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Ordinary differential equations

Doubly periodic wave solutions of some nonlinear evolution equations using sinh-Gordon equation expansion method

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Abstract

Using the sinh-Gordon expansion method, doubly periodic solutions in terms of Jacobian elliptic functions have been derived for a magma equation, a new Hamiltonian amplitude equation and a coupled nonlinear wave equation. All the solutions obtained are new ones.

1 Introduction

Investigation of exact solutions of nonlinear evolution equations[1)] has been a hot topic of research for several decades. The powerful methods used for this purpose are for example, Backlund transformation[2)], inverse scattering technique[3)], Hirota's direct method[4)], tanh method[5)], series method[6–7)], Jacobian elliptic function expansion method and its extension[8–10)], and the algebraic method[11)].

In this paper, we employ the method developed in [12,13] which proposes a transformation from the sinh- Gordon equation which reveals a relationship with nonlinear wave equations. The transformation and the sinh-Gordon equation are used to construct the doubly periodic Jacobian elliptic function solutions of nonlinear wave equations. We give a brief description of the method.

The travelling wave transformation $u(x,t) = u(\xi),\ \xi = k\,(x - \lambda t)$ reduces the sinh-Gordon equation

$$\frac{\partial^2 \phi}{\partial x\, \partial t} = \alpha \sinh \phi \tag{1.1}$$

to an ordinary differential equation

$$\frac{d^2 \phi}{d\,\xi^2} = -\frac{\alpha}{k\,\lambda} \sinh \phi, \tag{1.2}$$

where α is a constant, k and λ are the wave number and wave speed respectively.

Integrating (1.2) w.r.t. ξ once, we get

$$\left(\frac{d}{d\xi}\frac{\phi}{2}\right)^2 = -\frac{\alpha}{k\lambda} \sinh^2\left(\frac{\phi}{2}\right) + c, \tag{1.3}$$

where c is a constant of integration.

Setting $\phi = 2\,w,\ \ -\dfrac{\alpha}{k\lambda} = 1,$ (1.3) reduces to

$$\left(\frac{dw}{d\xi}\right)^2 = \sinh^2 w + c. \tag{1.4}$$

Taking $c = 1 - m^2$, where $m\,(0 < m < 1)$ is the modulus of the Jacobian elliptic functions, we can easily see that (1.4) has the general solution

$$\sinh\left[w(\xi)\right] = cs(\xi; m), \tag{1.5a}$$

or

$$\cosh\left[w(\xi)\right] = ns(\xi; m), \tag{1.5b}$$

which are two of the 12 Jacobian elliptic functions given by[14)]

$$cs(\xi;m) = \frac{cn(\xi,m)}{sn(\xi;m)}, \quad ns(\xi;m) = \frac{1}{sn(\xi;m)}, \tag{1.6}$$

with the properties

$$\begin{aligned} \frac{d\,cs(\xi;m)}{d\,\xi} &= -\,ns(\xi;m)\,ds(\xi;m), \\ \frac{d\,ns(\xi;m)}{d\,\xi} &= -\,cs(\xi;m)\,ds(\xi;m), \\ ns^2(\xi;m) &= 1 + cs^2(\xi;m) \end{aligned} \tag{1.7}$$

For a given nonlinear partial differential equation, we seek its travelling wave solution which reduces it to a nonlinear ordinary differential equation. By using the new variable $w = w(\xi)$, we assume that the ODE has a solution in the form

$$u(\xi) = u(w(\xi)) = A_0 + \sum_{i=1}^{s} \cosh^{i-1} w\,[A_i \sinh w + B_i \cosh w], \tag{1.8}$$

where $A_i\,(i = 0,1,2,...s)$, $B_j\,(j = 1,2,...s)$ are constants to be determined later. Substituting (1.8) in the reduced ODE and balancing the highest derivative term and the nonlinear term, we obtain the value of s.

In section 2, we derive some new doubly periodic solutions for magma equations with two sets of values of the physical parameters n and m, in section 3 we investigate the doubly periodic solutions for a new Hamiltonian amplitude equation and in section 4, we consider a coupled nonlinear wave equation for the Jacobian elliptic function solutions.

