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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A KIND OF THIRD-ORDER FUNCTIONAL DIFFERENTIAL EQUATION WITH A TIME-DELAY

A. M. MAHMOUD

Department of Mathematics, Faculty of Science, New Valley Branch, Assiut University. New Valley, El-Khargah 72111, Egypt.

Email address: math_ayman27@yahoo.com

Abstract

In this paper, by using the Mawhin coincidence degree theory and analysis techniques, we establish new results on the existence and uniqueness of T-periodic solutions for a kind of third-order functional differential equation (FDE) with a time-delay.

The obtained results are new and complement the related results of third-order (FDE) with a time-delay that have appeared in the literature. In the last section, we give an example to illustrate our main results of periodic solutions.

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1 Introduction

In recent years numerous methods have been developed to study the differential equations (DEs). Exact methods (quantitative), in which all the solutions are known and could be written in closed form in terms of elementary functions or sometime special functions. There are also some other type of methods, called analytical methods (qualitative), in which one can describe the behaviour of a differential equation's solution, such as existence of solutions, uniqueness, stability, instability, chaotic or asymptotic character, boundedness, periodicity, etc., without actually solving it exactly. This is an important and relatively new step in the theory of (DEs), because most of the (DEs) cannot be solved exactly.

The investigation of the qualitative properties of solutions of (FDEs) for higher-order of solutions play an important role in many real world phenomena related to the sciences and engineering technique fields. (FDEs) of higher-order can serve as excellent tools for description of mathematical modeling of systems and processes in population dynamics, stochastic processes, physics, control theory, neural networks, mechanics, etc.

Existence and uniqueness of periodic solutions for (FDEs) are of great interest in mathematics and its applications to the modeling of various practical problems.

In recent years, many books and papers dealt with the existence and uniqueness of periodic solutions for (FDEs) with applications and obtained many good results, for example, [3, 7, 8, 11, 12], etc.

Continuation theorem of coincidence degree theory plays a significant role in the investigation of the existence of periodic solutions for (FDEs) of higher-order.

The existence and uniqueness of periodic solutions of third-order (FDEs) with a or more deviating arguments have been widely investigated and are still being investigated, for example, (see [1, 2, 5, 6, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21], and the references therein).

In 2010, Tunç [15] established certain sufficient conditions by using the Lyapunov functional approach for the existence of a periodic solution of the nonlinear differential equation of the third-order with constant deviating argument $(\tau > 0)$

$$\dddot{x} + \psi(\dot{x})\,\ddot{x} + g(\dot{x}(t-\tau)) + f(x) = p(t, x, x(t-\tau), \dot{x}, \dot{x}(t-\tau), \ddot{x}).$$

Moreover, In 2011, Abou-El-Ela, Sadek and Mahmoud [1] studied the existence and uniqueness of periodic solutions to third-order delay differential equation with a deviating argument of the form

$$\dddot{x}(t) + f(t, x(t))\,\ddot{x}(t) + g(x(t))\,\dot{x}(t) + h(t, x(t - r(t))) = p(t).$$

The main objective of this work is to establish criteria to guarantee the existence and uniqueness of a T-periodic solution for third-order (FDE) with a time-delay as the following form

$$\dddot{x}(t) + f(\dot{x}(t))(\ddot{x}(t))^2 + g(t, x(t))\dot{x}(t) + h(t, x(t - \tau(t))) = p(t). \tag{1.1}$$

We can write Equation (1.1) in the following equivalent system

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = -f(y)z^2 - g(t, x(t))y - h(t, x(t - \tau(t))) + p(t), \end{aligned} \tag{1.2}$$

where $f, \tau, p : \mathbb{R} \to \mathbb{R}$ and $h, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions; $f(0) = 0$, τ and p are T-periodic, and g is T-periodic in the first argument with period $T > 0$.

By applying the continuation theorem of the coincidence degree theory, we obtain new results, which complement previously known results. An illustrative example is given in the last section.

The rest of this paper is organized as follows. In section 2 we state some preliminary results and other technical details and we shall prove the main existence results and in section 3 we shall establish main result s of uniqueness. We will give an example of an application in section 4.

