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Ordinary differential equations

Green's Function For Linear Differential Operators In One Variable

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Abstract

General formula for causal Green's function of linear differential operator of given degree in one variable, $(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k)$, is given according to coefficient functions of differential operator as a series of integrals. The solution also provides analytic formula for fundamental solutions of corresponding homogenous linear differential equation, $\big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\big)\, y(x) = 0$, as series of integrals. Furthermore, multiplicative property of causal Green's functions is shown and by which explicit formulas for causal Green's functions of some classes of decomposable linear differential operators are given. A method to find Green's function of general linear differential operator of given degree in one variable with arbitrary boundary condition according to coefficient functions of differential operator is demonstrated.

1 Causal Green's function for linear differential operators in one variable

Initial value problem for inhomogeneous linear differential equation of degree n in one variable,

$$\Big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\Big)\, y(x) = g(x), \tag{1}$$

can be converted to Volterra's integral equation of second kind [see [3]]. For initial condition $\partial_x^i y(a) = 0$ for $i = 0, 1, .., n-1$, the corresponding Volterra's equation is given by;

$$u(x) + \int_a^x dz\, K(x,z)\, u(z) = g(x),$$

where $K(x,z) = \big(\sum_{k=0}^{n-1} P_k(x)\frac{(x-z)^{n-k-1}}{(n-k-1)!}\big)$ and $y(x) = \int_a^x dz \frac{(x-z)^{n-1}}{(n-1)!} u(z)$. For $g(x) \in L^2[a,b]$, the condition $(\int_a^b \int_a^b dx\, dy\, |K(x,y)|^2) < \infty$ is sufficient condition for existence of unique solution in $L^2[a,b]$, given by iteration (e.g. see [2]). Clearly this conditions can be satisfied if $P_i(x)$ ($i = 0, 1, ..., n-1$) and $g(x)$ functions are taken to be continuous on $[a,b]$. Therefore we can state the following theorem;

Theorem 1 *The Green's function for inhomogeneous linear differential equation* $\big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\, \partial_x^k\big)\, y(x) = g(x)$, *where* $P_i(x)$ $(i = 0, 1, ..., n-1)$ *and* $g(x)$ *are in* $\mathbb{C}[a,b]$, *with the boundary condition;* $\partial_x^i y(a) = 0$ *for* $i = 0, 1, .., n-1$, *is given by;*

$$G(x,y) = \theta(x-y)\Big(\frac{(x-y)^{n-1}}{(n-1)!} + \int_y^x dz\, \frac{(x-z)^{n-1}}{(n-1)!}\, R(z,y)\Big), \tag{2}$$

where

$$R(x,y) = h(x,y) + \sum_{r=2}^{\infty} \int_y^x dz_1 \int_y^{z_1} dz_2 \cdots \int_y^{z_{r-2}} dz_{r-1}\, h(x,z_1)\, h(z_1,z_2) \cdots h(z_{r-1},y), \tag{3}$$

and $h(x,y) = -\sum_{k=0}^{n-1} P_k(x) \frac{(x-y)^{n-k-1}}{(n-k-1)!}$. *The solution to inhomogeneous linear differential equation (1) for* $x \in [a,b]$ *is then given by* $y(x) = \int_a^{\infty} dz\, G(x,z)\, g(z)$.

Proof. In order to prove (2) is Green's function of (1) it is enough to prove that for the two variables function;

$$T(x,y) = \Big(\frac{(x-y)^{n-1}}{(n-1)!} + \int_y^x dz\, \frac{(x-z)^{n-1}}{(n-1)!}\, R(z,y)\Big), \tag{4}$$

we have $\left(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\right) T(x,y) = 0$ and $(\partial_x^i T(x,y))|_{x=y} = 0$ for $i = 0, 1, .., n-2$ and $(\partial_x^{n-1} T(x,y))|_{x=y} = 1$ [e.g. see [1]]. This can be easily done by noting;

$$\partial_x^i T(x,y) = \Big(\frac{(x-y)^{n-i-1}}{(n-i-1)!} + \int_y^x dz\, \frac{(x-z)^{n-i-1}}{(n-i-1)!}\, R(z,y)\Big) \quad i = 0, 1, .., n-1 \quad (5)$$

$$\partial_x^n T(x,y) = R(x,y),$$

and therefore,

$$\begin{aligned}
(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k)\ T(x,y) = & \quad R(x,y) + \sum_{k=0}^{n-1} \frac{P_k(x)(x-y)^{n-k-1}}{(n-k-1)!} \\
& + \sum_{k=0}^{n-1} P_k(x) \int_y^x dz\, \frac{(x-z)^{n-k-1}}{(n-k-1)!} R(z,y) \\
= & \quad R(x,y) - h(x,y) - \int_y^x dz\, h(x,z)\, R(z,y) \\
= & \quad R(x,y) - h(x,y) - \big(R(x,y) - h(x,y)\big) = 0.
\end{aligned}$$