2 Magma equations

We consider the magma equation which describes the motion of melt in the Earth[15)]. The buoyancy force owing to the density difference of the liquid phase of melt and the solid phase of matrix causes the melt in the earth's mantle propagate through the partially molten rock. This flow of melt is like a porous flow. Assuming that the liquid phase of melt and the solid phase of matrix are fully connected and incompressible, neglecting the phase transition and allowing only vertical motions, Scott and Stevenson proposed an equation

$$u_t = \frac{\partial}{\partial x}\left[u^n\left\{\frac{\partial}{\partial x}(u^{-m}\,u_t) - 1\right\}\right], \tag{2.1}$$

where x is the vertical space coordinate and t is the time and $u(x,t)$ is the mean volume fraction of the liquid phase which should be nonnegative for any x and t. The exponents n and m denote the dependence of permeability and effective viscosity. It is suggested that the reasonable values of n and m are $2 \sim 5$ and $0 \sim 1$ respectively.

In this section we will consider two different sets of values of the parameters n and m.

Solutions for $n = 4$ and $m = 0$

For this choice of parameters, equation (2.1) reduces to

$$u_t = \big(u^4(u_{xt} - 1)\big)_x \tag{2.2}$$

We seek travelling wave solutions of (2.2) in the form $u(x,t) = u(z)$, $z = k(x - \lambda\, t)$ so that (2.2) reduces to

$$\frac{k^2\,\lambda}{2}\,u_z^2 + u + \frac{\lambda}{2}\,u^{-2} + A\,u^{-3} + B = 0, \tag{2.3}$$

where, A and B are constants of integration.

Using the independent variable transformation

$$\xi = \int^z u^{-3/2}\,dz, \tag{2.4}$$

equation (2.3) becomes

$$\frac{\lambda}{2} + C\,u^2 + 4\,u^3 + k^2\,\lambda\,u_{\xi\xi} = 0, \tag{2.5}$$

where C is the integration constant.

Assuming a solution in the form (1.8) for equation (2.5) and balancing the highest order derivative term with the nonlinear term, we get $3s = s + 2$ which gives $s = 1$.

Therfore, we have our solution in the form

$$u(\xi) = A_0 + A_1\,\sinh w(\xi) + B_1\,\cosh w(\xi), \tag{2.6}$$

where, A_0, A_1, B_1 are constants.

Substituting (2.6) in equation (2.5) and equating the coefficients of powers of $\cosh w \sinh w$ to zero, we get

$$4\,B_1^3 + 12\,A_1^2\,B_1 + 2\,k^2\,\lambda\,B_1 = 0, \tag{2.7}$$

$$4\,A_1^3 + 12\,A_1\,B_1^2 + 2\,k^2\,\lambda\,A_1 = 0, \tag{2.8}$$

$$C\,A_1^2 + C\,B_1^2 + 12\,A_0\,A_1^2 + 12\,A_0\,B_1^2 = 0, \tag{2.9}$$

$$2\,C\,A_0\,A_1 + 12\,A_0^2\,A_1 + 12\,A_1\,B_1^2 + k^2\,\lambda\,A_1 + k^2\,\lambda\,c\,A_1 = 0, \tag{2.10}$$

$$2\,C\,A_0\,B_1 - 12\,A_1^2\,B_1 + 12\,A_0^2\,B_1 - 2\,k^2\,\lambda\,B_1 + k^2\,\lambda\,c\,B_1 = 0, \tag{2.11}$$

$$2\,C\,A_1\,B_1 + 24\,A_0\,A_1\,B_1 = 0, \tag{2.12}$$

$$\frac{\lambda}{2} + C\,A_0^2 - C\,A_1^2 + 4\,A_0^3 - 12\,A_0\,A_1^2 = 0. \tag{2.13}$$

By solving the system of equations (2.7) – (2.13), we have

Case 1:

$$A_0 = -\frac{C}{12}, \quad A_1 = 0, \quad B_1 = \pm\sqrt{\frac{C^2}{24\,(1+m^2)}}, \quad k = \pm\sqrt{-\frac{C^2}{12\,\lambda\,(1+m^2)}}. \tag{2.14}$$

For (2.14) to be valid, λ should be negative.

