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Integrodifferential Equations

Controllability Results for general Integrodifferential Evolution Equations in Banach Space

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ABSTRACT. The sufficient conditions for controllability of general class of nonlinear evolution integrodifferential equations in Banach space are established. The results are obtained by using the resolvent operator and Schaefer fixed point theorem.

Keywords: Controllability, Nonlinear integrodifferential evolution equation, Resolvent operator, Schaefer's fixed point theorem.

Mathematics Subject Classification: 34G20, 93B05.

1. Introduction

Pazy [15] has discussed the existence and uniqueness of mild, strong and classical solution of semilinear evolution equations by using semigroup theory. The nonlocal problem for the same equation has been first studied by Byszewski [8]. Then it has been extensively studied by many authors, see for example, Byszewski and Acka [7] and Balachandran and Chandrasekaran [3]. Lin and Liu [13] studied the nonlocal Cauchy problem for semilinear integrodifferential equations by using resolvent operators. Balachandran and Ravikumar [4] studied the existence of mild solutions of nonlinear integrodifferential equations with time varying delays in Banach spaces.

The first step in the study of the problem of controllability is to determine if an objective can be reached by some suitable control function. The problem of controllability happens when a system described by a state $x(t)$ is controlled by a given law such as a differential equation $X' = G(t, x(t), u(t))$. We discuss the possibility of driving the solution for a given system from an initial state to a final state by an adequate choice of the control function u. Roughly speaking, controllability means, that is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. So, the notion of controllability is a great importance in mathematical control theory such as pole assignment, stabilization and optimal control may be solved under the assumption that the system is controllable.

Controllability of nonlinear systems represented by ordinary differential equations in infinite dimensional space has been extensively studied by several authors, [5, 6, 14]. Sakthivel, Choi and Anthoni [16] derived sufficient conditions for controllability of nonlinear neutral evolution integrodifferential systems in a Banach space by using the resolvent operators and Schaefer fixed point theorem. Sakthivel, Choi and Anthoni [17] establish the sufficient conditions for the controllability of nonlinear evolution integrodifferential system by using resolvent operator and Schaefer fixed point theorem. Atmania and Mazouri [1] established the controllability results for some semilinear integrodifferential system with nonlocal condition in a Banach space by using semigroup theory and Schaefer theorem. Kumar and Kumar [11, 12] established a set of sufficient conditions for the controllability of Sobolev type nonlocal impulsive mixed functional integrodifferential evolution systems with finite delay by using semigroup theory and fixed point theorem. Hazi and Bragdi [10] established the controllability results of fractional integrodifferential systems in Banach space by using fractional calculus, semigroup theory and fixed point theorem.

Motivated by the above mentioned works and the work of Sakthivel, Choi and Anthoni [16], Balachandaran and Kumar [4], K. Balachandran, N. Annapoorani and J.K. Kim [2] and Sathiyanathan and Gopal [18], we study the controllability for time varying delay integrodifferential evolution equation in Banach space by using the resolvent operators and Schaefer fixed point theorem.

2. Preliminaries

Consider the nonlinear delay integrodifferential equation with nonlocal condition of the form

$$\frac{d}{dt}\left[x(t)-G\left(t,x(t),\int_0^t g(t,s,x(s))ds\right)\right]=A(t)x(t)+\int_0^t Q(t,s)x(s)ds$$
$$+F\left(t,x(t),\int_0^t f_1(t,s,x(s))ds,\int_0^t f_2(t,s,x(s))ds,\dots,\int_0^t f_n(t,s,x(s))ds\right)+(Bu)(t);\ t\in J \tag{1}$$

$$x(0)+h(x)=x_0 \tag{2}$$

where the state $x(.)$ takes values in a Banach space with the norm $\|.\|$, and the control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Here $A(t)$ and $Q(t,s)$ are closed linear operators on X with dense domain $D(A)$ which is independent of t, B is a bounded linear operator from U into X, $f_i: J\times J\times X\to X$, $h: C(J,X)\to X$, $g: J\times J\times X\to X$, $F: J\times X^{n+1}\to X$ and $G: J\times X\times X\to X$ are given functions. Here $J=[0,T]$.

We shall make the following assumption [9].

(I) $A(t)$ generates a strongly continuous semigroup of evolution operators.

(II) Suppose Y is a Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $Q(t,s)$ are closed operators, it follows that $A(t)$ and $Q(t,s)$ are in the set of bounded operators from Y to X,

$B(Y,X)$, for $0 \le t \le T$ and $0 \le s \le t \le T$, respectively. $A(t)$ and $Q(t,s)$ are continuous on $0 \le t \le T$ and $0 \le s \le t \le T$, respectively, into $B(Y,X)$.