2 Preliminary Results

In this section we give some technical, yet elementary results, that will serve us well in the section that follows.

For ease of exposition throughout this article we shall adopt the following notation:

$$|x|_k = (\int_0^T |x(t)|^k dt)^{\frac{1}{k}}, k \geq 1, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|$$

and

$$|p|_\infty = \max_{t\in[0,T]} |p(t)|.$$

Let

$$X = \{x | x \in C^2(\mathbb{R}, \mathbb{R}), x(t + T) = x(t), \text{ for all } t \in \mathbb{R}\}$$

and

$$Y = \{y | y \in C(\mathbb{R}, \mathbb{R}), y(t + T) = y(t), \text{ for all } t \in \mathbb{R}\},$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty, |\dot{x}|_\infty, |\ddot{x}|_\infty\} \text{ and } \|y\|_Y = |y|_\infty.$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{x | x \in X, \dddot{x}(t) \in C(\mathbb{R}, \mathbb{R})\},$$

and for $x \in D(L)$

$$Lx = \dddot{x}(t). \tag{2.1}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx = -f(\dot{x}(t))(\ddot{x}(t))^2 - g(t, x(t))\dot{x}(t) - h(t, x(t - \tau(t))) + p(t). \tag{2.2}$$

Then we notice that

$$KerL = \mathbb{R} \text{ and } ImL = \{y | y \in Y, \int_0^T y(s)ds = 0\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projector $P : X \to KerL$ and the averaging projector $Q : Y \to Y$ by setting

$$Px(t) = \frac{1}{T}\int_0^T x(s)ds \text{ and } Qy(t) = \frac{1}{T}\int_0^T y(s)ds.$$

Hence $ImP = KerL$ and $KerQ = ImL$. Denoting by $L_P^{-1} : ImL \to D(L) \cap KerP$ the inverse of $L|_{D(L)\cap KerP}$, we have

$$L_P^{-1}y(t) = \int_0^T \left(\frac{(s - t - T/2)^3}{6T} - \frac{Ts}{24}\right)y(s)ds + \int_0^T \frac{(t - s)^2}{2}y(s)ds.$$

Therefore we can see from (2.2) and the above equation, that N is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X.

To prove the main result of existence and uniqueness of a T-periodic solution for the third-order functional differential equation with a deviating argument, we use the continuation theorem of coincidence degree theory and analysis techniques.

For convenience of use, we introduce the continuation theorem of coincidence degree theory formulated in [4] as follows.

Theorem 2.1 *let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X. In addition, if the following conditions hold:*

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$ *and* $\lambda \in (0,1)$*;*

(2) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap KerL$;

(3) $deg\{QN, \Omega \cap KerL, 0\} \neq 0$.

Then equation $Lx = Nx$ *has at least one solution on* $\bar{\Omega}$.

It is convenient to introduce the following assumption.

(A_0) Assume that there exist non-negative constants c_1 and c_2 such that

$$f \in C^1(\mathbb{R}, \mathbb{R}),\ f'(y) \leq 0,\ |f(y_1) - f(y_2)| \leq c_1|y_1 - y_2|,\ f(0) = 0,$$

for all $y, y_1, y_2 \in \mathbb{R}$; and $|g(t,x)| \leq c_2$, for all $t, x \in \mathbb{R}$.

In view of (2.1) and (2.2) the operator equation

$$Lx = \lambda Nx,$$

is equivalent to the following equation

$$\dddot{x}(t) + \lambda\{f(\dot{x}(t))(\ddot{x}(t))^2 + g(t, x(t))\dot{x}(t) + h(t, x(t - \tau(t)))\} = \lambda p(t), \lambda \in (0,1). \tag{2.3}$$

The following lemmas are very important to prove the existence of the most one T-periodic solution of (1.1).

Lemma 2.1 *Suppose that there exists a constant* $d > 0$ *such that*

(A_1) $x\{h(t,x) - p(t)\} < 0$, *for all* $t \in \mathbb{R}$ *and* $|x| \geq d$.