Case 2:

$$A_0 = -\frac{C}{12}, \quad B_1 = 0, \quad A_1 = \pm\sqrt{-\frac{C^2}{24\,(2-m^2)}}, \quad k = \pm\sqrt{\frac{C^2}{12\,\lambda\,(2-m^2)}}. \tag{2.15}$$

Equations (2.15) will be valid only for negative values of λ.

Thus we have two new Jacobian elliptic function solutions

$$u(\xi) = -\frac{C}{12} \pm \sqrt{\frac{C^2}{24\,(1+m^2)}}\;ns(\xi), \tag{2.16}$$

and

$$u(\xi) = -\frac{C}{12} \pm \sqrt{-\frac{C^2}{24\,(2-m^2)}}\;cs(\xi). \tag{2.17}$$

Solutions for $n = 3$ and $m = 0$

For this choice of parameters, equation (2.1) reduces to

$$u_t = \big(u^3(u_{xt} - 1)\big)_x \tag{2.18}$$

We seek travelling wave solutions of (2.18) in the form $u(x,t) = u(z),\ z = k(x - \lambda t)$ so that (2.18) reduces to

$$\frac{k^2\lambda}{2}u_z^2 + u + \frac{\lambda}{u} + A\,u^{-2} + B = 0, \tag{2.19}$$

where, A and B are constants of integration.

Using the independent variable transformation

$$\xi = \int^z u^{-1}\,dz, \tag{2.20}$$

equation (2.19) becomes

$$\lambda + C\,u + 3\,u^2 + k^2\lambda\,u_{\xi\xi} = 0, \tag{2.21}$$

where C is the integration constant.

Assuming a solution in the form (1.8) for equation (2.21) and balancing the highest order derivative term with the nonlinear term, we get $2s = s + 2$ which gives $s = 2$.

Therefore, we have our solution in the form

$$u(\xi) = A_0 + A_1\sinh w(\xi) + B_1\cosh w(\xi) + A_2\cosh w\,\sinh w + B_2\cosh^2 w, \tag{2.22}$$

where, A_0, A_1, B_1, A_2, B_2 are constants.

Substituting (2.22) into the equation (2.21) and equating the coefficients of powers of $\cosh w \sinh w$ to zero, we get

$$6\,k^2\lambda\,B_2 + 3\,A_2^2 + 3\,B_2^2 = 0, \tag{2.23}$$

$$2\,k^2\lambda\,B_1 + 6\,B_1\,B_2 + 6\,A_1\,A_2 = 0, \tag{2.24}$$

$$2\,k^2\lambda\,A_1 + 6\,A_1\,B_2 + 6\,A_2\,B_1 = 0, \tag{2.25}$$

$$6\,k^2\lambda\,A_2 + 6\,A_2\,B_2 = 0, \tag{2.26}$$

$$-8k^2\lambda B_2 + 4ck^2\lambda B_2 + CB_2 + 6A_0B_2 - 3A_2^2 + 3B_1^2 + 3A_1^2 = 0, \quad (2.27)$$

$$-2k^2\lambda B_1 + ck^2\lambda B_1 + CB_1 + 6A_0B_1 - 6A_1A_2 = 0, \quad (2.28)$$

$$k^2\lambda A_1 + ck^2\lambda A_1 + CA_1 + 6A_0A_1 + 6A_1B_2 + 6A_2B_1 = 0, \quad (2.29)$$

$$-5k^2\lambda A_2 + 4k^2\lambda cA_2 + 6A_1B_1 + 6A_0A_2 = 0, \quad (2.30)$$

$$2k^2\lambda B_2 - 2ck^2\lambda B_2 + CA_0 - 3A_1^2 + 3A_0^2 + \lambda = 0. \quad (2.31)$$

