*DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 1, 2011
Electronic Journal,
reg. N ФС77-39410 at 15.04.2010
ISSN 1817-2172*

*http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru*

Partial Differential Equations for Wave Pockets in the Minkowski 4-dimensional Spaces

Zoran Majkić
International Society for Research in Science and Technology
PO Box 2464 Tallahassee, FL 32316 - 2464 USA
email: majk.1234@yahoo.com,
http://zoranmajkic.webs.com/

Abstract

In this paper we presented the partial differential equations [4] for the wave-pockets in the Minkowski 4-dimensional spaces. In particular we considered the stationary cases for the matter wave-pockets, by considering that in all cases, the geometric form (the matter distribution Φ) of a particle in a given time-instance t depends on the particular boundary conditions for these differential equations as well.

In physics and mathematics, Minkowski space (or Minkowski time-space) is the mathematical setting in which Einstein's theory of special relativity is most conveniently formulated. In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional manifold for representing a time-space.
In theoretical physics, Minkowski space is often contrasted with Euclidean space. While a Euclidean space has only spacelike dimensions, a Minkowski space also has one timelike dimension. We define the basic time-space four mutually orthogonal vectors $e_j, 0 \leq j \leq 3$, by the following matrix:

$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \text{ with imaginary number } \ i = \sqrt{-1}.$$

Note that the matrix above is a particular case of the Minkowski tensor using a four-dimensional time-space, which combines the real dimension of time with the three imaginary dimensions of space.

Consequently, a vector of position in this space-time 4-dimensional system is given by

$\overrightarrow{\mathbf{r}_4} = cte_0 + xe_1 + ye_2 + ze_3 = cte_0 + \overrightarrow{\mathbf{r}}$, where t is the time (i.e., ct is the timelike component of $\overrightarrow{\mathbf{r}_4}$, where c is the velocity of light in the vacuum) and $\overrightarrow{\mathbf{r}} = xe_1 + ye_2 + ze_3$ is an ordinary Euclidean vector with x, y, z three spatial coordinates.

Its infinitesimal amount is defined by $d\overrightarrow{\mathbf{s}} = cdte_0 + dxe_1 + dye_2 + dze_3$, where dt, dx, dy and dz are infinitesimal amounts of time-space dimensions.

Thus, in this 4-dimensional system the time is real while the three orthogonal space coordinates are imaginary. This choice is adopted in order to have that the distance

$d\mathbf{s}^2 = d\overrightarrow{\mathbf{s}}d\overrightarrow{\mathbf{s}} = (cdt)^2 - dx^2 - dy^2 - dz^2$,

for all local time-space reference systems of observations of quantum events be the positive real value (where space dimensions are limited).

An angular wavenumber vector in this four-dimensional time-space is given by $\overrightarrow{\mathbf{k}_4} = k_t e_0 + k_x e_1 + k_y e_2 + k_z e_3$.

In what follows we will denote by $\overrightarrow{\mathbf{k}} = k_x e_1 + k_y e_2 + k_z e_3$ the spatial component of the angular wavenumber vector, with $k^2 = |\overrightarrow{\mathbf{k}}\overrightarrow{\mathbf{k}}| = k_x^2 + k_y^2 + k_z^2$, so that $k_4^2 = \overrightarrow{\mathbf{k}_4}\overrightarrow{\mathbf{k}_4} = k_t^2 - k^2$.

The mutually independent space-components are defined as usual by $k_x = \frac{2\pi}{\lambda_x}$, $k_y = \frac{2\pi}{\lambda_y}$, $k_z = \frac{2\pi}{\lambda_z}$, where $\lambda_x, \lambda_y, \lambda_z$ are spatial wavelengths w.r.t the axes x, y and z respectively, and $\lambda = \frac{2\pi}{k}$ is the (total) spatial wavelength. Let $\omega = 2\pi\nu$ be an angular frequency that depends on the space-components, $\nu = \frac{1}{T}$ with a time period T. Thus, $\lambda_t = cT$ is the time-like wavelength and $k_t = \frac{2\pi}{\lambda_t} = \frac{\omega(k)}{c}$ depends on the space-components in $\overrightarrow{\mathbf{k}_4}$, so that it holds that $dk_4 = dk = dk_x dk_y dk_z$, and $-\overrightarrow{\mathbf{k}_4}\overrightarrow{\mathbf{r}_4} = k_x x + k_y y + k_z z - \omega(k)t$.

Remark: from the relativistic theory we have that for each massive elementary particle (with rest mass m_0 greater than zero) it holds that $\omega(k) = \pm c\sqrt{k^2 + (m_0 c \setminus \hbar)^2}$, where $\hbar = \frac{h}{2\pi}$ is the Dirac's constant, for the Planck's

constant $h = 6.6210^{-34} Js$.
Note that we assume that ω can be positive or negative (clockwise or counter-clockwise angular frequency), so that the energy of the particle is $E = \hbar|\omega|$, where $|\_|$ denotes the absolute value. In the rest of this paper we will consider the cases when ω is positive.
Notice that if we represent an elementary particle, with energy $E = \hbar\omega$ and momentum $\overrightarrow{\mathbf{p}} = \hbar\overrightarrow{\mathbf{k}}$, by a single harmonic $A\mathrm{e}^{-i\overrightarrow{\mathbf{k}_4}\overrightarrow{\mathbf{r}_4}}$ in this four-dimensional space, where $\overrightarrow{\mathbf{k}_4} = \frac{\omega}{c}e_0 + \overrightarrow{\mathbf{k}} = \frac{E}{\hbar c}e_0 + \frac{\overrightarrow{\mathbf{p}}}{\hbar}$, then we obtain that $k_4^2 = (\frac{\omega}{c})^2 - k^2 = 0$ for particles with rest mass equal to zero (photons, gravitons, etc..), and $k_4^2 > 0$ for the massive particles (with rest mass m_0 greater than zero). In fact, we obtain that $|k_4| = \frac{\omega_0}{c}$ where $\omega_0 = \frac{m_0c^2}{\hbar}$ is the invariant angular frequency for particles (analog to the invariant rest mass m_0 of particles). Thus, similarly to the 3-dimensional angular wavenumber vector $\overrightarrow{\mathbf{k}}$ that in physics means the particle's momentum, the 4-dimensional angular wavenumber $\overrightarrow{\mathbf{k}_4}$ has a physical meaning as the particle's relativistically invariant angular frequency.
□
The plan of this paper is the following: In Section 1 we present an introduction to matter-events in the Minkowski's space, and we specified their stationary conditions for the massive and massless elementary particles. In Section 2 we specified the formal definition for matter-events as propagation of wave-pockets, and we presented an example for the propagation of photons (light). The main result is developed and presented in Section 3, in two propositions with proofs, for the first and for the second-order partial differential equations of these wave-pockets. Finally, in Section 4 are given applications of these results to the quantum mechanics, with two significant examples.

1 Introduction to matter-events

In any given instance of time t, any matter-event in this time-space is a particular time-space perturbation $\Psi(\overrightarrow{\mathbf{r}_4})$, that can be mathematically given by the following Fourier transformation:
$\Psi(\overrightarrow{\mathbf{r}_4}) = \Psi(x, y, z, t) = \int C(k_4)\mathrm{e}^{i(-\overrightarrow{\mathbf{k}_4}\overrightarrow{\mathbf{r}_4})}dk_4 =$
$= \int A(k)\mathrm{e}^{i(-\overrightarrow{\mathbf{k}}\overrightarrow{\mathbf{r}}-\omega(k)t)}dk =$ (where $A(k) = C(k_4) = C(\sqrt{(\frac{\omega(k)}{c})^2 - k^2}\ )$)
$= \int\int\int_{-\infty}^{+\infty} A(k)\mathrm{e}^{i(k_xx+k_yy+k_zz-\omega(k)t)}dk_xdk_ydk_z$.
It is a space-distribution of a particle in a given instance of time t, and it changes in time, that is, the amplitudes $A(k)$ are generally dependent on time