In the last line we used $\int_y^x dz\, h(x,z)\, R(z,y) = \big(R(x,y) - h(x,y)\big)$, which comes from definition of $R(x,y)$. By using (5) we have $(\partial_x^i T(x,y))|_{x=y} = 0$ for $i = 0, 1, .., n-2$ and $(\partial_x^{n-1} T(x,y))|_{x=y} = 1$.

The Green's function for (1) with mentioned boundary condition is called causal solution which by method of variation of parameters is given by;

$$G(x,y) = \big(\sum_{i=1}^{n} \frac{W_i(y)\, u_i(x)}{W(y)}\big)\theta(x-y), \quad (6)$$

where $u_1(x), u_2(x), ..., u_n(x)$ are fundamental solutions of corresponding homogeneous differential equation; $(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k)\, u_i(x) = 0$. $W(y)$ is the Wronskian and $W_i(y)$ is the Wronskian with its i^{th} column in determinant is replace by $(0, 0, .., 0, 1)$. Comparing this result with (2) we have the identity;

$$\sum_{i=1}^{n} \frac{W_i(y)\, u_i(x)}{W(y)} = \frac{(x-y)^{n-1}}{(n-1)!} + \int_y^x dz\, \frac{(x-z)^{n-1}}{(n-1)!}\, R(z,y). \quad (7)$$

For linear differential operator of first degree (n=1), like $\partial_x - P(x)$, the causal Green's function using [theorem 1] is equal to; $(\partial_x - P(x))^{-1} = \theta(x -$

$y)\Big(1+\sum_{k=1}^{\infty}\int_y^x dz_1\,\cdots\int_y^{z_{n-2}} dz_{k-1}\int_y^{z_{k-1}} dz_k\,P(z_1)\cdots P(z_{k-1})P(z_k)\Big) =$
$\theta(x-y)\Big(1+\frac{1}{k!}\,\sum_{k=1}^{\infty}\int_y^x\cdots\int_y^x\int_y^x dz_1\cdots dz_{k-1}dz_k\,P(z_1)\;\cdots P(z_{k-1})P(z_k)\Big) =$
$\theta(x-y)e^{\int_y^x dzP(z)}$. For linear differential operator of degree two in the form of; $(\partial_x^2-P(x))$, the causal Green's function by using [theorem 1] is given by $T_s(x,y)\theta(x-y)$ where;

$$T_s(x,y)=\Big\{(x-y)+\sum_{k=1}^{\infty}\Big(\int_y^x dz_1\cdots\int_y^{z_{k-2}} dz_{k-1}\int_y^{z_{k-1}} dz_k$$
$$(x-z_1)P(z_1)(z_1-z_2)P(z_2)(z_2-z_3)\cdots(z_{k-1}-z_k)P(z_k)(z_k-y)\Big)\Big\}. \qquad (8)$$

For example $(\partial_x^2-x)^{-1}=\theta(x-y)\,(x-y)+\theta(x-y)\int_y^x dz(x-z)z(z-y)+\theta(x-y)\int_y^x dt\int_y^t dz\,\big((x-t)t(t-z)z(z-y)\big)+\cdots=\theta(x-y)\Big(\big(x-y\big)+\big(\frac{x^4}{12}-\frac{(x^3y)}{6}+\frac{(xy^3)}{6}-\frac{y^4}{12}\big)+\big(\frac{x^7}{504}-\frac{(x^6y)}{180}+\frac{(x^4y^3)}{72}-\frac{(x^3y^4)}{72}+\frac{(xy^6)}{180}-\frac{y^7}{504}\big)+\cdots\Big)$, which is consistent with solution $(\partial_x^2-x)^{-1}=\theta(x-y)\big(\dfrac{-\mathrm{Ai}(x)\mathrm{Bi}(y)+\mathrm{Ai}(y)\mathrm{Bi}(x)}{\mathrm{Ai}(y)\mathrm{Bi}'(y)-\mathrm{Ai}'(y)\mathrm{Bi}(y)}\big)$ derived by (6).