By solving the system of equations (2.23) – (2.31), we have

$$A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad (2.32)$$

$$A_0 = \frac{4k^2\lambda(1+m^2) - C}{6}, \quad B_2 = \frac{C^2 - 16k^4\lambda^2(1+m^2)^2 - 12\lambda}{24k^2\lambda m^2},$$

$$k = \left[\frac{C^2 - 12\lambda}{8\lambda^2(2+m^2+2m^4)}\right]^{1/4}. \quad (2.33)$$

Therefore, the new Jacobian elliptic function solution for (2.21) is

$$u(\xi) = \frac{4k^2\lambda(1+m^2) - C}{6} + \frac{C^2 - 16k^4\lambda^2(1+m^2)^2 - 12\lambda}{24k^2\lambda m^2}\, ns^2(\xi), \quad (2.34)$$

with k given by (2.33_3).

3 New Hamiltonian amplitude equation

A new Hamiltonian amplitude equation[16)]

$$i\,u_x + u_{tt} + 2\sigma\,|u|^2 u - \epsilon\, u_{xt} = 0, \quad (3.1)$$

where $\sigma = \pm 1$, $\epsilon << 1$, was recently introduced by Wadati et al[17)]. This is an equation which governs certain instabilities of modulated wave trains, with the additional term $-\epsilon\, u_{xt}$ overcoming the ill-posedness of the unstable nonlinear Schrodinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinski equation which arises in dissipative systems and is apparently not integrable.

We let

$$u(x,t) = \phi(\xi)\, e^{i(Kx - \Omega t)}, \quad \xi = k(x - \lambda t). \quad (3.2)$$

Substituting equation (3.2) into equation (3.1), we get

$$k^2(\lambda^2+\epsilon\lambda)\,\phi'' + ik(1+2\lambda\Omega+\epsilon\lambda K+\epsilon\Omega)\,\phi' - (K+\Omega^2+\epsilon K\Omega)\,\phi + 2\sigma\phi^3 = 0, \quad (3.3)$$

with the prime meaning differentiation with respect to ξ.

If we take

$$\lambda = -\frac{1+\epsilon\Omega}{2\Omega+\epsilon K}, \quad (3.4)$$

equation (3.3) is transformed into

$$\alpha\phi'' + \beta\phi + \gamma\phi^3 = 0, \quad (3.5)$$

where

$$\alpha = k^2(\lambda^2+\epsilon\lambda), \quad \beta = -(K+\Omega^2+\epsilon K\Omega), \quad \gamma = 2\sigma. \quad (3.6)$$

Assuming a solution in the form (1.8) for equation (3.5) and balancing the highest order derivative term with the nonlinear term, we get $3s = s+2$ which gives $s = 1$.

Therefore, we have our solution in the form

$$\phi(\xi) = A_0 + A_1 \sinh w(\xi) + B_1 \cosh w(\xi), \quad (3.7)$$

where, A_0, A_1, B_1 are constants.

Substituting (3.7) into equation (3.5) and equating the coefficients of powers of $\cosh w \sinh w$ to zero, we get

$$2\,\alpha\,B_1 + \gamma\,B_1^3 + 3\,\gamma\,A_1^2\,B_1 = 0, \quad (3.8)$$

$$2\,\alpha\,A_1 + \gamma\,A_1^3 + 3\,\gamma\,A_1\,B_1^2 = 0, \quad (3.9)$$

$$3\,\gamma\,A_0\,A_1^2 + 3\,\gamma\,A_0\,B_1^2 = 0, \quad (3.10)$$

$$\alpha\,A_1 + \alpha\,c\,A_1 + \beta\,A_1 + 3\,\gamma\,A_0^2\,A_1 + 3\,\gamma\,A_1\,B_1^2 = 0, \quad (3.11)$$

$$-2\,\alpha\,B_1 + \alpha\,c\,B_1 + \beta\,B_1 - 3\,\gamma\,A_1^2\,B_1 + 3\,\gamma\,A_0^2\,B_1 = 0, \quad (3.12)$$

$$6\,A_0\,A_1\,B_1\,\gamma = 0, \quad (3.13)$$

$$\beta\,A_0 + \gamma\,A_0^3 - 3\,\gamma\,A_0\,A_1^2 = 0. \quad (3.14)$$

By solving the system of equations (3.8) – (3.14), we find that for this equation, the only solution possible is when $A_0 = 0$ and $A_1^2 = B_1^2$.