Definition 2.1. A resolvent operator for (1) and (2) is a bounded operator valued function $R(t,s) \in B(X)$, $0 \le s \le t \le T$ the space of bounded linear operators on X, having the following properties:

(i) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, $0 \le s \le T$, $\|R(t,s)\| \le Me^{\beta(t-s)}$ for some constants M and β.

(ii) $R(t,s)Y \subset Y$, $R(t,\ s)$ is strongly continuous in s and t on Y.

(iii) For $y \in Y$, $R(t,s)y$ is continuously differentiable in s and t, and for $0 \le s \le t \le T$,

$$\frac{\partial}{\partial t} R(t,s)y = A(t)R(t,s)y + \int_s^t B(t,r)R(r,s)ydr,$$

and $$\frac{\partial}{\partial s} R(t,s)y = -R(t,s)A(s)y - \int_s^t R(t,r)B(r,s)ydr,$$

with $\frac{\partial}{\partial t} R(t,s)y$ and $\frac{\partial}{\partial s} R(t,s)y$ are strongly continuous on $0 \le s \le t \le T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equation in Banach spaces. It will not, however, be an evolution operator because it will not satisfy an evolution or semigroup property. Because a number of results follow directly from the definition of the resolvent operator.

Definition 2.2. A continuous function $x(t)$ is said to be a mild solution of the nonlocal Cauchy problem (1) and (2), if

$$\begin{aligned} x(t) = {} & R(t,0)\big(x_0 - h(x) - G(0,x(0),0)\big) + G\left(t, x(t), \int_0^t g(t,s,x(s))ds\right) \\ & + \int_0^t R(t,s)\left[(Bu)(s) + F\left(s, x(s), \int_0^s f_1(s,\tau,x(\tau))d\tau, \dots, \int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right]ds \\ & - \int_0^t R(t,s)A(s)G\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau\right)ds \\ & - \int_0^t \left[\int_s^t R(t,\tau)Q(\tau,s)G\left(\tau, x(\tau), \int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds \end{aligned} \tag{3}$$

is satisfied.

Schaefer's Theorem [19]. Let E be a normed linear space. Let $F: E \to E$ be a completely continuous operator, i.e. it is continuous and the image of any bounded set is contained in a compact set, and let

$$\varsigma(F) = \{x \in E : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$$

Then either $\varsigma(F)$ is unbounded or F has a fixed point.

Definition 2.3. The system (1) is said to be controllable with nonlocal condition (2) on the interval J if for every $x_0, x_T \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(.)$ of (1) - (2) satisfies $x(0) + h(x) = x_0$ and $x(T) = x_T$.

To establish the result, we need the following additional hypothesis:

(H_1) The resolvent operator $R(t,s)$ is compact and there exists constant $M_1 > 0$ such that $\|R(t,s)\| \le M_1$

(H_2) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^T R(T,s)Bu(s)\,ds,$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$ and there exists a positive constants $a_1,\ a_2$ such that $\|B\| \le a_1$ and $\|W^{-1}\| \le a_2$.

(H_3) The function $G : C(J, X) \to X$ is completely continuous and there exists a constant $0 \le c_1 \le 1$ such that $\|G(t, x(t), y(t))\| \le c_1(\|x(t)\| + \|y(t)\|)$, for any $t \in J$ and is equicontinuous in (J, X).

(H_4) For each $t \in J$, the function $F(t,.) : X^{n+1} \to X$ is continuous and for each $(x_0, x_1, ..., x_n) \in X^{n+1}$ the function $F(., x_0, x_1, ..., x_n) : J \to X$ is strongly measurable.

(H_5) There exists an integrable function $m_i : J \times J \to [0,\infty)$ such that $\|f_i(t,s,x)\| \le m_i(t,s)\Omega_i(\|x\|)$, for any $t, s \in J, x \in X$, where $\Omega_i : [0,\ \infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H_6) There exists an integrable function $m_0 : J \times J \to [0,\infty)$ such that

$$\|g(t,s,x)\| \le m_0(t,s)\Omega_0(\|x\|),\ \text{for any } t, s \in J, x \in X,$$

where $\Omega_0 : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function.

(H_7) The function $F : C(J, X) \to X$ is completely continuous and there exists a constant $0 \le c_2 \le 1$ such that $\|F(t, x(t), y(t))\| \le c_2(\|x(t)\| + \|y(t)\|)$, for any $t \in J$ and is equicontinuous in (J, X).

