_n & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha E_n & -\alpha E_n \end{pmatrix} \tag{29}$$

is a block matrix with E_n being the $n \times n$-identity matrix.

The linear vector-functional $l : B = \mathbf{R}^{mn} \to \mathbf{R}^{\nu}$ will be independent of t:

$$l = (c_1, c_2, \dots, c_m), \tag{30}$$

where c_i are the $\nu \times n$- matrices in the representation (24) of the kernel of the delay operator, so that $lz = \sum_{i=1}^{m} c_i z_i (z_i \in \mathbf{R}^n)$.

The linear operators $p : \mathbf{R}^n \to B$ and $q : \mathcal{S} \to B$ will be defined as

$$px = \begin{pmatrix} \alpha x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{31}$$

and

$$q\varphi = \begin{pmatrix} (q\varphi)_1 \\ \dots \\ (q\varphi)_m \end{pmatrix}, \tag{32}$$

respectively, where

$$z_i(0) = (q\varphi)_i := (-1)^{i-1} \frac{\alpha^i}{(i-1)!} \int_{-\infty}^{0} \tau^{i-1} e^{\alpha\tau} \varphi(\tau) d\tau, \quad i = 1, ..., m. \tag{33}$$

These expressions also show that we can for instance choose the space $\mathcal{S}$ of initial functions to consist of all bounded measurable functions φ defined on the interval $(\infty, 0]$ (if $I_- = [-H, 0]$ we may assume that $\varphi(\tau) = 0$ for $\tau < -H$).

As the matrix A in (28) is quasi-diagonal, we can explicitly find (see e.g. [4, Ch. 5.3]) the fundamental solution $Z(t)$ of the homogeneous equation cor-

responding to the reference equation (28):

$$Z(t) = e^{At} = e^{-\alpha t} \begin{pmatrix} E_n & 0 & 0 & \dots & 0 \\ \alpha t E_n & E_n & 0 & \dots & 0 \\ \frac{(\alpha t)^2}{2!} E_n & \alpha t E_n & E_n & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{(\alpha t)^{m-1}}{(m-1)!} E_n & \frac{(\alpha t)^{m-2}}{(m-2)!} E_n & \dots & \alpha t E_n & E_n \end{pmatrix}, \tag{34}$$

so that the general solution of the reference equation (28) will be

$$z(t) = W^t(y, z_0) = W_0^t(y) + Z(t) z_0, \tag{35}$$

where

$$W_0^t(y) = \int_0^t Z(t-s) y(s) ds = \int_0^t e^{A(t-s)} y(s) ds.$$

To set up the W-transform of the solution $x(t)$ we notice that according to (33) and (34) we can write

$$z_0 = q\varphi = \int_{-\infty}^0 e^{A(-\tau)} (p\varphi)(\tau) d\tau,$$

as p drops all but the first coordinate of the block vector $z = (z_1, ..., z_m)^T$ (see the definition 31). As $x(\tau) = \varphi(\tau)$ for $\tau < 0$, we obtain

$$\begin{aligned} z(t) &= W^t(px, q\varphi) = \textstyle\int_0^t Z(t-s) px(s) ds + Z(t) q\varphi \\ &= \textstyle\int_0^t e^{A(t-s)} px(s) ds + \int_{-\infty}^0 e^{A(t-\tau)} (p\varphi)(\tau) d\tau \\ &= \textstyle\int_{-\infty}^t e^{A(t-s)} px(s) ds. \end{aligned} \tag{36}$$

>From (23)-(25) and (34) it is now clear that

$$\begin{aligned} (\Re x)(t) &= \textstyle\int_{-\infty}^t d_s R(t,s) x(s) \\ &= \textstyle\int_{-\infty}^t \left(\sum_{p=1}^m c_p G_\alpha^p (t-s) \right) x(s) ds \\ &= \textstyle\int_{-\infty}^t (c_1, c_2, \dots, c_m) \left(e^{A(t-s)} px(s) \right) ds \\ &= l \left(\textstyle\int_{-\infty}^t e^{A(t-s)} px(s) ds \right) \\ &= l W^t(px, q\varphi) \end{aligned}$$

due to Definition (30) of the vector-functional l.