as well.
Mathematically, these matter-events are complex functions, composed by one real and one imaginary component. The amplitudes $A(k)$ of the harmonics, in a given instance of time t, are given by inverse Fourier transformation,
$A(k) = \int\int\int_{-\infty}^{+\infty} \Psi(x,y,z,t)\mathrm{e}^{-i(k_x x + k_y y + k_z z - \omega(k)t)} dxdxdz.$
The elementary particles are pocket waves that propagate in this four-dimensional space.
Thus, for such particular *stationary* cases we have that $d\omega(k)/d\overrightarrow{\mathbf{k}}$ is constant (that is, it does not depend on $\overrightarrow{\mathbf{k}}$), equal to the particle's velocity $-\overrightarrow{\mathbf{v}} = -v_x e_1 - v_y e_2 - v_z e_3$ (negative sign is the consequence that $e_i, i \geq 1$ are imaginary, thus the scalar products of (only) spatial vectors are negative), that can depend on the time t as well.
Consequently, for any fixed instance of time t, by integration we obtain that,
(0) $\int_{\overrightarrow{\mathbf{k}_0}}^{\overrightarrow{\mathbf{k}}} d\omega = \omega(k) - \omega(k_0) = -\int_{\overrightarrow{\mathbf{k}_0}}^{\overrightarrow{\mathbf{k}}} \overrightarrow{\mathbf{v}} d\overrightarrow{\mathbf{k}} = -\overrightarrow{\mathbf{v}} \int_{\overrightarrow{\mathbf{k}_0}}^{\overrightarrow{\mathbf{k}}} d\overrightarrow{\mathbf{k}} = -\overrightarrow{\mathbf{v}}(\overrightarrow{\mathbf{k}} - \overrightarrow{\mathbf{k}_0}),$
where the constant $\overrightarrow{\mathbf{k}_0} = \frac{\overrightarrow{\mathbf{p}}}{\hbar}$ for a given momentum $\overrightarrow{\mathbf{p}} = p_x e_1 + p_y e_2 + p_z e_3$ of a particle that is collinear with the velocity $\overrightarrow{\mathbf{v}}$, that is, $\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}} = -pv$. Because of that we can write $\overrightarrow{\mathbf{p}} = p\overrightarrow{i_v}$, $\overrightarrow{\mathbf{v}} = v\overrightarrow{i_v}$, where $\overrightarrow{i_v}$ is unitary vector tangent to the trajectory of a particle (i.e., $\overrightarrow{i_v}\,\overrightarrow{i_v} = -1$). Thus,
$\omega(k) = \omega_0 + v_x(k_x - \frac{p_x}{\hbar}) + v_y(k_y - \frac{p_y}{\hbar}) + v_z(k_z - \frac{p_z}{\hbar}),$
where ω_0 denotes the constant $\omega(k_0)$ that does not depend on k but may depend on time as we will see in what follows.
The *phase velocity* of a particle's pocket-wave, observed in a given referential system, is defined by $\vartheta = \frac{\omega_0}{k_0}$.
The constant ω_0 is determined as follows in the following two cases, by considering that the angular frequency $\omega(k)$ for its particular values is correlated by De Broglie [1] to the total energy o particle $E = \hbar\omega(k)$:

- Case for *massive particles* (with rest mass $m_0 > 0$), denominated as mass-particles as well: when $\overrightarrow{\mathbf{v}} = 0$ then the energy of this particle is $E = m_0 c^2$, that is, the energy in the rest-state of this particle. Consequently, from (0) we have that $\omega_0 = \omega(k) = \frac{m_0 c^2}{\hbar}$.
  Consequently, for the total energy of these mass-particles that propagates with velocity $v = |\overrightarrow{\mathbf{v}}|$, with $\beta = \frac{v}{c}$, it holds that,
  $E = \sqrt{(m_0 c^2) + (pc)^2} = \sqrt{(\hbar\omega_0)^2 + (pc)^2} = \frac{m_0 c^2}{\sqrt{1-\beta^2}} = \hbar\omega_v,$
  where $\omega_v = \omega_0/\sqrt{1-\beta^2}$ is a computed angular frequency relative to the velocity v of this particle w.r.t. the reference system of an observer (for different observes in different referential systems, that move with different

velocities, this computed value of the *same* observed particle is different). For an observer in a given fixed position (the origin of its coordinate system, for example), this observed particle's frequency is by Lorentz low slowed down by the factor $\sqrt{1-\beta^2}$, so that the *really observed* particle's angular frequency of the observed wave-pocket $\Psi(\overrightarrow{\mathbf{r}_4})$ given above is constant and equal to $\omega_v\sqrt{1-\beta^2}=\omega_0$. Thus, ω_0 is the angular frequency of this massive particle equal in any inertial system (without acceleration), that is, an invariant as is the rest-mass m_0.

- Case for *massless particles* (with rest mass $m_0=0$): they propagate, as usual, with very high velocity $c\geq v>0$ equal to the maximal velocity of light if this particle propagates in the vacuum, thus the zero value of equation (0) we can obtain when $\overrightarrow{\mathbf{k}}=\overrightarrow{\mathbf{k}}_0=\frac{\overrightarrow{\mathbf{p}}}{\hbar}$. Consequently, the value of E is the *total* energy of this particle with the given momentum $\overrightarrow{\mathbf{p}}$, so that $\omega_0=\omega(k)=\omega(\frac{p}{\hbar})=\frac{E}{\hbar}$. The total energy of massless particles is defined by $E=\hbar\omega_o=(\hbar k_0)\vartheta=p\vartheta$. When a particle propagates in the vacuum then $\vartheta=c$, so that $E=pc$.
  When the total energy changes in time, to a fixed observer this angular frequency appears to change as well (for example, the relativistic effects for red-shifting of photons for a fixed observer). Thus, differently from massive particles where for a fixed observer $\frac{\partial\omega_0}{\partial t}=0$, here ω_0 can change in time, if a particle changes its total energy during the propagation.

2 Definition of the wave-pocket in the Minkowski pseudoeuclidean space

Consequently, from the introduction to matter-events given in the previous Section, for the wave-pocket of an elementary particle, and given reference system, we have that

(1) $\Psi(x,y,z,t)=\int A(k)\mathrm{e}^{i(-\overrightarrow{\mathbf{k}}\overrightarrow{\mathbf{r}}-\omega(k)t)}dk=$

$=(\int A(k)\mathrm{e}^{-i(\overrightarrow{\mathbf{k}}\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{v}}(\overrightarrow{\mathbf{k}}-\overrightarrow{\mathbf{k}_0})t)}dk)\mathrm{e}^{-i\omega_0 t}=$

$=(\int A(k)\mathrm{e}^{-i(\overrightarrow{\mathbf{k}}-\overrightarrow{\mathbf{k}_0})(\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{v}}t)}dk)\mathrm{e}^{i(-\overrightarrow{\mathbf{k}_0}\overrightarrow{\mathbf{r}}-\omega_0 t)}=$

$=\Phi(\overrightarrow{\mathbf{r}},t)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}=\Phi(\overrightarrow{\mathbf{r}},t)\mathrm{e}^{i\frac{p}{\hbar}(-\overrightarrow{i_v}\overrightarrow{\mathbf{r}}-\vartheta t)}.$

In what follows we introduce the spatial vector $\overrightarrow{\mathbf{u}}=\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{v}}t$, that is equal to zero for the time-space points of particle's trajectory.

The "corpuscular" geometric wave-pocket shape (matter's distribution) of a particle, that *appears to a fixed observer*, is given by