Therefore, we have

$$A_1 = \pm\sqrt{-\frac{\alpha}{2\,\gamma}}, \quad k = \pm\sqrt{-\frac{2\,(K + \Omega^2 + \epsilon\, K\,\Omega)}{(\lambda^2 + \epsilon\,\lambda)\,(2\,m^2 - 1)}}. \tag{3.15}$$

so that the solution is

$$\phi = \pm\sqrt{-\frac{\alpha}{2\,\gamma}}\,(cs(\xi) \pm ns(\xi)) \tag{3.16}$$

Thus the new doubly periodic solution for the equation (3.1) is,

$$u(x,t) = \pm\sqrt{-\frac{\alpha}{2\,\gamma}}\,[cs\,\{k(x-\lambda t)\} \pm ns\,\{k(x-\lambda t)\}]\,e^{i(Kx-\Omega t)}, \tag{3.17}$$

where k is given by (3.15_2).

4 Coupled nonlinear wave equation

We consider a coupled nonlinear wave equation[18)]

$$u_t + \alpha\,v^2\,v_x + \beta\,u^2\,u_x + \lambda\,u\,u_x + \gamma\,u_{xxx} = 0, \tag{4.1}$$

$$v_t + \delta\,(uv)_x + \epsilon\,v^3\,v_x = 0. \tag{4.2}$$

We seek travelling wave solutions in the form

$$u(x,t) = u(\xi),\ \ v(x,t) = v(\xi),\ \ \xi = k(x-\lambda t),$$

so that equation (4.2) reduces to

$$v^3 = \frac{4}{\epsilon}\,(\lambda - \delta\,u), \tag{4.3}$$

and we arrive at the equation for u in the form

$$k^2\,u_{\xi\xi} + P + Q\,u + R\,u^2 + S\,u^3 = 0, \tag{4.4}$$

where

$$P = \frac{4\,\alpha\,\delta}{3\,\gamma\,\epsilon} + \frac{C}{\gamma}, \quad Q = -\left(\frac{\lambda}{\gamma} + \frac{4\,\alpha\,\delta}{3\,\epsilon\,\gamma}\right), \quad R = \frac{\lambda}{2\,\gamma}, \quad S = \frac{\beta}{3\,\gamma}, \tag{4.5}$$

with C being an integration constant.

Assuming a solution in the form (1.8) for equation (4.4) and balancing the highest order derivative term with the nonlinear term, we get $3s = s + 2$ which gives $s = 1$.

Therefore, we have our solution in the form

$$u(\xi) = A_0 + A_1 \sinh w(\xi) + B_1 \cosh w(\xi), \tag{4.6}$$

where, A_0, A_1, B_1 are constants.

Thus we obtain the equations determining $A_0,\ A_1,\ B_1$ and k as

$$2\,k^2\,B_1 + S\,B_1^3 + 3\,S\,A_1^2\,B_1 = 0, \tag{4.7}$$

$$2\,k^2\,A_1 + S\,A_1^3 + 3\,S\,A_1\,B_1^2 = 0, \tag{4.8}$$

$$R\,A_1^2 + R\,B_1^2 + 3\,S\,A_0\,A_1^2 + 3\,S\,A_0\,B_1^2 = 0, \tag{4.9}$$

$$k^2\,A_1 + k^2\,c\,A_1 + P\,A_1 + 2\,R\,A_0\,A_1 + 3\,S\,A_0^2\,A_1 + 3\,S\,A_1\,B_1^2 = 0, \tag{4.10}$$

$$-\,2\,k^2\,B_1 + k^2\,c\,B_1 + P\,B_1 + 2\,R\,A_0\,B_1 - 3\,S\,A_1^2\,B_1 + 3\,S\,A_0^2\,B_1 = 0, \tag{4.11}$$

$$2\,R\,A_1\,B_1 + 6\,S\,A_0\,A_1\,B_1 = 0 \tag{4.12}$$

$$P + Q\,A_0 + R\,A_0^2 - R\,A_1^2 + S\,A_0^3 - 3\,S\,A_0\,A_1^2 = 0 \tag{4.13}$$