These calculations explain why the delay operator $\Re$ admits the required representation (20), which implies that Equation (7) can be reduced to the system of ordinary differential equation (21). At the same time the calculations show how the initial condition (8) can be rewritten as the initial conditions (22).

We have therefore shown that the classical linear chain trick is a particular case of the general W-transform described in the previous subsection. In what follows we will consider more examples of how the W-method can be used.

5 Another example: an alternative approach to stability

In this section we explain how the W-transform, described in Section 3, can be used in the stability theory. Not intending to analyze here all possible notions of stability, we just outline the algorithm and give its justification in two particular (though important) cases. We notice also that the method was used in the paper [3] to investigate stability of nonlinear integro-differential equations.

In the rest of the section we assume that $I_+ = (0, \infty)$ and that the reference equation (16) has the following representation of its solutions:

$$z(t) = \int_0^t Z(t,s)y(s)ds + Z(t,0)z_0, \tag{37}$$

where the kernel $Z(t,s)$ is bounded in B for any $t \geq s \geq 0$.

Theorem 2 *Assume that the operator $Z(t,s)$ in (37) admits the following exponential estimate*

$$\|Z(t,s)\| \leq Me^{-\gamma(t-s)}, \quad \gamma > 0, \quad t \geq s \geq 0. \tag{38}$$

The global uniform asymptotical (resp., global exponential) stability of solutions of Equation (7) is then equivalent to the global uniform asymptotical (resp., global exponential) stability of solutions of System (21).

Proof. Let us fix an arbitrary positive number K and assume that $\|\varphi\|_{\mathcal{S}} \leq K$.

Consider first the case of exponential stability. Clearly, we are only to check that if a solution $x(t)$ of Equation (7) satisfies

$$\|x(t)\| \leq Ce^{-ct}$$

for any φ such that $\|\varphi\|_{\mathcal{S}} \le K$, then its W-transform $z(t) = W^t(px, q\varphi)$ satisfies:

$$\|z(t)\| \le De^{-dt}, \tag{39}$$

where the constants d and D are independent of φ.

Using (38) we have

$$\begin{aligned} \|z(t)\| &\le M\|p\| \int_0^t e^{-\gamma(t-s)} e^{-cs} ds + KM\|q\| e^{-\gamma t} \\ &= \tfrac{1}{\gamma - c}(e^{-ct} - e^{-\gamma t}) + KM\|q\| e^{-\gamma t} \\ &\le De^{-dt}, \end{aligned}$$

where $d = \min\{c, \gamma\}$ and $D = \max\{KM\|q\|, |\gamma - c|^{-1}\}$.

Now, let us consider the case of the uniform asymptotic stability. Again it is sufficient to show that if $\overline{x}(t) := \sup_{\|\varphi\| \le K} |x(t)|$ goes to 0 as $t \to \infty$, then this also holds for $z(t) = W^t(px, q\varphi)$. As

$$\|z(t)\| \le M\|p\| \int_0^t e^{-\gamma(t-s)} \overline{x}(t) ds + KM\|q\| e^{-\gamma t},$$

the result will be implied from the following

Lemma 5.1 *Given $\gamma > 0$ and a scalar integrable function $v(t)$ such that $v(t) \to 0$ as $t \to \infty$, one has*

$$u(t) := \int_0^t e^{-\gamma(t-s)} v(s) ds \to 0 \tag{40}$$

as $t \to \infty$.

To prove the lemma we notice first that the function $u(t)$ satisfies the differential equation

$$u'(t) + \gamma u(t) = v(t) \tag{41}$$

and the initial condition $u(0) = 0$. The definition of the function $u(t)$ given in (40) shows that $u(t)$ is bounded on $[0, \infty)$ as $v(t)$ is bounded there. This implies, in particular, that there exists a sequence $t_k \to \infty$ such that $u'(t_k) \to 0$, $k \to \infty$. At the same time $v(t_k) \to 0$, and we obtain that $u(t_k) \to 0$, too. Now we use another integral representation of the solution $u(t)$:

$$u(t) = u(\sigma) + \int_\sigma^t e^{-\gamma(t-s)} v(s) ds, \quad t \ge \sigma.$$