$\Phi(\overrightarrow{\mathbf{r}},t) = \Phi(x,y,z,t) = \int A(k)\mathrm{e}^{-i(\overrightarrow{\mathbf{k}}-\overrightarrow{\mathbf{k}_0})(\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{v}}t)}dk =$
$= \int A(|\overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{k}_0}|)\mathrm{e}^{-i\overrightarrow{\mathbf{k}}\overrightarrow{\mathbf{u}}}dk \ =$
(here we denote by $B(k)$ the value $A(|\overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{k}_0}|)$),
$= \int\int\int_{-\infty}^{+\infty} B(k)\mathrm{e}^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk_xdk_ydk_z =$
$= \int\int\int_{-\infty}^{+\infty} B(k)\mathrm{e}^{i(k_xu_x+k_yu_y+k_zu_z)}dk_xdk_ydk_z.$
In the case when a particle propagates in the vacuum with a constant velocity $\overrightarrow{\mathbf{v}}$ (stationary case), then the coefficients $B(k)$ does not change in time, i.e. $\frac{\partial B(k)}{\partial t} = 0$, so that the "corpuscular" geometry (matter distribution) does not change in time and $\Phi(\overrightarrow{\mathbf{r}},t) = \Phi(\overrightarrow{\mathbf{u}}) = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t) = \Phi(x-v_xt, y-v_yt, z-v_zt)$ is a wave-pocket that propagates with a velocity $\overrightarrow{\mathbf{v}}$.
For instance, in such a stationary case, for a particle with $m_0 = 0$ (for example boson as photon, graviton, etc..) that propagates in the vacuum with a velocity equal to its phase velocity (when $c = |\overrightarrow{\mathbf{c}}| = \vartheta$), so that the total energy is $E = p\vartheta = pc = -\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{c}}$, we have that $\omega_0 = -\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{c}}/\hbar$, so that
$\Psi(x,y,z,t) = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{c}}t)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{c}}t)/\hbar} = \Psi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{c}}t).$
But in non-stationary cases, when a particle changes its velocity and total energy, its distribution ("corpuscular" geometry) Φ changes in time as well, because the coefficients $B(k)$ in the Fourier integral above change in time, that is, $\frac{\partial B(k)}{\partial t} \neq 0$. In such more complex cases with an acceleration of particles, we have both phenomena: particle changes its velocity, momentum and total energy (because of the interaction with another particles), and its corpuscular geometry as well.
There are particular cases, when a particle does not change its total energy and with $\frac{\partial}{\partial t}(\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}}) = 0$, with acceleration caused only by (constant) velocity that changes its direction, when Φ does not change in time its distribution w.r.t. its trajectory, so that it is of the form $\Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)$. For instance in the case of the stationary rotation of electrons around nucleus of an atom.
Thus, based on standard Fourier transformation, the function $\Phi(\overrightarrow{\mathbf{r}},t)$ is a *real* function, differently from $\Psi(x,y,z,t$ that is *complex*. The real and imaginary components of $\Psi(x,y,z,t)$ are determined by the oscillation of the complex oscillator component $\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0t)}$, that is an oscillation identical to the complex *plain wave* (like, for example, the complex electromagnetic plain wave). The amplitudes $B(k)$ of the harmonics can be obtained by the inverse Fourier transformation, for each given instance of time t, by:
$B(k) = \int\int\int_{-\infty}^{+\infty} \Phi(x,y,z,t)\mathrm{e}^{-i(k_xu_x+k_yu_y+k_zu_z)}dk_xdk_ydk_z.$
Thus, generally any particle is determined by the pocket-wave $\Psi(x,y,z,t)$ composed by two sub components: by the corpuscular matter distribution

$\Phi(x, y, z, t)$ that is a real function, and by the 'phase wave' $e^{i(-\frac{\vec{\mathbf{p}}\,\vec{\mathbf{r}}}{\hbar}-\omega_0 t)}$ that is a complex function.

Example 1: speed of light.

When light propagates through a material, it travels slower than the vacuum speed. This is a change in the phase velocity ϑ of the light and is manifested in physical effects such as refraction. The ratio between c and the speed ϑ at which light travels in a material is called the refractive index n of the material ($n = c/\vartheta$). Refraction occurs when light waves travel from a medium with a given refractive index to a medium with another at an angle. At the boundary between the media, the wave's phase velocity is altered, usually causing a change in direction. Its wavelength increases or decreases but its frequency remains constant (thus photons change their momentum and velocity but does not change their energy). This reduction in speed is quantified by the refractive index of the material.

For example, for visible light the refractive index of glass is typically around 1.5, meaning that light in glass travels at $c/1.5 \approx 200,000 km/s$; the refractive index of air for visible light is about 1.0003, so the speed of light in air is very close to c.

Certain materials have an exceptionally low *group* velocity for light waves, a phenomenon called slow light. In 1999, a team of scientists led by Lene Hau were able to slow the speed of a light pulse to about 17 meters per second (61 km/h; 38 mph) [3], they were able to momentarily stop a beam [2].

□

It can be shown that, in the case of these massless elementary particles that propagate in vacuum with the velocity of light c, we have that this "corpuscular" wave pocket in the *stable*(general) state corresponds to the Dirac function (with $A(k) = B(k) = 1/(2\pi)^3$), $\delta(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{c}} t) = \delta(x - c_x t, y - c_y t, z - c_z t) = \delta(x - c_x t)\delta(y - c_y t)\delta(z - c_z t)$. It is reasonable assumption that the volume of a distribution Φ (where it is greater than zero) of a massive particle (with rest mass m_0 greater than zero) is always greater than zero, so that it is a reason that such particles can not reach the limit velocity of light. In the analog way, the massless particles (with rest mass equal to zero) must have, in their stable state, this volume equal to zero, so that their distribution Φ is equal to Dirac function above, and they are able to propagate with the velocity of light in the vacuum. The non stable states of particles with rest mass $m_0 = 0$ can have more complex wave-pocket forms and it happen only in a very short instances of time, when the particle enters in strongly unsymmetric space region, as will be explained in what follows. In such situations its velocity of propagation becomes less than the velocity of light in the vacuum so that this particle can have a similar behavior as massive particles with Φ that occupies a limited but nonzero volume (so called spatial explosion of excited bosons). This unstable state of the particles with $m_0 = 0$ tends to come back into the stable state with Dirac function geometry for distribution Φ.

For any matter-perturbation of an elementary particle that propagates in the 3-dimensional space with a velocity that changes in the time, because of external forces that influence this particle, the 3-dimensional wave-pocket distribution $\Phi(x, y, z, t)$ changes as well, but it must satisfy the *conservation matter* principle, that is, at each given time instance t it must be satisfied the following *invariance* property:

(2) $\mathbf{1}_\Phi = \int\int\int_{-\infty}^{+\infty} \Phi(x, y, z, t)dxdydz = \int\int\int_{-\infty}^{+\infty} \Phi(x, y, z, 0)dxdydz > 0$,

where $\mathbf{1}_\Phi$ is a time-invariant constant value of an elementary particle (not necessarily equal to 1), and $V(t)$ is a finite cube (or sphere) which contains the whole "corpuscular" wave-pocket in a given instance of time t, and $dV = dxdydz$.

We define the minimal (limit) cube $V_m(t) = lim(2^3 \triangle X \triangle Y \triangle Z)$, such that in this time-instance t, $\Phi(x, y, z, t) = 0$ for ($x \leq - \triangle X$ or $x \geq \triangle X$ or $y \leq - \triangle Y$ or $y \geq \triangle Y$ or $z \leq - \triangle Z$ or $z \geq \triangle Z$).

The real function $\Phi(x, y, z, t)$ is the "corpuscular" geometric wave-pocket form of a particle that propagates in the ordinary 3-dimensional space with a velocity $\overrightarrow{\mathbf{v}} = v_x e_1 + v_y e_2 + v_z e_3$). In the stationary case, when it propagates in the vacuum with a constant velocity, it has constant distribution, that propagates as pocket-wave $\Phi(\overrightarrow{\mathbf{u}}) = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t) = \Phi(x - v_x t, y - v_y t, z - v_z t)$.

The interactions between any two pocket-waves (particles) can be obtained only by their local collisions, and depending on their energy and velocities they can produce a kind of Compton effects (elastic collisions) where they survive the collisions by chaining their momentum and energy (with conservation of total momentum and energy), or can make total fusion between them with possible creation of new stable particles (in Feynman's diagrams). In order to be able for two pocket-waves to have a collision, and mutual interference, at least one of them must have a volume $V_m(t)$ (in a given instance of time of mutual collision) greater than zero. So, from this point of view, it can not happen that the distance between any two particles becomes equal to zero, so that we avoid classic infinitary problems of gravitational and electronic fields and forces where the particles are pointlike, so that it is possible to have the distances between particles equal to zero with, consequently, infinite values of gravitational (or electric) forces.

The particles with $V_m(t)$ equal to zero are, for example, the particles with

$\Phi(x-v_xt, y-v_yt, z-v_zt)$ equal to the Dirac function $\delta(x-v_xt, y-v_yt, z-v_zt) = \delta(x-v_xt)\delta(y-v_yt)\delta(z-v_zt)$.

Thus, for any two particles with the "corpuscular" form given by Dirac function, it is impossible to have the collisions in their stable states, but only when they are excited and are involved in their temporary "spatial explosions". Such explosions can happen also when two stable particles are very close one to another so that the ideal spatial symmetry for a free particle in the vacuum does not hold more: it explains why, for example, photons can interact with gravitons (i.e., gravitational field) and may have the gravitational redshifts. Because of that, it will be natural consequence that the massless particles, as bosons (gravitons, photons, etc..) in their stable states, will have the volume $V_m(t)$ equal to zero (with Dirac function for their distribution Φ). In that case they can be used as intermediators between the massive particles (that have the rest mass and the volume $V_m(t)$ greater than zero), that is, to be the quantum-correspondence for the "fields" (the statistical events as gravitational, electromagnetic, etc., that are statistical results of actions of a high number of bosons), by avoiding in more common situations the significant interference between themselves.