If $x(t)$ *is a* T*-periodic solution of* (2.3), *then*

$$|x|_\infty \leq d + \frac{1}{2}\sqrt{T}|\dot{x}|_2. \tag{2.4}$$

Proof. Let $x(t)$ be a T-periodic solution of (2.3) for a certain $\lambda \in (0,1)$. Set

$$x(t_1) = \max_{t \in \mathbb{R}} x(t), \quad x(t_2) = \min_{t \in \mathbb{R}} x(t), \text{ where } t_1, t_2 \in \mathbb{R},$$

therefore we obtain

$$\dot{x}(t_1) = 0,\ \ddot{x}(t_1) \leq 0,\ \dddot{x}(t_1) \leq 0, \text{ and } \dot{x}(t_2) = 0,\ \ddot{x}(t_2) \geq 0,\ \dddot{x}(t_2) \geq 0.$$

It follows from (2.3) that

$$h(t_1, x(t_1 - \tau(t_1))) - p(t_1) \geq 0, \ \text{ and } \ h(t_2, x(t_1 - \tau(t_2))) - p(t_2) \leq 0.$$

Taking this together with (A_1) as appropriate we have

$$x(t_1 - \tau(t_1)) < d \quad \text{and} \quad x(t_2 - \tau(t_2)) > -d.$$

Since $x(t - \tau(t))$ is a continuous function on $\mathbb{R}$, this implies that there exists $\xi \in \mathbb{R}$ such that

$$|x(\xi - \tau(\xi))| \leq d.$$

let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$ and m be an integer then we obtain

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} \dot{x}(s)ds| \leq d + \int_{t_0}^{t} |\dot{x}(s)|ds, \ t \in [t_0, t_0 + T],$$

and

$$|x(t)| = |x(t-T)| = |x(t_0) - \int_{t-T}^{t_0} \dot{x}(s)ds| \leq d + \int_{t-T}^{t_0} |\dot{x}(s)|ds, \ t \in [t_0, t_0 + T].$$

Combining the above two inequalities, we have

$$\begin{aligned} |x|_\infty &= \max_{t \in [t_0, t_0+T]} |x(t)| \\ &\leq \max_{t \in [t_0, t_0+T]} \left\{ d + \frac{1}{2} \left(\int_{t_0}^{t} |\dot{x}(s)|ds + \int_{t-T}^{t_0} |\dot{x}(s)|ds \right) \right\} \\ &= \max_{t \in [t_0, t_0+T]} \left\{ d + \frac{1}{2} \int_{t-T}^{t} |\dot{x}(s)|ds \right\} = d + \frac{1}{2} \int_{0}^{T} |\dot{x}(s)|ds \leq d + \frac{1}{2}\sqrt{T}|\dot{x}|_2. \end{aligned}$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Suppose that (A_0) and (A_1) hold, and the following condition is satisfied:*

(A_2) *There exists a non-negative constant b such that*

$$|h(t, x_1) - h(t, x_2)| \leq b|x_1 - x_2|, \ \textit{for all} \ \ t, x_1, x_2 \in \mathbb{R}$$

and

$$c_2 \frac{T^2}{4} + b\frac{T^3}{8} < 1.$$

If $x(t)$ is a T-periodic solution of (1.1), then

$$|\ddot{x}|_\infty \leq \frac{1}{2} \frac{[bd + \max\{|h(t,0)| : 0 \leq t \leq T\} + |p|_\infty]T}{1 - (c_2\frac{T^2}{4} + b\frac{T^3}{8})} := D.$$

Proof. Let $x(t)$ be a T-periodic solution of (1.1). Multiplying (1.1) by $\dddot{x}(t)$ and then integrating it over $[0,T]$, implies

$$\begin{aligned}\int_0^T |\dddot{x}(t)|^2 dt &= -\int_0^T f(\dot{x}(t))(\ddot{x}(t))^2\dddot{x}(t)dt - \int_0^T g(t,x(t))\dot{x}(t)\dddot{x}(t)dt \\ &\quad - \int_0^T h(t,x(t-\tau(t)))\dddot{x}(t)dt + \int_0^T p(t)\dddot{x}(t)dt \\ &= \frac{1}{3}\int_0^T f'(\dot{x}(t))(\ddot{x}(t))^4 dt - \int_0^T g(t,x(t))\dot{x}(t)\dddot{x}(t)dt \\ &\quad - \int_0^T h(t,x(t-\tau(t)))\dddot{x}(t)dt + \int_0^T p(t)\dddot{x}(t)dt.\end{aligned}$$