>From this we get the following estimate:

$$|u(t) - u(\sigma)| \le \gamma^{-1} \sup_{t \ge \sigma} |v(t)|. \tag{42}$$

On the other hand, since $v(t) \to 0$ as $t \to \infty$ and $u'(t_k) \to 0$ as $k \to \infty$, then, given a positive ε, there exists a natural number N for which

$$|u(t_N)| < \frac{\varepsilon}{2} \quad \text{and} \quad |v(t)| < \frac{\varepsilon\gamma}{2} \quad \text{for} \quad t \ge t_N.$$

Setting $\sigma = t_N$ in (42) we obtain that $|u(t)| < \varepsilon$ for $t \ge t_N$. The lemma and hence Theorem 2 are proved. □

6 The Hale-Krasovskii form of delay equations

By the Hale-Krasovskii form (we do not insist on this terminology) of the delay equation (7) we mean the following representation:

$$x'(t) = \tilde{f}(t, x_t), \tag{43}$$

where $x_t(\tau) = x(t+\tau)$ and $\tilde{f}(t,\psi) = f\left(t, \int_{I_-} d_\tau R(t, t+\tau)\psi(\tau)\right)$. The initial condition for this equation looks as usual, i.e. as in (8).

This form itself is **not** an ordinary differential equation, but it is well-known that the solutions of (43) have a number of properties that are typical for solutions of ordinary differential equations. The aim of this section is to show that the representation (43) **can** be rewritten as an ordinary differential equation in the space $\mathcal{S}$ of initial functions. This result can for instance be used to explain rigorously why the solutions of (43) have the same properties as solutions of ordinary differential equations. Also in this section we are again going to apply a properly chosen W-transform. Let us only stress that it is not the aim of this paper to treat the Hale-Krasovskii representation in its general form. Nor will we treat general spaces of initial functions. Our intension is much more modest: we just want to show how one can derive the Hale-Krasovskii form from the delay equation (7) by making use of the W-method, what kind of ordinary differential equation one will get, and how one should define an appropriate reference equation.

We start with the Banach space B. Informally speaking, this space will consist of bounded functions φ, defined on I_-, that are continuous for all

$t \in I_-$ except probably $t = 0$, where φ is supposed to have a left-hand limit $\varphi(-0)$. Formally we can write

$$B = C(I_-, \mathbf{R}^n) \times \mathbf{R}^n, \tag{44}$$

meaning that the values at $t = 0$ are taken from the second factor. The first factor is endowed with the uniform norm.

Such a B is quite a natural choice as in the Hale-Krasovskii paradigm it also serves as the space $\mathcal{S}$ of initial functions, so that we, in general, do not have continuity at $t = 0$ and have therefore to split the initial condition as it is described in Remark 3.1. Of course, continuous initial functions are included in this representation as a special case.

The operator $p : \mathbf{R}^n \to B$ just drops the first component (making it zero) in (44) and by this isometrically maps the space $\mathbf{R}^n$ onto the second factor in this representation. The operator

$$q : \mathcal{S} (= B) \to B$$

is simply the identity.

The family l^t of linear vector functionals $l^t : B \to \mathbf{R}^\nu$ will be defined as follows:

$$l^t\psi = \int_{I_-} d_\tau R(t, t+\tau)\psi(\tau) = \int_{I_-} d_\tau R(t, t+\tau)\tilde{\psi}(\tau) + R(t,t)\psi_0, \tag{45}$$

where $\tilde{\psi} \in C(I_-, \mathbf{R}^n)$ and $\psi_0 \in \mathbf{R}^n$. This is a crucial step in the Hale-Krasovskii approach. The kernel $R(t, s)$ is assumed here to make this functional bounded (see for instance Properties **R1-R2** from Section 1).

In what follows we will also use τ as the variable of the functions belonging to the space B as in the Hale-Krasovskii approach it is always assumed that $\mathcal{S} = B$. Thus, functions $z : I_+ \to B$ can be regarded as functions of two variables t and τ, as any such function $z(t, \tau)$ generates a function $t \mapsto z(t, \cdot)$ mapping I_+ into B.