This situation can be obtained in the quantum level only if the collisions between gravitons and photons, for example, are practically improbable. Consequently, a number of gravitons and photons can coexist in the same small region of space without any significant direct interference between them during the contemporary collisions with particles with rest mass and volume $V_m(t)$ greater than zero. Also in such situation we can have the cases of the interference between a graviton and a photon, in the situation when they are very, very close in a given instance of time, so that they are involved in temporary spatial explosion (with their volume of distribution greater than zero in that instance of time). In normal situations, these interferences statistically can be neglected, while in the cases of very strong field interactions (when the local density of photons and gravitons is very high) these inter-boson's interactions are significant, so that the gravitation field has strong interactions with the electromagnetic field in a given local space region.

3 Partial differential equations for the wave-pockets in the Minkowski space

Let $\nabla = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}$ be the gradient, so that the Laplasian is defined by $\triangle = -\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Then the derivation of the wave-pocket along its trajectory with the unitary tangent vector $\overrightarrow{i_v}$ on the trajectory, collinear with the vector of its velocity $\overrightarrow{\mathbf{v}} = v\overrightarrow{i_v} = \overrightarrow{i_v}\sqrt{v_x^2 + v_y^2 + v_z^2}$, is denoted by the operator $\overrightarrow{i_v}\nabla$.

In what follows, by $Re()$ and $Im()$ we will denote the real and imaginary component of complex expressions, so that $\gamma = Re(\gamma) + iIm(\gamma)$ for a complex number γ.

Proposition 1 *The geometric distribution $\Phi(\overrightarrow{\boldsymbol{r}}, t)$ of the wave-pockets with the independent space-time variables $\overrightarrow{\boldsymbol{r}}$ and t, of an elementary particle given in (1), is determined by the following differential equations:*

(e.1) $\quad e^{i\omega_0 t}\frac{\partial \Phi e^{-i\omega_0 t}}{\partial t} = -i\frac{\partial(\omega_0 t)}{\partial t})\Phi + \overrightarrow{\boldsymbol{v}_1}\nabla\Phi + \Phi_D(\overrightarrow{\boldsymbol{r}}, t),$ *where,* $\overrightarrow{\boldsymbol{v}_1} = \frac{\partial}{\partial t}(\overrightarrow{\boldsymbol{v}}t)$, $\Phi_D(\overrightarrow{\boldsymbol{r}}, t) = \int_{-\infty}^{+\infty}\frac{\partial B(k)}{\partial t}e^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk_xdk_ydk_z$ *is equal to zero when this particle is in a stationary state, that is, when* $\frac{\partial B(k)}{\partial t} = 0$. *Thus,*

(e.2) $\quad \frac{\partial\Psi}{\partial t} = -i\omega_p\Psi + \overrightarrow{\boldsymbol{v}_1}\nabla\Psi + \Psi_D(\overrightarrow{\boldsymbol{r}}, t),$

where $\Psi_D(\overrightarrow{\boldsymbol{r}}, t) = \Phi_D(\overrightarrow{\boldsymbol{r}}, t)e^{i(-\frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{r}}}{\hbar}-\omega_0 t)}$, *with* $\omega_p = \omega_1 + \frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{v}_1}}{\hbar}$ *and* $\omega_1 = \frac{\partial}{\partial t}(\frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{r}}}{\hbar} + \omega_0 t) = \frac{\overrightarrow{\boldsymbol{r}}}{\hbar}\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t} + \frac{\partial}{\partial t}(\omega_0 t)$, *where the particle's velocity* $\overrightarrow{\boldsymbol{v}}$, *momentum* $\overrightarrow{\boldsymbol{p}}$, *and* ω_0 *in the case of massless particles, may change in time.*

Proof: From the fact (1) we have that
$\frac{\partial}{\partial t}(\Phi e^{-i\omega_0 t}) = -i\frac{\partial(\omega_0 t)}{\partial t}\Phi e^{-i\omega_0 t} + (\frac{\partial\Phi}{\partial t})e^{-i\omega_0 t}$.
Let us show that $\frac{\partial\Phi}{\partial t} = \frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t}\nabla\Phi + \Phi_D$. We can make the following derivation:

(a.0) $\quad \frac{\partial\Phi}{\partial t} = \frac{\partial}{\partial t}(\int\int\int_{-\infty}^{+\infty}B(k)e^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk_xdk_ydk_z)$
$= \int_{-\infty}^{+\infty}(-B(k)i(\frac{\partial(v_xt)}{\partial t}k_x + \frac{\partial(v_yt)}{\partial t}k_y + \frac{\partial(v_zt)}{\partial t}k_z) + \frac{\partial B(k)}{\partial t})e^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk$
$= -\frac{\partial(v_xt)}{\partial t}\int\int\int_{-\infty}^{+\infty}(ik_x)B(k)e^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk_xdk_ydk_z - ...$
$- \frac{\partial(v_zt)}{\partial t}\int\int\int_{-\infty}^{+\infty}(ik_z)B(k)e^{i(k_x(x-v_xt)+k_y(y-v_yt)+k_z(z-v_zt))}dk_xdk_ydk_z + \Phi_D$
$= -\frac{\partial(v_xt)}{\partial t}\frac{\partial\Phi}{\partial x} - \frac{\partial(v_yt)}{\partial t}\frac{\partial\Phi}{\partial y} - \frac{\partial(v_zt)}{\partial t}\frac{\partial\Phi}{\partial z} + \Phi_D = \frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t}\nabla\Phi + \Phi_D$.

Consequently, we obtain,

(a.1) $\quad \frac{\partial}{\partial t}(\Phi e^{-i\omega_0 t}) = (-i\omega_0\Phi - it\frac{\partial\omega_0}{\partial t}\Phi + \frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t}\nabla\Phi + \Phi_D)e^{-i\omega_0 t}$,

and the differential equation (e.1).

From the fact that,

(a.2) $\quad \nabla(\Phi e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}}) = (\nabla\Phi)e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}} + \Phi\frac{i}{\hbar}(p_xe_1 + p_ye_2 + p_ze_3)e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}} =$
$= (\nabla\Phi + \frac{i}{\hbar}\overrightarrow{\mathbf{p}}\Phi)e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}}$,

we obtain that $(\nabla\Phi)e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}} = \nabla(\Phi e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}}) - \frac{i}{\hbar}\overrightarrow{\mathbf{p}}\Phi e^{-i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}}$, and

(a.3) $\quad (\nabla\Phi)e^{i(-\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} = \nabla\Psi - \frac{i}{\hbar}\overrightarrow{\mathbf{p}}\Psi$,

and by substitution in (e.1) we obtain (e.2).

□

Notice that $\frac{\partial \omega_0}{\partial t} \neq 0$ only for the massless particles in their unstable states and very-very short interval of times $\triangle t \approx 0$, when they change their total energy $E = \hbar\omega_0$ during the collisions with another particles (the Compton effects). In what follows we will denote by $\mathcal{E} = \hbar\omega_p$ the energy associated to this angular frequency ω_p.

Example 2: The *stationary case* is obtained when a particle propagates with constant total energy E and constant value $\mathcal{E} = \hbar\omega_p$. Thus, in such a stationary case we have that $\Phi_D(\overrightarrow{\mathbf{r}}, t) = 0, \Psi_D(\overrightarrow{\mathbf{r}}, t) = 0$. It is easy to verify that the stationary case is one, for example, of the following two cases:

1. When a particle propagates with constant momentum $\overrightarrow{\mathbf{p}}$, velocity $\overrightarrow{\mathbf{v}}$ (thus, $\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t} = \frac{\partial \overrightarrow{\mathbf{p}}}{\partial t} = 0$), and total energy E (in that case ω_0 is constant for massless particles as well). Thus, without any acceleration. In that case ω_p is constant as well, with constant $\mathcal{E} = \hbar(\frac{\overrightarrow{\mathbf{r}}}{\hbar}\frac{\partial \overrightarrow{\mathbf{p}}}{\partial t} + \frac{\partial}{\partial t}(\omega_0 t) + \frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}) = \hbar\omega_0 + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}} = \hbar\omega_0 - pv$.