By using condition (A_0) we find

$$\begin{aligned}|\dddot{x}(t)|_2^2 &\le \int_0^T \{|h(t,x(t-\tau(t))) - h(t,0)| + |h(t,0)|\}|\dddot{x}(t)|dt \\ &\quad + c_2\int_0^T |\dot{x}(t)||\dddot{x}(t)|dt + \int_0^T |p(t)||\dddot{x}(t)|dt.\end{aligned}$$

Then from condition (A_2), we obtain

$$\begin{aligned}|\dddot{x}(t)|_2^2 &\le b\int_0^T |x(t-\tau(t))||\dddot{x}(t)|dt + \int_0^T |h(t,0)||\dddot{x}(t)|dt \\ &\quad + c_2\int_0^T |\dot{x}(t)||\dddot{x}(t)|dt + \int_0^T |p(t)||\dddot{x}(t)|dt \\ &\le c_2\int_0^T |\dot{x}(t)||\dddot{x}(t)|dt + b|x|_\infty\int_0^T |\ddot{x}(t)|dt \\ &\quad + \max\{|h(t,0)| : 0\le t\le T\}\int_0^T |\dddot{x}(t)|dt + |p|_\infty\int_0^T |\dddot{x}(t)|dt.\end{aligned}$$

Thus from (2.4) and by using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\dddot{x}|_2^2 &\le [bd + \max\{|h(t,0)| : 0\le t\le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2 \\ &\quad + c_2|\dot{x}|_2|\dddot{x}|_2 + \frac{1}{2}bT|\dot{x}|_2|\dddot{x}|_2.\end{aligned} \tag{2.5}$$

Since $x(0)=x(T)$ there exists a constant $\xi\in[0,T]$ such that $\dot{x}(\xi)=0$ and

$$|\dot{x}(t)| = |\dot{x}(\xi) + \int_\xi^t \ddot{x}(s)ds| \le \int_\xi^t |\ddot{x}(s)|ds, \quad t\in[\xi, T+\xi]. \tag{2.6}$$

Again

$$\begin{aligned}|\dot{x}(t)| &= |\dot{x}(\xi+T) + \int_{\xi+T}^{t} \ddot{x}(s)ds| \\ &\leq |\dot{x}(\xi+T)| + \int_{t}^{\xi+T} |\ddot{x}(s)|ds = \int_{t}^{\xi+T} |\ddot{x}(s)|ds, \ \ t \in [0,T].\end{aligned} \tag{2.7}$$

The inequalities (2.6) and (2.7) imply that

$$2|\dot{x}(t)| \leq \int_{\xi}^{t} |\ddot{x}(s)|ds + \int_{t}^{\xi+T} |\ddot{x}(s)|ds = \int_{0}^{T} |\ddot{x}(s)|ds, \ \ t \in [0,T].$$

Therefore by using Cauchy-Schwarz inequality we have

$$|\dot{x}(t)| \leq \frac{1}{2}\sqrt{T}(\int_{0}^{T} |\ddot{x}(s)|^2 ds)^{\frac{1}{2}}, \ \text{ for all } \ t \in [0,T], \tag{2.8}$$

so

$$|\dot{x}|_\infty \leq \frac{1}{2}\sqrt{T}|\ddot{x}|_2, \tag{2.9}$$

$$|\dot{x}|_2 \leq \sqrt{T} \max_{t\in[0,T]} |\dot{x}(s)| \leq \frac{1}{2}T(\int_{0}^{T} |\ddot{x}(s)|^2 ds)^{\frac{1}{2}} = \frac{1}{2}T|\ddot{x}|_2. \tag{2.10}$$