Now, the reference equation (16) will be defined via a unbounded, linear and local operator $\mathcal{L}$ acting on functions $z : I_+ \to B$, treated as functions of two variables $z(t, \tau)$, in the following way:

$$(\mathcal{L}z)(t,\tau) = \begin{cases} \frac{\partial z(t,\tau)}{\partial t} - \frac{\partial z(t,\tau)}{\partial \tau}, & \text{if } t \in I_+, \tau \in I_-, \tau < 0, \\ \\ z(t,0), & \text{if } t \in I_+, \tau = 0. \end{cases} \tag{46}$$

It is important to stress once again that according to the definition of the underlying space B the values $z(t,0)$ for all t are taken from the second factor in the representation 44.

A shorter way to define the operator $\mathcal{L}$ reads as follows

$$\mathcal{L}z := \chi_{\{\tau<0\}}\left(\frac{\partial z}{\partial t} - \frac{\partial z}{\partial \tau}\right) + z\chi_{\{\tau=0\}}, \tag{47}$$

where $\chi_A(s) = 1$ for $s \in A$ and $\chi_A(s) = 0$ otherwise. s:

It is straightforward that $\mathcal{L}$ is a local operator in the sense of Definition 2.

A linear subspace where $\mathcal{L}$ is formally defined consists of z for which $\mathcal{L}z$ belongs to the space $L^1_{loc}(I_+, B)$. As we will see, this unbounded linear operator generates a bounded W-transform which can be described explicitly.

Summing up and using the definition of the operator p gives the following form of the reference equation:

$$\begin{aligned} &\tfrac{\partial z(t,\tau)}{\partial t} - \tfrac{\partial z(t,\tau)}{\partial \tau} = y(t,\tau)\chi, \ \text{ if } t \in I_+, \tau \in I_-, \ \tau < 0, \\ &z(t,0) = y(t,0), \qquad\qquad \text{if } t \in I_+, \tau = 0. \end{aligned} \tag{48}$$

It is well-known that the general solution to the partial differential equation $\frac{\partial z(t,\tau)}{\partial t} = \frac{\partial z(t,\tau)}{\partial \tau}$ is given by $z(t,\tau) = \Phi(t+\tau)$ for some Φ. Of course, one has to interpret this solution as generalized if Φ is not smooth.

To set up the associated W-transform we put $y = px$ which gives

$$\begin{aligned} &\tfrac{\partial z(t,\tau)}{\partial t} - \tfrac{\partial z(t,\tau)}{\partial \tau} = 0, \ \text{ if } t \in I_+, \tau \in I_-, \ \tau < 0, \\ &z(t,0) = x(t), \qquad \text{if } t \in I_+, \tau = 0. \end{aligned} \tag{49}$$

Initial conditions to (49) can be obtained from the definition of the operator q in this section:

$$\begin{aligned} &z(0,\tau) = \varphi(\tau), \qquad\qquad \tau \in I_-, \ \tau < 0, \\ &z(0,0) = x_0\,(= \varphi(0)), \qquad \tau = 0. \end{aligned} \tag{50}$$

In order to find Φ in the general representation $z(t,\tau) = \Phi(t+\tau)$ we use the second equation in (49) and the initial conditions (50). This results in two equalities: $\Phi(t) = x(t)$ for $t \in I_+$ and $\Phi(\tau) = \varphi(\tau)$ for $\tau \in I_-$.

Thus, we obtain the W-transform of the solution $x(t)$ of Equation (7):

$$z(t,\tau) := W^t(px, q\varphi)(\tau) = x(t+\tau), \tag{51}$$

where $z(t,\tau)$ and $x(t+\tau)$ should be understood as functions of the variable t with the values in the functional space B of functions of the variable τ, i. e. as the functions $z(t,\cdot)$ and $x(t+\cdot)$, respectively. One should also remember that in (51) one has $x(\tau) = \varphi(\tau)$ for $\tau \le 0$.