2. When a particle routes with constant radius R around a fixed center, with constant angular velocity $\nu = \frac{|\overrightarrow{\mathbf{v}}|}{R} = \frac{v}{R}$, constant value of the momentum $p = |\overrightarrow{\mathbf{v}}|$, and total energy E. In this case we can obtain ω_p constant in a particular coordinate system: coordinate center of the reference system x, y of the plain in which this particles routes. Then, the position of the trajectory of this particle in a given moment t is equal to $\overrightarrow{\mathbf{r}} = R\overrightarrow{i_\theta}$, where $\overrightarrow{i_\theta}$ is a unitary radial vector with angle $\theta = \nu t$ w.r.t the axis x. In this case the acceleration $\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t} = -|\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t}|\overrightarrow{i_\theta}$, and $\frac{\partial \overrightarrow{\mathbf{p}}}{\partial t} = -|\frac{\partial \overrightarrow{\mathbf{p}}}{\partial t}|\overrightarrow{i_\theta}$ are radial vectors that have the constant values and are orthogonal to the vectors of velocity $\overrightarrow{\mathbf{v}}$ and momentum $\overrightarrow{\mathbf{p}}$. So that, $\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}} = 0$, $\overrightarrow{\mathbf{p}}\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t} = 0$, and
   $\omega_p = \frac{\partial}{\partial t}(\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar} + \omega_0 t) + \frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar} = +\omega_0 + \frac{1}{\hbar}\overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{v}} + t\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t}) = \omega_0 + \frac{1}{\hbar}\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}} =$
   $= \omega_0 - \frac{pv}{\hbar}$.
   Here both ω_0 and pv are constant, and, consequently, we obtained the stationarity condition $\frac{\partial \omega_p}{\partial t} = 0$ in all points of the trajectory of this particle, with constant $\mathcal{E} = \hbar\omega_0 + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}} = \hbar\omega_0 - pv$. This case will be applied for the stationary orbits of electrons in an atom, in Example 4.

Notice that in both stationary cases above we obtained that this particular constant energy is equal to $\mathcal{E} = \hbar\omega_p = \hbar\omega_0 + \overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{v}} = \hbar\omega_0 - pv$.

If $\omega_p = \omega_1 + \frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}$ is computed for current space-time positions on the particle's trajectory, then, in this particular case, ω_1 is taken as a derivation $\frac{\partial}{\partial t}$ of the current particle's phase $-\varphi = \frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar} + \omega_0 t$ (we have that $\Psi(\overrightarrow{\mathbf{r}}, t) = \Phi(\overrightarrow{\mathbf{r}}, t)e^{i\varphi}$)

and express the particle's phase-changing on its trajectory. □
Notice that the energy changes only during collisions with another particles (Compton effects; we consider a "field" as a statistical result of the iterations with bosons of this particular field), so that after it this particle continue to propagate again as a stationary particle, but with new values of total energy E, velocity $\overrightarrow{\mathbf{v}}$, momentum $\overrightarrow{\mathbf{p}}$, and new stable wave-pocket geometry (distribution) $\Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)$, so that it can be described by the simpler stationary-case (defined in Example 2) differential equations:
(e.1.1) $\quad e^{i\omega_0 t}\frac{\partial \Phi e^{-i\omega_0 t}}{\partial t} = (-i\omega_0 + \frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t}\nabla)\Phi$
(e.2.1) $\quad \frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\mathcal{E}\Psi + \frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t}\nabla\Psi,$
where, when a velocity $\overrightarrow{\mathbf{v}}$ is constant, we have that $\frac{\partial(\overrightarrow{\mathbf{v}}t)}{\partial t} = \overrightarrow{\mathbf{v}} + t\frac{\partial \overrightarrow{\mathbf{v}}}{\partial t} = \overrightarrow{\mathbf{v}}$. Notice that, as in Example 2 above, we have particular stationary cases when $\overrightarrow{\mathbf{v}}$ is not constant, as for instance, for a stationary electron that rotates around the nucleus of an atom with a constant radial acceleration (in that case $\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}}$ and total energy of this electron are constant, thus $\mathcal{E} = E + \overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}}$ is constant as well).
From the quantum state point of view, the external "field's" forces that interfere with the particles are based on the collisions between them and a particular kind of "field's" particles with rest mass equal to zero (bosons): it is possible because the mass-particles (or excited massless particles) have a wave-pocket's distribution Φ with a finite space volume, and, consequently, they may mutually collide with another particles (as bosons of this "field").
The bosons with rest-mass m_0 equal to zero can interfere between themselves only when they are excited, i.e., when their space volume distribution of Φ is greater than zero, so that they can collide. During these collisions, a particle changes its total energy as well, thus its geometric form (distribution) Φ as well: in such cases in the equations (e.1) and (e.2), the component,
$\Phi_D(\overrightarrow{\mathbf{r}}, t) = \int\int\int_{-\infty}^{+\infty} \frac{\partial B(k)}{\partial t} e^{i(k_x(x-v_x t)+k_y(y-v_y t)+k_z(z-v_z t))} dk_x dk_y dk_z,$
becomes dominant in these unstationary (unstable) short-time intervals states, caused by the time-changing of the harmonic amplitudes $\frac{\partial B(k)}{\partial t} \neq 0$. During these collisions a particle does not change only its velocity $\overrightarrow{\mathbf{v}}$, but also its geometry (distribution) can continuously evolve. For example, as it was discussed about the changing of massive particle's geometry and resulting changing of the relativistic mass, or in "particle's explosions". This kind of explosion happens when, for instance, a massless particle that propagates in the vacuum with velocity of light come up against a barrier with a small slit (where it will pass), so that this barrier interrupts drastically the space symmetry and the bound-

ary conditions for the differential equations (e.1) and (e.2) that determine the movement and geometry of the particles.
From this point of view, the "fields" are only statistical results of the influence that high number of bosons produce to the mass-particles. For example, the electromagnetic vectorial field is a statistic result of movements of high number of photons, as it will be explained in the rest of this paper, and, consequently, Maxwell's lows are only the statistical lows.
The collisions of one mass-particle with bosons produce an external force $\overrightarrow{F}$, generated by the interacting bosons ("field") with this particle, that determines the propagation and trajectory of this mass-particle. In order to be able to consider the application of these external forces to the mass-particles, we need to obtain the following second-order differential equations:

Proposition 2 *The propagation and geometric form of the wave-pocket* $\Psi(x,y,z,t) = \Phi(\overrightarrow{\boldsymbol{r}},t)\mathrm{e}^{i(-\frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{r}}}{\hbar}-\omega_0 t)}$*, of an elementary particle that propagates with velocity* $\overrightarrow{\boldsymbol{v}}$*, are defined by the following second-order differential equations:*
(e.3) $\quad (\frac{v_1}{c})^2\triangle\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - ((\frac{\omega_p}{c})^2 + i\frac{1}{c^2}\frac{\partial\omega_p}{\partial t} - i\frac{\overrightarrow{\boldsymbol{v}_1}}{c^2\hbar}\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t})\Psi - (i\frac{2\omega_p}{c^2}\overrightarrow{\boldsymbol{v}_1} - \frac{1}{c^2}\frac{\partial\overrightarrow{\boldsymbol{v}_1}}{\partial t})\nabla\Psi =$
$= \Upsilon_D(\overrightarrow{\boldsymbol{r}},t)$,
where $\omega_p = \frac{\mathcal{E}}{\hbar} = \omega_1 + \frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{v}_1}}{\hbar}$*, with* $\omega_1 = \frac{\partial}{\partial t}(\frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{r}}}{\hbar} + \omega_0 t) = \frac{\overrightarrow{\boldsymbol{r}}}{\hbar}\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t} + \frac{\partial}{\partial t}(\omega_0 t)$, $\overrightarrow{\boldsymbol{v}_1} = \frac{\partial(\overrightarrow{\boldsymbol{v}}t)}{\partial t}$,
$v_1 = \sqrt{-\overrightarrow{\boldsymbol{v}_1}\overrightarrow{\boldsymbol{v}_1}}$*, can change in time during the propagation. The right side of (e.3),* $\Upsilon_D(\overrightarrow{\boldsymbol{r}},t) = \frac{1}{c^2}(\overrightarrow{\boldsymbol{v}_1}\nabla\Phi_D - i2\omega_1\Phi_D + \frac{\partial\Phi_D}{\partial t})\mathrm{e}^{i(-\frac{\overrightarrow{\boldsymbol{p}}\overrightarrow{\boldsymbol{r}}}{\hbar}-\omega_0 t)}$*, is different from zero only in unstationary cases when* $\Phi_D(\overrightarrow{\boldsymbol{r}},t) \neq 0$.
In the stationary case, when ω_p *and* $\omega_1 = \omega_0$ *are two constants, this equation can be given in a simpler D'Alambert-like form:*
(e.4) $\quad (\frac{v_1}{c})^2\triangle\Psi_1 - \frac{1}{c^2}\frac{\partial^2\Psi_1}{\partial t^2} = -\frac{1}{c^2}\frac{\partial\overrightarrow{\boldsymbol{v}_1}}{\partial t}\nabla\Psi_1 - i\frac{\overrightarrow{\boldsymbol{v}_1}}{c^2\hbar}\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t}\Psi_1$,
where $\Psi_1 = \mathrm{e}^{i\omega_p t}\Psi(\overrightarrow{\boldsymbol{r}},t)$.
Thus, an external force $\overrightarrow{F}$ *influences a particle* Ψ *as follows:*
1. Differential equation for massive particles, non-relativistic case: then we substitute the acceleration $\frac{\partial\overrightarrow{\boldsymbol{v}}}{\partial t}$ *in the equation (e.3) by the expression* $\frac{\overrightarrow{F}}{m_0}$.
2. Differential equation for relativistic case when $\hbar\omega_0$ *is a constant: then we substitute* $\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t}$ *in the equation (e.4) by* $\overrightarrow{F}$.
3. Differential equation for a massless particle (with rest mass $m_0 = 0$*) during its unstable state, when it changes the velocity* $\overrightarrow{\boldsymbol{v}}$ *(with* $c > v > 0$*) and momentum* $\overrightarrow{\boldsymbol{p}}$*, but does not change the total energy* $E = \hbar\omega_0$*: then we substitute* $\frac{\partial\overrightarrow{\boldsymbol{p}}}{\partial t}$ *by* $\overrightarrow{F}$, ω_p *by* $\frac{\overrightarrow{\boldsymbol{r}}}{\hbar}\overrightarrow{F} + \omega_0 + \frac{\overrightarrow{\boldsymbol{v}_1}\overrightarrow{\boldsymbol{p}}}{\hbar}$ *and* $\frac{\partial\omega_p}{\partial t}$ *by* $\frac{1}{\hbar}(\overrightarrow{\boldsymbol{r}}\frac{\partial\overrightarrow{F}}{\partial t} + \overrightarrow{\boldsymbol{v}_1}\overrightarrow{F} + \overrightarrow{\boldsymbol{p}}\frac{\partial\overrightarrow{\boldsymbol{v}_1}}{\partial t})$*, in the equation (e.3).*

Proof: From $\Psi(x,y,z,t) = \Phi(\overrightarrow{\mathbf{r}},t)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}$, we obtain that:

(b.0) $\frac{\partial}{\partial t}(\nabla\Phi) = \nabla(\frac{\partial\Phi}{\partial t}) = \nabla(\overrightarrow{\mathbf{v}_1}\nabla\Phi + \Phi_D)$ (from (a.0), proof of Proposition 1)

$= -\overrightarrow{\mathbf{v}_1}\triangle\Phi + \nabla\Phi_D$.

(b.1) $\nabla(\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}\nabla\Phi) =$

$= (\frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3)((\frac{\partial\Phi}{\partial x}e_1 + \frac{\partial\Phi}{\partial y}e_2 + \frac{\partial\Phi}{\partial z}e_3)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) =$

$= -(\frac{ip_x}{\hbar}\frac{\partial\Phi}{\partial x} + \frac{\partial^2\Phi}{\partial x^2} + \frac{ip_y}{\hbar}\frac{\partial\Phi}{\partial y} + \frac{\partial^2\Phi}{\partial y^2} + \frac{ip_z}{\hbar}\frac{\partial\Phi}{\partial z} + \frac{\partial^2\Phi}{\partial z^2})\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar} =$

$= -(-\frac{i\overrightarrow{\mathbf{p}}}{\hbar}\nabla\Phi + \triangle\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}$, and

(b.2) $\triangle\Psi = -\nabla\nabla\Psi = -\mathrm{e}^{-i\omega_0 t}\nabla(\nabla(\Phi\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar})) =$

$= -\mathrm{e}^{-i\omega_0 t}\nabla((\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) =$

$= -\mathrm{e}^{-i\omega_0 t}(\nabla(\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}\nabla\Phi) + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\nabla(\Phi\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar})) =$ (from (b.1))

$= \mathrm{e}^{-i\omega_0 t}((-\frac{i\overrightarrow{\mathbf{p}}}{\hbar}\nabla\Phi + \triangle\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar} - \frac{i\overrightarrow{\mathbf{p}}}{\hbar}(\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) =$

$= \mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}(\triangle\Phi - \frac{i2\overrightarrow{\mathbf{p}}}{\hbar}\nabla\Phi + \frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{p}}}{\hbar^2}\Phi) =$

$= \mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}\triangle\Phi - \frac{i2\overrightarrow{\mathbf{p}}}{\hbar}(\nabla\Psi - \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Psi) + \frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{p}}}{\hbar^2}\Psi$.

Thus, from (b.2) we obtain that,

(b.3) $\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}\triangle\Phi = \triangle\Psi + \frac{i2\overrightarrow{\mathbf{p}}}{\hbar}\nabla\Psi + \frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{p}}}{\hbar^2}\Psi$. Consequently,

(b.4) $\overrightarrow{\mathbf{v}_1}\frac{\partial}{\partial t}\nabla\Psi = \overrightarrow{\mathbf{v}_1}\mathrm{e}^{-i\omega_0 t}(\frac{\partial}{\partial t}(\nabla(\Phi\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) - i\frac{\partial(\omega_0 t)}{\partial t}\nabla(\Phi\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar})) =$

$= \overrightarrow{\mathbf{v}_1}\mathrm{e}^{-i\omega_0 t}(\frac{\partial}{\partial t}((\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) - i\frac{\partial(\omega_0 t)}{\partial t}(\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar}) =$

(from (a.2) in the prof of Proposition 1)

$= \overrightarrow{\mathbf{v}_1}\mathrm{e}^{-i\omega_0 t}((\frac{\partial}{\partial t}\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\frac{\partial\Phi}{\partial t} + \frac{i}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Phi)\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar} + (\frac{\partial}{\partial t}\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar} - i\frac{\partial(\omega_0 t)}{\partial t}\mathrm{e}^{-i\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}/\hbar})$
$(\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi)) =$

$= \overrightarrow{\mathbf{v}_1}(\frac{\partial}{\partial t}\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\frac{\partial\Phi}{\partial t} + \frac{i}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Phi - i\omega_1(\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi))\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

(from (b.0), and (a.0) in the proof of Proposition 1)

$= \overrightarrow{\mathbf{v}_1}(-\overrightarrow{\mathbf{v}_1}\triangle\Phi + \nabla\Phi_D + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}(\overrightarrow{\mathbf{v}_1}\nabla\Phi + \Phi_D) + \frac{i}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Phi - i\omega_1(\nabla\Phi + \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Phi))\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

(from (b.2), and (a.3) in the proof of Proposition 1)

$= \overrightarrow{\mathbf{v}_1}(-\overrightarrow{\mathbf{v}_1}(\triangle\Psi + \frac{i2\overrightarrow{\mathbf{p}}}{\hbar}\nabla\Psi + \frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{p}}}{\hbar}\Psi) + \frac{i\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}(\nabla\Psi - \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Psi) + \frac{i}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi - i\omega_1(\nabla\Psi - \frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Psi +$
$\frac{i\overrightarrow{\mathbf{p}}}{\hbar}\Psi)) + (\overrightarrow{\mathbf{v}_1}\nabla\Phi_D + i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}\Phi_D)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

$= v_1^2\triangle\Psi - i(\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar} + \omega_1)\overrightarrow{\mathbf{v}_1}\nabla\Psi + i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi + (\overrightarrow{\mathbf{v}_1}\nabla\Phi_D + i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}\Phi_D)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

$= v_1^2\triangle\Psi - i\omega_p\overrightarrow{\mathbf{v}_1}\nabla\Psi + i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi + (\overrightarrow{\mathbf{v}_1}\nabla\Phi_D + i\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}_1}}{\hbar}\Phi_D)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}$.