(H_8) There are functions $H_1(.),\ H_2(.), \mu_1(.),\ \mu_2(.) : J \to [0,\ \infty)$ such that

$$\|A(t)R(t,s)\| \le H_1(t)\mu_1(s); \quad \|Q(t,s)R(t,s)\| \le H_2(t)\mu_2(s).$$

(H_9) The function

$$\widehat{m}(t)=\max\left\{\frac{1}{1-c_1}M_1c_2,\ \frac{M_1c_2}{1-c_1}\sum_{i=1}^{n}m_i(t,s),\ \frac{c_1}{1-c_1}m_0(t,s),\ \frac{c_1}{1-c_1}H_1(t)\mu_1(t),\right.$$

$$\left.\frac{c_1}{1-c_1}H_1(t)\mu_1(t)m_0(t,s),\ \frac{c_1}{1-c_1}\int_s^t H_2(s)\mu_2(s)ds,\ \frac{c_1}{1-c_1}\int_s^t\int_0^s H_1(t)\mu_1(t)m_0(t,s)dt\,ds\right\}$$

such that $\int_0^T \widehat{m}(s)ds<\int_c^\infty \frac{ds}{3s+2\Omega_0(s)+\sum_{i=1}^{n}\Omega_i(s)}$; see [7]

(H_{10}) The function $h:C(J,X)\to X$ is continuous and there exists a constant $M_2>0$ such that $\|h(x)\|\le M_2$ for any $x\in X$.

Then the system (1) – (2) has a mild solution of the following form

$$\begin{aligned}x(t)&=R(t,0)\big(x_0-h(x)-G(0,x(0),0)\big)+G\left(t,x(t),\int_0^t g(t,s,x(s))ds\right)\\&+\int_0^t R(t,s)\left[(Bu)(s)+F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,\dots,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right]ds\\&-\int_0^t R(t,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds\\&-\int_0^t\left[\int_s^t R(t,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds\end{aligned}\tag{4}$$

3. Controllability Result

Theorem 3.1. If the hypotheses $(H_1)-(H_9)$ are satisfied, then the system (1) - (2) is controllable on J.

Proof. Using the hypotheses (H_2) for an arbitrary function $x(.)$, we define the control

$$\begin{aligned}u(t)=W^{-1}\Bigg[&x_T-R(T,0)\{x_0-h(x)-G(0,x(0),0)\}-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\\&-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,\dots,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds\\&+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds\\&+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\Bigg](t)\end{aligned}\tag{5}$$

We now show that when using the control $u(t)$, the operator $\varphi:Z\to Z$, defined by

$$
\begin{aligned}
(\varphi x)(t) &= R(t,0)\left(x_0 - h(x)\right) - R(t,0)\left(G\left(0,x(0),0\right)\right) + G\left(t,x(t),\int_0^t g\left(t,s,x(s)\right)ds\right) \\
&-\int_0^t R(t,s)A(s)G\left(s,x(s),\int_0^s g\left(s,\tau,x(\tau)\right)d\tau\right)ds \\
&-\int_0^t\left[\int_s^t R(t,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g\left(\tau,\zeta,x(\zeta)\right)d\zeta\right)d\tau\right]ds \\
&+\int_0^t R(t,s)F\left(s,x(s),\int_0^s f_1\left(s,\tau,x(\tau)\right)d\tau,\dots,\int_0^s f_n\left(s,\tau,x(\tau)\right)d\tau\right)ds \\
&+\int_0^t R(t,\eta)BW^{-1}\left[x_T - R(T,0)\left\{x_0 - h(x) - G\left(0,x(0),0\right)\right\} - G\left(T,x(T),\int_0^T g\left(T,s,x(s)\right)ds\right)\right. \\
&-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1\left(s,\tau,x(\tau)\right)d\tau,\dots,\int_0^s f_n\left(s,\tau,x(\tau)\right)d\tau\right)\right\}ds \\
&+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g\left(s,\tau,x(\tau)\right)d\tau\right)ds \\
&\left.+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g\left(\tau,\zeta,x(\zeta)\right)d\zeta\right)d\tau\right)ds\right](\eta)\,d\eta \qquad (6)
\end{aligned}
$$

has a fixed point. This fixed point is then a solution of equation (1) – (2).

Clearly $x(T) = x_T$, which means that the control u steers the system (1) – (2) from the initial state x_0 to x_T in time T, provided we can obtain a fixed point of the nonlinear operator φ.

In order to study the controllability problem of (1) – (2), we introduce a parameter $\lambda(0,1)$ and consider the following system

$$
\begin{aligned}
\frac{d}{dt}\left[x(t) - \lambda G\left(t,x(t),\int_0^t g\left(t,s,x(s)\right)ds\right)\right] &= A(t)x(t) + \lambda\int_0^t Q(t,s)x(s)\,ds + \lambda(Bu)(t) \\
&+\lambda F\left(t,x(t),\int_0^t f_1\left(t,s,x(s)\right)ds,\dots,\int_0^t f_n\left(t,s,x(s)\right)ds\right);\ t \in J \qquad (7)
\end{aligned}
$$