The W-transformed delay equation (7) becomes then

$$\begin{aligned}
&\tfrac{\partial z(t,\tau)}{\partial t} - \tfrac{\partial z(t,\tau)}{\partial \tau} = 0, && \text{if } t \in I_+, \tau \in I_-,\ \tau < 0,\\
&z(t,0) = x(t), && \text{if } t \in I_+, \tau = 0,\\
&x'(t) = f\left(t, \textstyle\int_{I_-} d_\tau R(t,t+\tau) x(t+\tau)\right)
\end{aligned} \tag{52}$$

(see also the general description of the W-transformed system given in (21)). The last equation in this system (without delay) is the Hale-Krasovskii form as it is defined in (43).

Also the initial conditions to (52) can be obtained from the general formulas (22). In particular, the component $z(t)$ satisfies (50), which in this case immediately implies that $x(\tau) = \varphi(\tau)$ if $\tau < 0$ and $x(0) = x_0$.

7 Singular perturbed delay differential equations in biological applications

Singular perturbed systems of delay differential equations arise in many applications. In the paper we demonstrate how Azbelev's transform can be applied to a system of delay equations coming from gene regulatory networks.

Assume we are given a system of delay differential equations of the following form

$$\dot{x} = f(x, Z) = F(Z) + G(Z)x. \tag{53}$$

Following [8] we assume F and G to be the vectors of multilinear polynomials (i.e. linear in each variable), and Z to be a vector of the Hill-type switching functions $S_i(\Re(x_i), \theta_i, q_i)$ with values in $[0,1]$. Here $\theta_i > 0$ is a threshold of $\Re(x_i)$, $\Re$ is a time delay operator and q_i is a parameter characterizing the

steepness of the switching functions (when $q_i \to 0$ then S_i approaches the Heaviside function).

As the real switches occur in short but nonzero time intervals we consider system (53) in the case where $0 < q \ll 1$. Of a particular interest is the case when $q \to 0$.

As an example of (53) one can consider a system

$$\begin{aligned} \dot{x}_1 = \;& S(\Re_1(x_1), \theta_1, q) + S(\Re_2(x_2), \theta_2, q) \\ & -2S(\Re_1(x_1), \theta_1, q)S(\Re_2(x_2), \theta_2, q) - \gamma_1 x_1 \qquad (54) \\ \dot{x}_2 = \;& S(\Re_1(x_1), \theta_1, q)S(\Re_2(x_2), \theta_2, q)) - \gamma_2 x_2. \end{aligned}$$

It is shown in [8] that even in the case of no delays involved ($\Re_i x = x$) it is necessary to use two time-scales to define a solution for the limit equation ($q = 0$), as there occur the so-called "sliding motions" parallel to switching domains ($x_i = \theta_i$ in this example). In the same paper it was shown how this difficulty can be tackled for ordinary differential equations.

Here we propose to use the W-transform to overcome the difficulties that arise in this model. Below we illustrate our approach by looking at a special delay function. More general functions can easily treated, too, but we omit this analysis for the sake of simplicity. Let us only remark that considering distributed delays for the operator $\Re$ is more realistic from the biological point of view.

It is also natural from the biological point of view to assume that the delay operator is regular, i. e.

$$d_s\, R(t, s) = G(t - s)ds, \qquad (55)$$

for some kernel ("the memory function") $G(u) \geq 0$, defined on $[0, +\infty)$. This form of delay preserves smoothness of solutions on the whole time interval, which can break down even in the case of "simple" delays like $x(t - h)$.

Usually, the kernel is assumed to be normalized:

$$\int_0^\infty G(u)du = 1.$$

A simple (though of practical importance) example of such a kernel is given by

$$G(u) = ae^{-au}, \qquad (56)$$

where $a > 0$.

If one uses two time scales: slow and fast, then it is easy to check how the delay operator $(\Re x)(t) = \int_{-\infty}^{t} G(t-s)x(s)ds$ with the kernel (56) behaves under the transition from the fast to the slow time $\tau = qt$ with a small q. Intuitively, it will give a "big" delay. Indeed, $\Re$ becomes $(\Re_h x)(t) = \int_{-\infty}^{\tau} G_q(\tau - \sigma)x(\sigma)d\sigma$, where $G_q(u) = aqe^{-aqu}$. The normalization condition $\int_0^{\infty} G_q(u)du = 1$ holds true for all q, but the memory function $G(u)$ will go to 0 slower, so that the time-delay will be bigger.