It can be seen from (2) that if $P_i(x)$ ($i=0,1,...,n-1$) functions are smooth on $[a,b]$ then $T(x,y)$ is smooth function on $[a,b]\times[a,b]$, in which case we state the following theorem;

Theorem 2 *If $T_1(x,y)\theta(x-y)$ and $T_2(x,y)\theta(x-y)$ are causal Green's functions for linear differential operators $\mathcal{O}_1(x,\partial_x)=\big(\partial_x^n+\sum_{k=0}^{n-1}P_k(x)\partial_x^k\big)$ and $\mathcal{O}_2(x,\partial_x)=\big(\partial_x^m+\sum_{k=0}^{m-1}q_k(x)\partial_x^k\big)$ respectively (assuming $P_i(x)$'s and $q_i(x)$'s functions are in $\mathbb{C}^\infty[a,b]$) then $T_3(x,y)\theta(x-y)$ where,*

$$T_3(x,y)=\int_y^x dz\,T_2(x,z)\,T_1(z,y), \qquad (9)$$

is the causal Green's function for linear differential operator $\mathcal{O}_3(x,\partial_x)=\mathcal{O}_1(x,\partial_x).\mathcal{O}_2(x,\partial_x)$

Proof. By assumption; $\mathcal{O}_1(x,\partial_x)\,T_1(x,y)=0$ and $(\partial_x^iT_1(x,y))|_{x=y}=0$ for $i=0,1,..,n-2$ and $(\partial_x^{n-1}T_1(x,y))|_{x=y}=1$ also $\mathcal{O}_2(x,\partial_x)\,T_2(x,y)=0$ and $(\partial_x^iT_2(x,y))|_{x=y}=0$ for $i=0,1,..,m-2$ and $(\partial_x^{m-1}T_2(x,y))|_{x=y}=1$, therefore we have;

$$\partial_x^i\,T_3(x,y)=\int_y^x dz\,(\partial_x^i(T_2(x,z)))\,T_1(z,y), \quad i=0,1,..,m-1 \qquad (10)$$

$$\partial_x^m T_3(x,y) = T_1(x,y) + \int_y^x dz\,(\partial_x^m(T_2(x,z)))\,T_1(z,y), \tag{11}$$

$$\partial_x^i T_3(x,y) = \partial_x^{i-m}(T_1(x,y)) + \partial_x^{i-m}\Big(\int_y^x dz \quad (\partial_x^m(T_2(x,z)))\,T_1(z,y)\Big),$$
$$i = m+1,..,m+n-1. \tag{12}$$

Concentrating on the second term in (12), we have for $k = 1,2,..,n-1$;

$$\begin{aligned}\partial_x^k\Big(\int_y^x dz\,(\partial_x^m(T_2(x,z)))\,T_1(z,y)\Big) &= \Big\{ \textstyle\sum_{j=0}^{k-1} \partial_x^j\Big((\partial_x^{m+k-1-j}T_2(x,z))|_{z=x} \\ & T_1(x,y)\Big)\Big\} + \Big(\textstyle\int_y^x dz\,(\partial_x^{m+k}(T_2(x,z)))\,T_1(z,y)\Big) \\ &= \{\textstyle\sum_{j=0}^{k-1}\sum_{r=0}^{j}\binom{j}{r}\Big((\partial_x^{m+k-1-j+r}T_2(x,z))|_{z=x}\,\partial_x^{j-r}T_1(x,y)\Big)\} \\ &\quad +\Big(\textstyle\int_y^x dz\,(\partial_x^{m+k}(T_2(x,z)))\,T_1(z,y)\Big)\end{aligned} \tag{13}$$