$$
x_0 = \lambda\left(x_0 - h(x)\right) \qquad (8)
$$

First we obtain a priori bounds for the mild solution of (7) – (8). Then from

$$
\begin{aligned}
x(t) &= \lambda R(t,0)\left(x_0 - h(x) - G\left(0,x(0),0\right)\right) + \lambda G\left(t,x(t),\int_0^t g\left(t,s,x(s)\right)ds\right) \\
&-\lambda\int_0^t R(t,s)A(s)G\left(s,x(s),\int_0^s g\left(s,\tau,x(\tau)\right)d\tau\right)ds
\end{aligned}
$$

$$-\lambda\int_0^t\left[\int_s^t R(t,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds$$

$$+\lambda\int_0^t R(t,s)F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)ds$$

$$+\lambda\int_0^t R(t,\eta)BW^{-1}\left[x_T-R(T,0)\{x_0-h(x)-G(0,x(0),0)\}-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\right.$$

$$-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds$$

$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$

$$\left.+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\right](\eta)d\eta.$$

We have

$$\|x(t)\|\le M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|\right)+c_1\left(\|x(t)\|+\int_0^t m_0(t,s)\Omega_0\left(\|x(s)\|\right)ds\right)$$

$$+c_1\int_0^t H_1(s)\mu_1(s)\left[\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0\left(\|x(\tau)\|\right)d\tau\right]ds$$

$$+c_1\int_0^t\left[\int_s^t H_2(\tau)\mu_2(\tau)\left\{\|x(\tau)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0\left(\|x(\zeta)\|\right)d\zeta\right\}d\tau\right]ds$$

$$+M_1c_2\int_0^t\left[\|x(t)\|+\sum_{i=1}^n m_i(t,s)\Omega_i\left(\|x(s)\|\right)\right]ds$$

$$+\int_0^t\|R(t,\eta)\|a_1a_2\left[\|x_T\|+M_1\{\|x_0\|+M_2+\|G(0,x(0),0)\|\}+c_1\left(\|x(T)\|+\int_0^T m_0(T,s)\Omega_0\left(\|x(s)\|\right)ds\right)\right.$$

$$+M_1c_2\int_0^T\left\{\|x(T)\|+\sum_{i=1}^n m_i(T,s)\Omega_i\left(\|x(s)\|\right)\right\}ds$$

$$+c_1\int_0^T H_1(s)\mu_1(s)\left\{\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0\left(\|x(\tau)\|\right)d\tau\right\}ds$$

$$\left.+c_1\int_0^T\left\{\int_s^T H_2(\tau)\mu_2(\tau)\left\{\|x(s)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0\left(\|x(\zeta)\|\right)d\zeta\right\}d\tau\right\}ds\right]d\eta.$$

$$\le M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|\right)+c_1\left[\|x(t)\|+\int_0^t m_0(t,s)\Omega_0\left(\|x(s)\|\right)ds\right]$$

$$+c_1\int_0^t H_1(s)\mu_1(s)\left[\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0\left(\|x(\tau)\|\right)d\tau\right]ds$$

$$+c_1\int_0^t\left[\int_s^t H_2(\tau)\mu_2(\tau)\left\{\|x(s)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0(\|x(\zeta)\|)d\zeta\right\}d\tau\right]ds$$

$$+M_1c_2\int_0^t\left[\|x(t)\|+\sum_{i=1}^n m_i(t,s)\Omega_i(\|x(s)\|)\right]ds+NM_1T$$

where

$$N=a_1a_2\left(\|x_T\|+M_1\left\{\|x_0\|+M_2+\|G(0,x(0),0)\|\right\}\right)+c_1\left(\|x(T)\|+\int_0^T m_0(T,s)\Omega_0(\|x(s)\|)ds\right)$$

$$+M_1c_2\int_0^T\left\{\|x(T)\|+\sum_{i=1}^n m_i(T,s)\Omega_i(\|x(s)\|)\right\}ds$$

$$+c_1\int_0^T H_1(s)\mu_1(s)\left\{\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0(\|x(\tau)\|)d\tau\right\}ds$$

$$+c_1\int_0^T\left\{\int_s^T H_2(\tau)\mu_2(\tau)\left(\|x(s)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0(\|x(\zeta)\|)d\zeta\right)d\tau\right\}ds\ ;$$

$$(1-c_1)\|x(t)\|\le M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|\right)+c_1\int_0^t m_0(t,s)\Omega_0(\|x(s)\|)ds$$

$$+c_1\int_0^t H_1(s)\mu_1(s)\left[\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0(\|x(\tau)\|)d\tau\right]ds$$

$$+c_1\int_0^t\left[\int_s^t H_2(\tau)\mu_2(\tau)\left\{\|x(s)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0(\|x(\zeta)\|)d\zeta\right\}d\tau\right]ds$$

$$+M_1c_2\int_0^t\left[\|x(t)\|+\sum_{i=1}^n m_i(t,s)\Omega_i(\|x(s)\|)\right]ds+M_1NT$$