Solving the system of equations (4.7) – (4.13), we have

Case 1:

$$\begin{gathered} A_0 = -\frac{R}{3\,S}, \quad A_1 = 0, \\ B_1 = \pm\sqrt{\frac{2\,(R^2 - 3\,P\,S)}{3\,S^2\,(1+m^2)}}, \\ k = \pm\sqrt{-\frac{(R^2 - 3\,P\,S)}{3\,S\,(1+m^2)}}. \end{gathered} \tag{4.14}$$

Case 2:

$$A_0 = -\frac{R}{3\,S}, \;\; B_1 = 0,$$
$$A_1 = \pm\sqrt{-\frac{2\,(R^2 - 3\,P\,S)}{3\,S^2\,(2-m^2)}}, \tag{4.15}$$
$$k = \pm\sqrt{\frac{R^2 - 3\,P\,S}{3\,S\,(2-m^2)}}.$$

Case 3:

$$A_0 = -\frac{R}{3S}, \;\; A_1^2 = B_1^2,$$
$$A_1 = \pm\sqrt{\frac{R^2 - 3\,P\,S}{3\,S^2\,(2m^2-1)}}, \tag{4.16}$$
$$k = \pm\sqrt{-\frac{2(R^2 - 3\,P\,S)}{3\,S\,(2m^2-1)}}.$$

Therefore, we have 3 new doubly periodic Jacobian elliptic function solutions such as

$$u_1(\xi) = -\frac{R}{3S} \pm \sqrt{\frac{2(R^2 - 3\,P\,S)}{3\,S^2\,(1+m^2)}}\; ns(\xi), \tag{4.17}$$
$$u_2(\xi) = -\frac{R}{3S} \pm \sqrt{-\frac{2(R^2 - 3\,P\,S)}{3\,S^2\,(2-m^2)}}\; cs(\xi), \tag{4.18}$$
$$u_3(\xi) = -\frac{R}{3S} \pm \sqrt{\frac{R^2 - 3\,P\,S}{3\,S^2\,(2m^2-1)}}\; \left(cs(\xi) \pm ns(\xi)\right). \tag{4.19}$$

and

$$v_1(\xi) = \left[\frac{4}{\epsilon}\left\{\lambda + \frac{\delta R}{3S} \mp \delta\sqrt{\frac{2(R^2 - 3\,P\,S)}{3\,S^2\,(1+m^2)}}\; ns(\xi)\right\}\right]^{1/3}, \tag{4.20}$$
$$v_2(\xi) = \left[\frac{4}{\epsilon}\left\{\lambda + \frac{\delta R}{3S} \mp \delta\sqrt{-\frac{2(R^2 - 3\,P\,S)}{3\,S^2\,(2-m^2)}}\; cs(\xi)\right\}\right]^{1/3}, \tag{4.21}$$

$$v_3(\xi) = \left[\frac{4}{\epsilon}\left\{\lambda + \frac{\delta R}{3S} \mp \delta\sqrt{\frac{R^2 - 3\,P\,S}{3\,S^2\,(2m^2-1)}}\ (cs(\xi) \pm ns(\xi))\right\}\right]^{1/3}. \quad (4.22)$$

5 Conclusion

The sinh-Gordon equation expansion method has been used to derive doubly periodic solutions in terms of Jacobian elliptic functions. For the magma equation we have derived two new Jacobian elliptic function solutions in the case of $n = 4$, $m = 0$ and one new solution in the case of $n = 3$, $m = 0$. A new solution as a combination of $cs(\xi; m)$ and $ns(\xi; m)$ has been derived for the new Hamiltonian amplitude equation. Also, we have derived 3 new solutions for a coupled nonlinear wave equation in terms of Jacobian elliptic functions.

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