From (10),(11), (12) and (13) we have $(\partial_x^i T_3(x,y))|_{x=y} = 0$ for $i = 0,1,...,m+n-3, m+n-2$ and $(\partial_x^{m+n-1}T_3(x,y))|_{x=y} = 1$. On the other hand;

$$\begin{aligned}\mathcal{O}_2(x,\partial_x)T_3(x,y) &= \Big(\partial_x^m + \sum_{k=0}^{m-1} q_k(x)\partial_x^k\Big)\int_y^x dz\,T_2(x,z)\,T_1(z,y) \\ &= \partial_x\Big(\int_y^x dz\,\partial_x^{m-1}T_2(x,z)\,T_1(z,y)\Big) \\ &\quad + \int_y^x dz\,\Big(\sum_{k=0}^{m-1} q_k(x)\partial_x^k T_2(x,z)\Big)\,T_1(z,y) \\ &= T_1(x,y) + \int_x^y dz\,\mathcal{O}_2(x,\partial_x)T_2(x,z)T_1(z,y) \\ &= T_1(x,y).\end{aligned}$$

Therefore $\mathcal{O}_1(x,\partial_x).\mathcal{O}_2(x,\partial_x)T_3(x,y) = \mathcal{O}_1(x,\partial_x)T_1(x,y) = 0$.

The following corollary comes as a consequence;

Corollary 1 *Causal Green's function for differential operator,*

$$\mathcal{O}(x,\partial_x) = \big(\partial_x - p_1(x)\big)\,\big(\partial_x - p_2(x)\big)\,\cdots\big(\partial_x - p_n(x)\big), \tag{14}$$

where $p_i(x) \in \mathbb{C}^\infty[a,b]$ *(for* $i = 1, ..., n$*)is given by;*

$$G(x,y) = \theta(x-y) \quad \int_y^x dz_1 \int_y^{z_1} dz_2 \cdots \int_y^{z_{r-2}} dz_{n-1} \Big(e^{\int_{z_1}^x dt_n\, p_n(t_n)} \\ e^{\int_{z_2}^{z_1} dt_{n-1}\, p_{n-1}(t_{n-1})} \cdots e^{\int_y^{z_{n-1}} dt_1\, p_1(t_1)} \Big) \tag{15}$$

For example for differential operator $\mathcal{O}(x,\partial_x) = \partial_x^2 + 3x\partial_x + (2x^2+2)$, since $(\partial_x + x)(\partial_x + 2x) = \partial_x^2 + 3x\partial_x + (2x^2+2)$, by using result (15) one gets $G(x,y) = \sqrt{\frac{\pi}{2}}\; e^{y^2 - \frac{x^2}{2}} \{\mathrm{Erf}(\frac{x}{\sqrt{2}}) - (\mathrm{Erf}(\frac{y}{\sqrt{2}}))\}\; \theta(x-y)$.

Corollary 2 *Causal Green's function for linear differential operator ;*

$$\mathcal{O}(x,\partial_x) = \sum_{k=0}^{n} \alpha_k \partial_x^k\,, \tag{16}$$

where $\alpha_k \in \mathbb{C}$ *and* $\alpha_n \neq 0$*, is given by;*

$$G(x,y) \;=\; \frac{\theta(x-y)}{\alpha_n} \quad \int_y^x dz_1 \int_y^{z_1} dz_2 \cdots \int_y^{z_{n-3}} dz_{n-2} \\ \int_y^{z_{n-2}} dz_{n-1}\; e^{\Big(\beta_1(x-z_1)+\beta_2(z_1-z_2)\cdots+\beta_n(z_{n-1}-y)\Big)}, \tag{17}$$

where $\beta_1,\ \beta_2 \cdots \beta_n$ *are* n *complex roots of equation* $\sum_{i=0}^{n} \alpha_i X^i = 0$.

Proof. Differential operator $\mathcal{O}(x,\partial_x) = \sum_{i=0}^n \alpha_i \partial_x^i$, according to *Fundamental theorem of algebra*, can be written as, $\sum_{i=0}^n \alpha_i \partial_x^i = \alpha_n(\partial_x - \beta_1)(\partial_x - \beta_2)\cdots(\partial_x - \beta_n)$. Therefore by using (15) the result is proved.

For example $(\partial_x^2 - \omega^2)^{-1} = \theta(x-y)\int_y^x dz_1 e^{(\omega(x-z_1)-\omega(z_1-y))} = \frac{\sinh\omega(x-y)}{\omega}\theta(x-y)$ and also $(\partial_x^3 - i\alpha\partial_x^2 - \omega^2\partial_x + i\alpha\omega^2)^{-1} = \theta(x-y)\int_y^x dz_1(\int_y^{z_1} dz_2 e^{(\omega(x-z_1)-\omega(z_1-z_2)+i\alpha(z_2-y))}) = \theta(x-y)\big(\frac{e^{\omega(x-y)} - e^{i\alpha(x-y)}}{\alpha^2+\omega^2} - \frac{\sinh[\omega(x-y)]}{i\,\alpha\,\omega+\omega^2}\big)$.