Implies that

$$\|x(t)\|\le\frac{1}{1-c_1}\left\{M_1\left(\|x(0)\|+M_2+\|G(0,x(0),0)\|+NT\right)+c_1\int_0^t m_0(t,s)\Omega_0(\|x(s)\|)ds\right.$$

$$+c_1\int_0^t H_1(s)\mu_1(s)\left(\|x(s)\|+\int_0^s m_0(s,\tau)\Omega_0(\|x(\tau)\|)d\tau\right)ds$$

$$+c_1\int_0^t\left[\int_s^t H_2(\tau)\mu_2(\tau)\left(\|x(s)\|+\int_0^\tau m_0(\tau,\zeta)\Omega_0(\|x(\zeta)\|)d\zeta\right)d\tau\right]ds$$

$$\left.+M_1c_2\int_0^t\left[\|x(t)\|+\sum_{i=1}^n m_i(t,s)\Omega_i(\|x(s)\|)\right]ds\right\}$$

Denoting by $v(t)$ the right hand side of the above inequality as $v(t)$. Then

$$\|x(t)\|\le v(t),$$

$$v(0)=c=\frac{1}{1-c_1}M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|+NT\right),$$

$$v'(t)=\frac{1}{1-c_1}\Bigg\{c_1 m_0(t,s)\Omega_0(v(t))+c_1 H_1(t)\mu_1(t)\left(v(t)+\int_0^t m_0(t,s)\Omega_0(v(s))ds\right)$$
$$+c_1\left[\int_s^t H_2(s)\mu_2(s)\left(v(t)+\int_0^s m_0(s,\tau)\Omega_0(v(\tau))d\tau\right)ds\right]$$
$$+M_1c_2\left(v(t)+\sum_{i=1}^n m_i(t,s)\Omega_i(v(t))\right)\Bigg\}$$
$$\le \widehat{m}(t)\left(2\Omega_0(v(t))+3v(t)+\sum_{i=1}^n\Omega_i(v(t))\right).$$

This implies

$$\int_{v(0)}^{v(t)}\frac{ds}{3s+\sum_{i=1}^n\Omega_i(s)+2\Omega_0(s)}\le\int_0^T\widehat{m}(s)ds<\int_c^\infty\frac{ds}{3s+\sum_{i=1}^n\Omega_i(s)+2\Omega_0(s)},0\le t\le T$$

where $c=\frac{1}{1-c_1}M_1\left(\|x(0)\|+M_2+\|G(0,x(0),0)\|+NT\right)$.

This inequality implies that there is a constant k such that $v(t)\le k, t\in J$ and hence we have $\|x\|=\sup\{|x(t)|:t\in J\}\le k$, where k depends only on T and on the functions $\widehat{m},\Omega_0$ and Ω_i.

Next we must prove that the operator F is a completely continuous operator. Let $B_k=\{x\in Z:\|x\|\le k\}$ for some $k\ge1$. We first show that φ maps B_k into an equicontinuous family. Let $x\in B_k$ and $t_1,t_2\in[0,T]$. Then if $0<t_1<t_2<T$, we have

$$\|(\varphi x)(t_1)-(\varphi x)(t_2)\|\le\|(R(t_1,0)-R(t_2,0))(x_0-h(x)-G(0,x(0),0))\|$$
$$+\left\|\left(G\left(t_1,x(t_1),\int_0^{t_1}g(t_1,s,x(s))ds\right)-G\left(t_2,x(t_2),\int_0^{t_2}g(t_2,s,x(s))ds\right)\right)\right\|$$
$$+\left\|\int_0^{t_1}\left[(R(t_1,s)-R(t_2,s))A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)\right]ds\right\|$$
$$+\left\|\int_0^{t_1}\left[\int_s^{t_1}(R(t_1,\tau)-R(t_2,\tau))Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds\right\|$$
$$+\left\|\int_0^{t_1}(R(t_1,s)-R(t_2,s))F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,\dots,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)ds\right\|$$
$$+\left\|\int_{t_1}^{t_2}\left[R(t_2,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)\right]ds\right\|$$
$$+\left\|\int_{t_1}^{t_2}\left[\int_s^{t_2}\left(R(t_2,s)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)\right)d\tau\right]ds\right\|$$

$$+\left\|\int_{t_1}^{t_2}\left[R(t_2,s)F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right]ds\right\|$$

$$+\left\|\int_0^{t_1}(R(t_1,\eta)-R(t_2,\eta))BW^{-1}\left[x_T-R(T,0)\{x_0-h(x)-G(0,x(0),0)\}-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\right.\right.$$

$$-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds$$

$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$

$$\left.\left.+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\right](\eta)d\eta\right\|$$