Finally, $\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial}{\partial t}(-i\omega_p\Psi + \overrightarrow{\mathbf{v}_1}\nabla\Psi + \Psi_D) =$

$= -i\frac{\partial\omega_p}{\partial t}\Psi - i\omega_p\frac{\partial\Psi}{\partial t} + \frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi + \overrightarrow{\mathbf{v}_1}\frac{\partial}{\partial t}\nabla\Psi + \frac{\partial\Psi_D}{\partial t} =$ (from (e.2))

$= -i\frac{\partial\omega_p}{\partial t}\Psi - i\omega_p(-i\omega_p\Psi + \overrightarrow{\mathbf{v}_1}\nabla\Psi + \Psi_D) + \frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi + \overrightarrow{\mathbf{v}_1}\frac{\partial}{\partial t}\nabla\Psi + (\frac{\partial\Phi_D}{\partial t} -$
$i\omega_1\Phi_D)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\,\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

(from (b.4))

$= v_1^2\triangle\Psi - \omega_p^2\Psi - i2\omega_p\overrightarrow{\mathbf{v}_1}\nabla\Psi - i\frac{\partial\omega_p}{\partial t}\Psi + \frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi + i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi + (\overrightarrow{\mathbf{v}_1}\nabla\Phi_D - i2\omega_1\Phi_D + \frac{\partial\Phi_D}{\partial t})\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}$.

Thus, by dividing with c^2 we obtain the differential equation (e.3).

When ω_p and ω_0 are two constants, then we obtain (e.4) as follows:

$v_1^2\triangle\Psi_1 - \frac{\partial^2}{\partial t^2}\Psi_1 = v_1^2\triangle(\Psi\mathrm{e}^{i\omega_p t}) - \frac{\partial^2}{\partial t^2}(\Psi\mathrm{e}^{i\omega_p t}) =$

$= (v_1^2\triangle\Psi)\mathrm{e}^{i\omega_p t} - \frac{\partial}{\partial t}((\frac{\partial}{\partial t}\Psi)\mathrm{e}^{i\omega_p t} + i\omega_p\Psi\mathrm{e}^{i\omega_p t}) =$

$= (v_1^2\triangle\Psi - \frac{\partial^2}{\partial t^2}\Psi - \omega_p^2\Psi - i2\omega_p\overrightarrow{\mathbf{v}_1}\nabla\Psi)\mathrm{e}^{i\omega_p t} =$ (from (e.3))

$= -(\frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi + i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi)\mathrm{e}^{i\omega_p t} = -\frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi_1 - i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi_1$.

□

The differential equation (e.3) describes the changing (in time) of the geometry Φ of a the "corpuscular" particle (wave-pocket) and its trajectory and velocity, caused by a given external force $\overrightarrow{F}$ (that is, caused by a total sum of given external "fields", as gravitational, electromagnetic, etc..).

4 Application to quantum mechanics

The *Compton effect* between any two particles with their distributions $\Phi_1(x, y, z, t)$ and $\Phi_2(x, y, z, t)$, happens when there exists a space-time point (x_1, y_1, z_1, t_1) such that $\Phi_1(x_1, y_1, z_1, t_1) \cdot \Phi_2(x_1, y_1, z_1, t_1) \neq 0$. This collision of these two particles is elastic if their kinetic energies are relatively small, with the well known lows of conservation of the total energy and total momentum for elastic collisions. In the high-energy collisions we obtain a fusion of these two particles and a creation of another, as in well-known Feynman diagrams.
In the *stationary cases*, the basic equation (e.4) can be divided into following three cases:

- When the velocity $v = 0$, we obtain a simple equation:
  (e.4.0) $\frac{\partial^2\Psi_1}{\partial t^2} = 0$, that is, $\frac{\partial^2\Psi}{\partial t^2} = -\omega_0^2\Psi$,
  with a simple solution, $\Psi(x, y, z, t) = \Phi(x, y, z)\mathrm{e}^{-\omega_0 t}$.
- When the velocity $\overrightarrow{\mathbf{v}}$ and the momentum $\overrightarrow{\mathbf{p}}$ are constant vectors during the propagation, different from zero, then $\overrightarrow{\mathbf{v}_1} = \overrightarrow{\mathbf{v}}$, so we obtain D'Alambert equation where v is a constant value:
  (e.4.1) $\triangle\Psi_1 - \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2} = 0$,
  with the solution $\Psi_1(x, y, z, t) = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)\mathrm{e}^{-i\frac{\overrightarrow{\mathbf{p}}}{\hbar}(\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{v}}t)}$, thus,
  $\Psi(x, y, z, t) = \Psi_1\mathrm{e}^{-\omega_p t} = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)\mathrm{e}^{i(-\frac{\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{r}}}{\hbar}-\omega_0 t)}$.
- The case when the velocity $\overrightarrow{\mathbf{v}}$ with $v > 0$ and the momentum $\overrightarrow{\mathbf{v}}$ change

only the direction (they are collinear vectors in each instance of time) during a propagation, but not their values, so that $-\frac{\partial(\overrightarrow{\mathbf{p}}\overrightarrow{\mathbf{v}})}{\partial t} = \frac{\partial(pv)}{\partial t} = 0$ and $v_1 = v$ is constant as well (this case includes the second case in Example 2 as well: when the velocity of massive particle with rest mass m_0 is $\overrightarrow{\mathbf{v}} = v(-\cos\theta e_1 + \sin\theta e_2)$, where $\theta = \nu t = \frac{v}{R}t$ is the angle w.r.t the axis x of a particle that routes around the coordinate center with constant angular velocity $\nu = \frac{v}{R}$ on circular orbit with constant radius R).
Thus, from (e.4) we obtain an extended D'Alambert equation where $v^2 = |\overrightarrow{\mathbf{v}}\overrightarrow{\mathbf{v}}| > 0$ is a constant value:
(e.4.2) $\triangle\Psi_1 - \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2} = \Theta(\overrightarrow{\mathbf{r}}, t)$,
where $\Theta(\overrightarrow{\mathbf{r}}, t) = -\frac{1}{v^2}(\frac{\partial\overrightarrow{\mathbf{v}_1}}{\partial t}\nabla\Psi_1 + i\frac{\overrightarrow{\mathbf{v}_1}}{\hbar}\frac{\partial\overrightarrow{\mathbf{p}}}{\partial t}\Psi_1)$. Thus, we obtain a general solution $\Psi_1(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t) = \Phi(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)\mathrm{e}^{-i\frac{\overrightarrow{\mathbf{p}}}{\hbar}(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{v}}t)}$.

Notice that the massive particles that does not change the total energy E during a propagation are always in the stationary states.
In all cases, the geometric form (the distribution Φ) of a particle in a given time-instance t depends on the particular boundary conditions for the differential equations as well. In the case when they are far from another massive particles (usually it can be considered if another particles are far from this particle in order of one millimeter), then Φ is symmetric w.r.t. the direction of propagation. Otherwise, the boundary conditions for these differential equations can drastically change, with the result that Φ can become enormously bigger than in normal situations, that is, they can instantaneously "explode", because the single harmonics of the Fourier representation of $\Phi(x, y, z)$ in a given time-instance t are contemporarily present in the whole 3-dimensional Euclidean space.
Let us consider the following well known example:
Example 3: One-dimensional particle in a box problem.
We assume that that the electron can move freely between two infinitely high potential barriers, along the x axes between $-\frac{a}{2}$ and $\frac{a}{2}$. The appropriate potential is $V(x) = 0$ for $-\frac{a}{2} \leq x \leq \frac{a}{2}$ and $V(x)$ equal to the infinity otherwise, that is, there are infinitely high walls at $x = -\frac{a}{2}$ and $x = \frac{a}{2}$, and the particle is trapped between them. This turns out to be quite a good approximation for electrons in a long molecule, and the three-dimensional version is a reasonable picture for electrons in metals. Consequently, the field, which is the gradient of this potential is equal to zero between these walls and directed into the box at the barriers, so that in the box we have not any field and any acceleration of the electron, so that the equation (e.4) has a simple D'Alambert form