Let us consider a differential operator in the form of

$$\mathcal{O}(x,\partial_x) = -\partial_x^2 + v(x). \tag{18}$$

By decomposing it into two firs degree differential operators; $-(\partial_x^2 - v(x)) = -(\partial_x - p(x))(\partial_x - q(x))$, we have consequently $q(x) = -p(x)$ and $p(x)^2 - \partial_x p(x) = v(x)$. Therefore according to (15) the causal the Green's function is given by;

$$\big(- \partial_x^2 + v(x)\big)^{-1} = -\theta(x-y)\int_y^x dz e^{(-\int_z^x dt p(t) + \int_y^z dt' p(t'))}, \tag{19}$$

where $p(x)$ is solution for first order nonlinear differential equation $p\,(x)^2 - \partial_x p\,(x) = v(x)$. This is just Riccati equation, thus the answers to $p\,(x)^2 - \partial_x p\,(x) = v(x)$ are given by solutions of homogenous differential equation $(-\partial_x^2 + v(x))u_{1,2}(x) = 0$ where $p(x) = -(\frac{u'_{1,2}}{u_{1,2}})$. Inserting $p(x) = -(\frac{u'_1}{u_1})$ into solution (19), we have; $\big(- \partial_x^2 + v(x)\big)^{-1} = -\theta(x-y)u_1(x)u_1(y)\int_y^x dz(\frac{1}{u(z)^2})$. Considering the relation $u_2(z) = u_1(z)\int dz \frac{1}{u_1(z)^2}$ (valid for homogenous differential equation $(-\partial_x^2 + v(x))u_{1,2}(x) = 0$) the solution (19) becomes the standard solution, $\big(- \partial_x^2 + v(x)\big)^{-1} = -\theta(x-y)\Big(u_2(x)u_1(y) - u_1(x)u_2(y)\Big)$.

Considering [theorem 2] one can introduce the following infinite non-abelian group of operators on a subspace of $\mathbb{C}^\infty[a,b]$. We call it "Lalescu Group";

- **Lalescu Group.** *The Group of differential operators of the form;* $\big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\big)$ *of all finite order,* $n \geq 0$*, where* $P_k(x) \in \mathbb{C}^\infty[a,b]$ *(for k=0,1,..,n-1) and their corresponding causal Green's functions* $G(x,y) = T(x,y)\theta(x-y)$ *(given by (2)), on subspace of* $\mathbb{C}^\infty[a,b]$ *consisting of functions which themselves and their derivatives to all orders are zero at* $x = a$*, creates non-abelian group with operators multiplication.*

Beside all differential operators $\mathcal{O}(x,\partial_x) = \big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\big)$ and their causal Green's functions $G(x,y) = T(x,y)\theta(x-y)$, the group also contains integro-differential operators and their inverses, coming from mixing these two groups of operators. For example $\mathcal{O}_1(x,\partial_x).T_2(x,y)\theta(x-y)$, acting on $\phi(x)$ in the function space as $\mathcal{O}_1(x,\partial_x)(\int_a^x dz\, T_2(x,z)\phi(z))$, and its inverse $\mathcal{O}_2(x,\partial_x).T_1(x,y)\theta(x-y)$ (where $\mathcal{O}_1^{-1}(x,\partial_x) = T_1(x,y)\theta(x-y)$ and $\mathcal{O}_2^{-1}(x,\partial_x) = T_2(x,y)\theta(x-y)$).