$$+\left\|\int_{t_1}^{t_2}R(t_2,\eta)BW^{-1}\left[x_T-R(T,0)\{x_0-h(x)-G(0,x(0),0)\}-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\right.\right.$$

$$-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds$$

$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$

$$\left.\left.+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\right](\eta)d\eta\right\|$$

$$\le\left\|(R(t_1,0)-R(t_2,0))(x_0-h(x)-G(0,x(0),0))\right\|$$

$$+\left\|\left(G\left(t_1,x(t_1),\int_0^{t_1}g(t_1,s,x(s))ds\right)-G\left(t_2,x(t_2),\int_0^{t_2}g(t_2,s,x(s))ds\right)\right)\right\|$$

$$+\left\|\int_0^{t_1}\left[(R(t_1,s)-R(t_2,s))A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)\right]ds\right\|$$

$$+\left\|\int_0^{t_1}\left[\int_s^{t_1}(R(t_1,\tau)-R(t_2,\tau))Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds\right\|$$

$$+\left\|\int_0^{t_1}(R(t_1,s)-R(t_2,s))F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)ds\right\|$$

$$+c_1\int_{t_1}^{t_2}H_1(s)\mu_1(s)\left[k+\int_0^s m_0(s,\tau)\Omega_0(k)d\tau\right]ds$$

$$+c_1\int_{t_1}^{t_2}\left[\int_s^{t_2}H_2(\tau)\mu_2(\tau)\left(k+\int_0^s m_0(s,\zeta)\Omega_0(k)d\zeta\right)d\tau\right]ds+M_1c_2\int_{t_1}^{t_2}\left(k+\sum_{i=1}^n m_i(t,s)\Omega_i(k)\right)ds$$

$$+\int_0^{t_1}\left\|(R(t_1,\eta)-R(t_2,\eta))\right\|a_1a_2\left\|x_T-R(T,0)(x_0-h(x)-G(0,x(0),0))-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\right.$$

$$-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds$$
$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$
$$+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\Bigg\|d\eta$$
$$+\int_{t_1}^{t_2} M_1a_1a_2\left\{\|x_T\|+M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|\right)+c_1\left(\|x(T)\|+\int_0^T m_0(t,s)\Omega_0(k)ds\right)\right.$$
$$+M_1c_2\int_0^T\left(k+\sum_{i=1}^n m_i(T,s)\Omega_i(k)\right)ds+c_1\int_0^T H_1(s)\mu_1(s)\left(k+\int_0^s m_0(s,\tau)\Omega_0(k)d\tau\right)ds$$
$$\left.+c_1\int_0^T\left(\int_s^T H_2(\tau)\mu_2(\tau)\left(k+\int_0^\tau m_0(\tau,\zeta)\Omega_0(k)d\zeta\right)d\tau\right)ds\right\}d\eta$$

The right hand side tends to zero as $t_2-t_1\to 0$, since f is completely continuous and by (H_1), $R(t,s)$ for $t>s$ is continuous in the uniform operator topology. Thus φ maps B_k into an equicontinuous family of functions. It is easy to see that the family φB_k is uniformly bounded.

Next, we show $\overline{\varphi B_k}$ is compact. Since we have shown φB_k is an equicontinuous collection, it suffice by the Arzela – Ascoli theorem to show $\{(\varphi x)(t): x\in B_k\}$ is precompact in X for any $t\in[0,T]$. Let $0<t\le T$ be fixed and ε a real number satisfying $0<\varepsilon<t$. For $x\in B_k$ we define

$$(\varphi_\varepsilon x)(t)=R(t,0)\left(x_0-h(x)-G(0,x(0),0)\right)+G\left(t,x(t),\int_0^{t-\varepsilon} g(t,s,x(s))ds\right)$$
$$-\int_0^{t-\varepsilon} R(t,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$
$$-\int_0^{t-\varepsilon}\left[\int_s^{t-\varepsilon} R(t,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right]ds$$
$$+\int_0^{t-\varepsilon} R(t,s)F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)ds$$
$$+\int_0^{t-\varepsilon} R(t,\eta)BW^{-1}\left\{x_T-R(T,0)\left(x_0-h(x)-G(0,x(0),0)\right)-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)\right.$$
$$-\int_0^T R(T,s)\left\{F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,...,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\}ds$$
$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$

$$+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\Bigg\}(\eta)d\eta$$

Since $R(t,s)$ is a compact operator, the set $Y_\varepsilon(t)=\{(\varphi_\varepsilon x)(t):x\in B_k\}$ is precompact in X for every ε, $0<\varepsilon<t$. Moreover, for every $x\in B_k$ we have

$$\|(\varphi x)(t)-(\varphi_\varepsilon x)(t)\|\le\int_{t-\varepsilon}^t\left\|R(t,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)\right\|ds$$

$$+\int_{t-\varepsilon}^t\left\|\int_s^{t-\varepsilon}R(t,\tau)Q(\tau,s)\left(G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)\right)d\tau\right\|ds$$