(e.4.1) $\triangle\Psi_1 - \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2} = 0,$
with the solution (we consider case when the momentum and the velociries are collinear with x axes) $\Psi_1(x, y, z, t) = \Phi(x - vt, y, z)\mathrm{e}^{i(\frac{p}{\hbar}(x-vt)}$, where $\Phi(x - vt, y, z)$ is the geometric form of electron during its propagation (notice that this electron is at position x in the time-instance t when $x - vt = 0$), and the constant velocity v can be positive and negative (opposite direction w.r.t. the x axes). The boundary condition at the walls is that in these points the velocity of the electron has to be equal to zero, and the stationary solution of the D'Alambert equation above has to satisfy that (after one complete oscillation, when it passes the distance $2a$), $\Psi_1(x, y, z, t) = \Psi_1(x, y, z, t + \frac{2a}{v})$, so that,
$\mathrm{e}^{i\frac{p}{\hbar}(x-vt)} = \mathrm{e}^{i\frac{p}{\hbar}(x-v(t+\frac{2a}{v})}$, that is,
$\frac{p}{\hbar}(x - vt) = 2n\pi + \frac{p}{\hbar}(x - v(t + \frac{2a}{v})$, for $n = 0, \pm1, \pm2, ..$
so that we obtain that $p = \frac{\pi\hbar}{a}n$, where $n = 0, \pm1, \pm2, ..$, exactly as in the solution of the Schrödinger equation. The positive and negative values for p corresponds to the propagation on the left or on the right (opposite direction) of this electron. Consequently we obtain a constant discrete set of possible velocities $v = \frac{p}{m_0}$ (we consider only non relativistic case when $v << c$).
The (non relativistic) kinetic energy of this electron can be one of the following discrete values:
$E = \frac{m_0 v^2}{2} = \frac{p^2}{2m_0} = \frac{(\pi\hbar)^2}{2m_0 a^2}n^2$, $n = 0, 1, 2, ..$
Thus, the wave-pocket of this electron in the box that oscillates along the axis x, is given by:
$\Psi(x, y, z, t) = \Phi(x + la - vt, y, z)\mathrm{e}^{i(\frac{p(x+la)}{\hbar} - \omega_0 t)}$, for $t \in [(l - \frac{1}{2})\triangle t, (l + \frac{1}{2})\triangle t]$,
or, when it propagates in opposite direction,
$\Psi(x, y, z, t) = \Phi(x - (l+1)a + vt, y, z)\mathrm{e}^{i(\frac{-p(x-(l+1)a)}{\hbar} - \omega_0 t)}$, for $t \in [(l+1-\frac{1}{2})\triangle t, (l + 1 + \frac{1}{2})\triangle t]$,
where $l = 0, 1, 2, ...$, $\triangle t = \frac{a}{v}$, and $v \in \{n\frac{4\pi\hbar}{m_0 a} \mid n = 0, 1, 2, ...\}$.
□
The D'Alambrt-like equation (e.4) is very important for the particles with the rest mass greater than zero as well, and especially to their stationary cases as well, when $\omega_p \neq 0$, as for example in the stationary orbits of electrons in the atoms. Based on the stationary solutions of the D'Alambert-like differential equation for electrons, directly from (e.4) we can derive the 3rd Bohr postulate, as follows:

Example 4: 3rd Bohr postulate for the electrons.

The electrons in the atoms have constant velocity $v > 0$ and momentum p, so that the non relativistic equation for electron's wave-pocket Ψ and its propagation given by (e.4.2), can be obtained by substituting $\frac{\partial \vec{\mathbf{v}}}{\partial t}$ with $\overrightarrow{F}/m_0$, with the radial force of the electric field $F = |\overrightarrow{F}| = \frac{Ze^2}{4\pi\epsilon_0 R^2}$, where Z is a number of protons in the atom's nucleus, e is the electrical charge of the electron, ϵ_0 the electric constant and R is the distance of the electron from the nucleus. Thus, we obtain the stationary case for the electron's propagation:

(e.5) $\quad \triangle\Psi_1 - \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2} = -\frac{2}{m_0 v^2}\overrightarrow{F}\nabla\Psi_1,$

with a general solution $\Psi_1(\vec{\mathbf{r}} - \vec{\mathbf{v}}t) = \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}t)\mathrm{e}^{-i\frac{\vec{\mathbf{p}}}{\hbar}(\vec{\mathbf{r}}-\vec{\mathbf{v}}t)}$. The stationary solutions are those for which the wave-pocket Ψ_1 obtain the same identical value after every full rotation of electron around the nucleus, That is when $\Psi_1(x,y,z,t) = \Psi_1(x,y,z,t+\frac{2\pi R}{v})$, where $\frac{2\pi R}{v}$ is the time period for one complete rotation of the wave-pocket Ψ_1 around the nucleus.

Thus, $\Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}(t+\frac{2\pi R}{v})) = \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}t)$, where $\frac{2\pi R}{v}$ is the period of time for one revolution of electron around the nucleus, and we obtain that:

$\Psi_1(x,y,z,t+\frac{2\pi R}{v}) = \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}(t+\frac{2\pi R}{v}))\mathrm{e}^{-i\frac{\vec{\mathbf{p}}}{\hbar}(\vec{\mathbf{r}}-\vec{\mathbf{v}}(t+\frac{2\pi R}{v}))} =$

$= \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}t)\mathrm{e}^{-i\frac{\vec{\mathbf{p}}}{\hbar}(\vec{\mathbf{r}}-\vec{\mathbf{v}}t)}\mathrm{e}^{+i\frac{\vec{\mathbf{p}}}{\hbar}\vec{\mathbf{v}}\frac{2\pi R}{v}} =$

$= \Psi_1(x,y,z,t)\mathrm{e}^{-i\frac{2\pi pR}{\hbar}} = \Psi_1(x,y,z,t)$, if $\frac{2\pi pR}{\hbar} = 2n\pi$, $n = 1,2,3,..$, that is, when $pR = n\hbar$, as in the 3rd Bohr postulate. Consequntly, for a given momentum of the electron, the radius R can be only one of the following values $R = n\frac{\hbar}{p}$, $n = 1,2,..$

Analogously, for a fixed radius R of an electron in a stationary states, its momentum p can have only the following discrete values $p = n\frac{\hbar}{R}$, with spatial wavelengths $\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi R}{n}$, $n = 1,2,...$

Thus, for a given radius R, the wave-pocket for an electron will be given by:

$\Psi(x,y,z,t) = \mathrm{e}^{-i\omega_p t}\Psi_1 = \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}t)\mathrm{e}^{i(-\frac{\vec{\mathbf{p}}\vec{\mathbf{r}}}{\hbar}-\omega_0 t)} =$

$= \Phi(\vec{\mathbf{r}} - \vec{\mathbf{v}}t)\mathrm{e}^{i(-\frac{n}{R}\overrightarrow{i_v}(t)\vec{\mathbf{r}}-\omega_0 t)}$,

where, if we take the plain x,y for rotation of this electron with a center in (0,0) and initial position of electron in $t = 0$ equal to $\vec{\mathbf{r}} = Re1 + 0e2$, then $\overrightarrow{i_v}(t) = -\cos(\frac{v}{R}t)e_1 + \sin(\frac{v}{R}t)e_2$ is the unit vector of the velocity, that is, tangent vector to the circle with radius R.

Consequently, these stationary cases correspond to the following values of ω_p:

$\omega_p = \omega_0 - \frac{pv}{\hbar} = \omega_0 - n\frac{v}{R}$, $n = 1,2,...$,

where $\omega_0 = \frac{m_0c^2}{\hbar}$, and the stationary values for the energies $\mathcal{E} = \hbar\omega_p$ are equal to $\mathcal{E} = m_0c^2 - n\frac{\hbar v}{R}$, $n = 1,2,....$

As a consequence, for mass-particles we can have a number of different stationary (stable) states for ω_p.

□

References

[1] L. Broglie. Recherches sur la theorie des quanta. *Ann. de Phys.,* 10^e *serie, t. III (Janvier-Fervier)*, 1925.

[2] C.Liu, Z.Dutton, C.Behroozi, and L.Hau. Observation of coherent optical information storage in an atomic medium using halted light pulses. *Nature 409 (6819)*, pages 490–493, 2001.

[3] L.Hau, S.Harris, Z.Dutton, and C.Behroozi. Light speed reduction to 17 metres per second in an ultracold atomic gas. *Nature 397*, pages 594–598, 1999.

[4] Z.Majkić. Partial differential equations for wave pockets in the Minkowski 4-dimensional spaces. *Technical Report 02-10, Compet Research, http://zoranmajkic.webs.com*, 2010.