2 Green's function for linear differential operators in one variable with other boundary conditions

In previous part the causal Green's function was derived according to coefficient functions of linear differential operator, here in this part it is shown that Green's function of general linear differential operator, for other boundary conditions on $[a, b]$ (e.g. Sturm-Liouville problem), can also be derived according to coefficient functions of differential operator. First we note that homogeneous linear differential equation of degree n in one variable, $\left(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\partial_x^k\right) u(x) = 0$, for initial condition $\partial_x^i u(a) = c_i$ for $i = 0, 1, .., n-1$ can be converted to Volterra's integral equation of second kind (see [3]) as; $\left(\mu(x) + \int_a^x dz\, K(x,z)\,\mu(z)\right) = S(x)$, where $K(x,z) = \left(\sum_{k=0}^{n-1} P_k(x)\frac{(x-z)^{n-k-1}}{(n-k-1)!}\right)$, $u(x) = D(x) + \int_a^x dz \frac{(x-z)^{n-1}}{(n-1)!}\mu(z)$, $D(x) = \sum_{k=0}^{n-1} \frac{c_k(x-a)^k}{k!}$ and $S(x) = -\sum_{i=0}^{n-1}\sum_{k=0}^{i} c_i \frac{P_k(x)(x-a)^{i-k}}{(i-k)!}$. Therefore we state the following theorem;

Theorem 3 *The solution of homogeneous linear differential equation* $\big(\partial_x^n + \sum_{k=0}^{n-1} P_k(x)\,\partial_x^k\big)\, u(x) = 0$, *where* $P_i(x)$ $(i = 0, 1, ..., n-1)$ *are in* $\mathbb{C}[a, b]$, *with initial condition;* $\partial_x^i u(a) = c_i$ *for* $i = 0, 1, .., n-1$, *is given by*

$$u(x) = D(x) + \int_a^x dz\, T(x,z) S(z), \tag{20}$$

where $D(x) = \sum_{k=0}^{n-1} \frac{c_k(x-a)^k}{k!}$ *and* $S(x) = -\sum_{i=0}^{n-1}\sum_{k=0}^{i} c_i \frac{P_k(x)(x-a)^{i-k}}{(i-k)!}$.

It is easy to check that for boundary condition; $\partial_x^i u(b) = c_i$ for $i = 0, 1, .., n-1$ the answer for homogenous differential equation in $\mathbb{C}[a, b]$ is given by (20) in which a is replaced by b. One can derive solutions of homogenous differential equation, with specific boundary conditions on either a or b, by using (20). By substituting solutions of homogenous differential equation with suitable boundary conditions (derived by (20)) in to the expressions for Green's functions given by method of variation of parameters, one can derive the Green's function according to coefficient functions of differential operator. For example the Green's function for Sturm-Lioville problem;

$$(\partial_x^2 - P(x))y(x) = g(x), \qquad \text{B.C.} \quad y(a) = y(b) = 0, \tag{21}$$

by method of variation of parameters is given by; $G(x,y) = (W(y))^{-1}$ $\big(u_1(x)u_2(y)\,\theta(y-x) + u_2(x)u_1(y)\ \theta(x-y)\big)$, where $u_1(x)$ and $u_2(x)$ are answers of corresponding homogenous differential equation, $W(y)$ is the Wronskian and we have $u_1(a) = u_2(b) = 0$. If we take B.C. $u_1(a) = 0$ and $\partial_x u_1(a) = 1$ then according to (20); we have $u_1(x) = (x-a) + \int_a^x dz T_s(x,z)P(z)(z-a) = T_s(x,a)$ (The function $T_s(x,y)$ is given by (8)). On the other hand by taking B.C. $u_2(b) = 0$ and $\partial_x u_2(b) = 1$ we have $u_2(x) = T_s(x,b)$. The Wronskian of $u_1(x)$ and $u_2(x)$ is constant and can be calculated easily at point $x = a$ as; $W(y) = -T_s(a,b)$ therefore;

$$G(x,y) = (-T_s(a,b))^{-1}\Big(T_s(x,a)T_s(y,b)\theta(y-x) + (x \leftrightarrow y)\Big). \tag{22}$$

3 Conclusion

Causal Green's function for general linear differential operator in one variable was given by [theorem 1] as series of integrals. Multiplicative property of causal Green's functions is shown by [theorem 2]. For differential operators which are equal to multiplications of first order linear differential operators, explicit formula (15), was given for causal Green's functions. An infinite non-abelian group of operators on a subspace of $\mathbb{C}^\infty[a,b]$ is introduced. Analytic formula for fundamental solutions of homogenous linear differential equation in one variable was given via equation (7) and for specific boundary condition via equation (20) as series of integrals. By using equation (20) and the method of variation of parameters a way to derive Green's functions with arbitrary boundary conditions in one variable according to coefficient functions of differential operators was given.

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