Let us explain our approach by considering a 2×2-system generalizing (54):

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, Z_1, Z_2) \\ \dot{x_2} &= f_2(x_1, x_2, Z_1, Z_2). \end{aligned} \tag{57}$$

We study a piece of the trajectory $(x_1(t), x_2(t))$ in the vicinity of the threshold $t = \theta_2$. This means that the variable x_2 can be subject to rapid changes (we say that it is a switching variable), while x_1 is a regular variable, as nothing dramatic can happen to it near the threshold $t = \theta_2$.

We denoted $Z_i = S(\Re_i x_i, \theta_i, q)$, where $S(x_i, \theta_i, q)$ are the Hill functions (see [8]). As $q \to 0$, the Hill function $S(x, \theta, q)$ converges to the Heaviside function with the unit jump which occurs at $t = \theta$.

The delay operators $\Re_i$ are given by

$$\Re_i(x) = \int_{-\infty}^{t} G_i(t-s)x(s)ds = a_i \int_{-\infty}^{t} e^{-a_i(t-s)}x(s)ds,$$

where $a_i > 0$, $i = 1, 2$ are two given constants.

Applying the linear chain trick to (57) one gets the following system

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, Z_1(y_1), Z_2(y_2)) \\ \dot{x_2} &= f_2(x_1, x_2, Z_1(y_1), Z_2(y_2)) \\ \dot{y_1} &= -a_1 y_1 + x_1 \\ \dot{y_2} &= -a_2 y_2 + x_2. \end{aligned} \tag{58}$$

To simplify the notation we put $S_i = S(x_1, \theta_i, q)$, $q > 0$. Then, changing

variables $Z_2 = S_2(x_2)$ yields the system

$$\begin{aligned} \dot{x_1} &= f_1(x_1, S_2^{-1}(Z_2), Z_1(y_1), Z_2) \\ \dot{y_1} &= -a_1 y_1 + x_1 \\ \dot{y_2} &= -a_2 y_2 + S_2^{-1}(Z_2) \\ q\dot{Z}_2 &= \tfrac{Z_2(1-Z_2)}{S_2^{-1}(Z_2)} f_2(x_1, S_2^{-1}(Z_2), Z_1(y_1), Z_2). \end{aligned} \tag{59}$$

Now we let $q \to 0$. This results in two different systems which complete each other. The first system ("the reduced equations") is obtained from (59) by setting $q = 0$. This gives the dynamics of the regular variable $x_(t)$ near the threshold θ_2. According to the classical Tikhonov's theory this passage to the limit is possible if, roughly speaking, there exists an asymptotically stable stationary point of the last equation in (59). This is the case for the system (54).

Another system is derived from (59) by making use of the stretching transformation $t = q\tau$ producing the corresponding boundary layer system:

$$\begin{aligned} x_1{}' &= q f_1(x_1, S_2^{-1}(Z_2), Z_1(y_1), Z_2) \\ y_1{}' &= q(-a_1 y_1 + x_1) \\ y_2{}' &= q(-a_2 y_2 + S_2^{-1}(Z_2)) \\ Z_2{}' &= \tfrac{Z_2(1-Z_2)}{x_2} f_2(x_1, S_2^{-1}(Z_2), Z_1(y_1), Z_2) \end{aligned} \tag{60}$$

where the derivative is taken with respect to the "slow" time τ.

Although this system involves two more equations, as compared to the original system (58), all the additional equations include the small parameter q. This means that setting $q = 0$ gives the boundary layer equation of the same dimension as if the system were without delays. This boundary layer equation is identical with the last equation in (60), where x_2 is replaced by θ_2.

As it is shown in [8] the combination of the boundary layer equation and the reduced equations gives a consistent description of the system's dynamics when $q = 0$. The aim of our presentation in this paragraph was to show that a similar result under similar assumptions is true for delay systems with regular kernels.

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