$$+\int_{t-\varepsilon}^t\left\|R(t,s)F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,\dots,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right\|ds$$

$$+\int_{t-\varepsilon}^t\Bigg\|R(t,\eta)BW^{-1}\Bigg\{x_T-R(T,0)\left(x_0-h(x)-G(0,x(0),0)\right)-G\left(T,x(T),\int_0^T g(T,s,x(s))ds\right)$$

$$-\int_0^T R(T,s)\left(F\left(s,x(s),\int_0^s f_1(s,\tau,x(\tau))d\tau,\dots,\int_0^s f_n(s,\tau,x(\tau))d\tau\right)\right)ds$$

$$+\int_0^T R(T,s)A(s)G\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\right)ds$$

$$+\int_0^T\left(\int_s^T R(T,\tau)Q(\tau,s)G\left(\tau,x(\tau),\int_0^\tau g(\tau,\zeta,x(\zeta))d\zeta\right)d\tau\right)ds\Bigg\}(\eta)\Bigg\|d\eta$$

$$\le c_1\int_{t-\varepsilon}^t H_1(s)\mu_1(s)\left(k+\int_0^s m_0(s,\tau)\Omega_0(k)d\tau\right)ds$$

$$+c_1\int_{t-\varepsilon}^t\left[\int_s^{t-\varepsilon}H_2(\tau)\mu_2(\tau)\left(k+\int_0^\tau m_0(\tau,\zeta)\Omega_0(k)d\zeta\right)d\tau\right]ds+M_1c_2\int_{t-\varepsilon}^t\left(k+\sum_{i=1}^n m_i(t,s)\Omega_i(k)\right)ds$$

$$+M_1a_1a_2\int_{t-\varepsilon}^t\Bigg\{\|x_T\|+M_1\left(\|x_0\|+M_2+\|G(0,x(0),0)\|\right)+c_1\left(\|x(T)\|+\int_0^T m_0(t,s)\Omega_0(k)ds\right)$$

$$+M_1c_2\int_0^T\left(k+\sum_{i=1}^n m_i(T,s)\Omega_i(k)\right)ds+c_1\int_0^T H_1(s)\mu_1(s)\left(k+\int_0^s m_0(s,\tau)\Omega_0(k)d\tau\right)ds$$

$$+c_1\int_0^T\left(\int_s^T H_2(\tau)\mu_2(\tau)\left(k+\int_0^\tau m_0(\tau,\zeta)\Omega_0(k)d\zeta\right)d\tau\right)ds\Bigg\}d\eta$$

Therefore there are precompact sets arbitrarily close to the set $\{(\varphi x)(t):x\in B_k\}$. Hence, the set $\{(\varphi x)(t):x\in B_k\}$ is precompact in X.

It remains to show that $\varphi:Z\to Z$ is continuous. Let $\{x_n\}_0^\infty\subseteq Z$ with $x_n\to x$ in Z. Then there is an integer q such that $\|x_n(t)\|\le q$ for all n and $t\in J$, so $x_n\in B_q$ and $x\in B_q$. By (H_4)

$$F\left(t,x_n(t),\int_0^t f_1(t,s,x_n(s))ds,\int_0^t f_2(t,s,x_n(s))ds,\dots,\int_0^t f_n(t,s,x_n(s))ds\right)$$
$$\to F\left(t,x(t),\int_0^t f_1(t,s,x(s))ds,\int_0^t f_2(t,s,x(s))ds,\dots,\int_0^t f_n(t,s,x(s))ds\right), \text{ for each } t\in J$$

and since

$$\left\|F\left(t,x_n(t),\int_0^t f_1(t,\tau,x_n(\tau))d\tau,\int_0^t f_2(t,\tau,x_n(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,x_n(\tau))d\tau\right)\right.$$
$$\left.-F\left(t,x(t),\int_0^t f_1(t,\tau,x(\tau))d\tau,\int_0^t f_2(t,\tau,x(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,x(\tau))d\tau\right)\right\|\le 2c_2m_i(t,\tau)\Omega_i(q);$$
$$\left\|G\left(t,x_n(t),\int_0^t g(t,\tau,x_n(\tau))d\tau\right)-G\left(t,x(t),\int_0^t g(t,\tau,x(\tau))d\tau\right)\right\|\le 2c_1m_0(t,\tau)\Omega_0(q)\ ;$$
$$\left\|A(\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-A(\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right\|\le 2c_1H_1(t)\mu_1(t)q$$

and

$$\left\|B(t,\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-B(t,\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right\|$$
$$\le 2c_1H_2(\tau)\mu_2(\tau)q;\ 0\le c_1\le 1,\ 0\le c_2\le 1.$$

Now by dominated convergence theorem, we have

$$\|\varphi x_n-\varphi x\|=\sup_{t\in J}\left\|\left[G\left(t,x_n(t),\int_0^t g(t,\tau,x_n(\tau))d\tau\right)-G\left(t,x(t),\int_0^t g(t,\tau,x(\tau))d\tau\right)\right]\right.$$
$$+\int_0^t\left[A(\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-A(\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right]ds$$
$$+\int_0^t\left[\int_0^s\left\{B(t,\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\zeta,x_n(\zeta))d\zeta\right)-B(t,\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\zeta,x(\zeta))d\zeta\right)\right\}d\tau\right]ds$$
$$+\int_0^t R(t,s)\left[F\left(t,x_n(t),\int_0^t f_1(t,\tau,x_n(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,x_n(\tau))d\tau\right)-F\left(t,x(t),\int_0^t f_1(t,\tau,x(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,x(\tau))d\tau\right)\right]ds$$
$$+\int_0^t R(t,\eta)BW^{-1}\left[G\left(T,x_n(T),\int_0^T g(T,\tau,x_n(\tau))d\tau\right)-G\left(T,x(T),\int_0^T g(T,\tau,x(\tau))d\tau\right)\right.$$
$$+\int_0^T R(T,s)\left\{F\left(t,x_n(t),\int_0^t f_1(t,\tau,x_n(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,x_n(\tau))d\tau\right)-F\left(t,\,x(t),\int_0^t f_1(t,\tau,\,x(\tau))d\tau,\dots,\int_0^t f_n(t,\tau,\,x(\tau))d\tau\right)\right\}ds$$
$$+\int_0^T\left\{A(\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-A(\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right\}ds$$

$$+\int_0^T\left\{\int_0^s\left\{B(t,\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\zeta,x_n(\zeta))d\zeta\right)-B(t,\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\zeta,x(\zeta))d\zeta\right)\right\}d\tau\right\}ds\Bigg](\eta)d\eta\Bigg\|$$

$$\leq\left\|G\left(t,x_n(t),\int_0^t g(t,\tau,x_n(\tau))d\tau\right)-G\left(t,x(t),\int_0^t g(t,\tau,x(\tau))d\tau\right)\right\|$$

$$+\int_0^T\left\|A(\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-A(\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right\|ds$$

$$+\int_0^T\left[\int_0^s\left\|B(t,\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\zeta,x_n(\zeta))d\zeta\right)-B(t,\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\zeta,x(\zeta))d\zeta\right)\right\|d\tau\right]ds$$

$$+\int_0^T\left\|R(t,s)\left[F\left(t,x_n(t),\int_0^t f_1(t,\tau,x_n(\tau))d\tau,...,\int_0^t f_n(t,\tau,x_n(\tau))d\tau\right)-F\left(t,x(t),\int_0^t f_1(t,\tau,x(\tau))d\tau,...,\int_0^t f_n(t,\tau,x(\tau))d\tau\right)\right]\right\|ds$$

$$+\int_0^T\|R(t,\eta)\|\|B\|\|W^{-1}\|\Bigg[\left\|G\left(T,x_n(T),\int_0^T g(T,\tau,x_n(\tau))d\tau\right)-G\left(T,x(T),\int_0^T g(T,\tau,x(\tau))d\tau\right)\right\|$$

$$+\int_0^T\left\|R(T,s)\left\{F\left(t,x_n(t),\int_0^t f_1(t,\tau,x_n(\tau))d\tau,..,\int_0^t f_n(t,\tau,x_n(\tau))d\tau\right)-F\left(t,x(t),\int_0^t f_1(t,\tau,x(\tau))d\tau,..,\int_0^t f_n(t,\tau,x(\tau))d\tau\right)\right\}\right\|ds$$

$$+\int_0^T\left\|A(\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\tau,x_n(\tau))d\tau\right)-A(\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\tau,x(\tau))d\tau\right)\right\|ds$$

$$+\int_0^T\left[\int_0^s\left\|B(t,\tau)R(t,\tau)G\left(\tau,x_n(\tau),\int_0^\tau g(t,\zeta,x_n(\zeta))d\zeta\right)-B(t,\tau)R(t,\tau)G\left(\tau,x(\tau),\int_0^\tau g(t,\zeta,x(\zeta))d\zeta\right)\right\|d\tau\right]ds\Bigg]d\eta$$

$\to 0$ as $n\to\infty$.

Thus φ is continuous. This completes the proof that φ is completely continuous.

Finally the set $\varsigma(\varphi)=\{x\in Z : x=\lambda\varphi x, \lambda\in(0,1)\}$ is bounded, as we prove in the first step. Consequently, by Schaefer's theorem, the operator φ has a fixed point in Z. This means that any fixed point of φ is a mild solution of (1) and (2) on J satisfying $(\varphi x)(t)=x(t)$.

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