Since $x(t)$ is periodic function for $t \in [0,T]$ and by using the above similar technique we find

$$|\ddot{x}(t)| \leq \frac{1}{2}\int_{0}^{T} |\dddot{x}(t)|dt.$$

Which together with Cauchy-Schwarz inequality implies

$$|\ddot{x}|_\infty \leq \frac{1}{2}\sqrt{T}(\int_{0}^{T} |\dddot{x}(s)|^2 ds)^{\frac{1}{2}} = \frac{1}{2}\sqrt{T}|\dddot{x}|_2, \tag{2.11}$$

$$|\ddot{x}|_2 \leq \sqrt{T} \max_{t\in[0,T]} |\ddot{x}(s)| \leq \frac{1}{2}\sqrt{T}\int_{0}^{T} |\dddot{x}(s)|ds \leq \frac{1}{2}T|\dddot{x}|_2. \tag{2.12}$$

By substituting from (2.12) in (2.10) we get

$$|\dot{x}|_2 \leq \frac{1}{4}T^2|\dddot{x}|_2. \tag{2.13}$$

By substituting from (2.12) in (2.9) we have

$$|\dot{x}|_\infty \leq \frac{1}{4}T^{\frac{3}{2}}|\dddot{x}|_2. \tag{2.14}$$

From (2.4) and (2.13) we obtain

$$|x|_\infty \le d + \frac{1}{8}T^{\frac{5}{2}}|\dddot{x}|_2. \tag{2.15}$$

Then by substituting from (2.13) in (2.5) we find

$$|\dddot{x}|_2^2 \le (c_2\frac{T^2}{4} + b\frac{T^3}{8})|\dddot{x}|_2^2 + [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2. \tag{2.16}$$

Thus we get

$$(1 - c_2\frac{T^2}{4} - b\frac{T^3}{8})|\dddot{x}|_2^2 \le [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2.$$

Therefore we find

$$|\dddot{x}|_2 \le \frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}}{1 - c_2\frac{T^2}{4} - b\frac{T^3}{8}}. \tag{2.17}$$

By substituting from (2.17) in (2.14) and (2.11) we obtain

$$|\dot{x}|_\infty \le \frac{1}{4}\frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]T^2}{1 - (c_2\frac{T^2}{4} + b\frac{T^3}{8})} := \frac{T}{2}D, \tag{2.18}$$

and

$$|\ddot{x}|_\infty \le \frac{1}{2}\frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]T}{1 - (c_2\frac{T^2}{4} + b\frac{T^3}{8})} := D. \tag{2.19}$$

This completes the proof of Lemma 2.2.

Theorem 2.2 *Suppose that assumption (A_1) holds, and the following condition is satisfied:*

(A_3) *Assume that (A_0) holds, $g(t,x) \equiv g(t)$ for all $t, x_1, x_2 \in \mathbb{R}$ and $h(t,x)$ is a strictly monotone decreasing function in x such that*

$$c_2\frac{T^2}{4} + c_1D^2\frac{T^4}{8} + c_1D^2\frac{T^2}{4} + b\frac{T^3}{8} < 1,$$

and

$$|h(t,x_1) - h(t,x_2)| \le b|x_1 - x_2|, \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

Then equation (1.1) has at most one T-periodic solution.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1), then we have

$$\dddot{x}_1(t) - \dddot{x}_2(t) + f(\dot{x}_1(t))(\ddot{x}_1(t))^2 - f(\dot{x}_2(t))(\ddot{x}_2(t))^2 + g(t)\dot{x}_1(t) - g(t)\dot{x}_2(t) \\ + h(t, x_1(t-\tau(t))) - h(t, x_2(t-\tau(t))) = 0.$$

Let $u(t) = x_1(t) - x_2(t)$. Then we get

$$\begin{aligned} &\dddot{u}(t) + f(\dot{x}_1(t))(\ddot{x}_1(t))^2 - f(\dot{x}_2(t))(\ddot{x}_2(t))^2 + g(t)\{\dot{x}_1(t) - \dot{x}_2(t)\} \\ &\quad + h(t, x_1(t-\tau(t))) - h(t, x_2(t-\tau(t))) = 0. \end{aligned} \tag{2.20}$$

Set

$$u(\bar{t_1}) = \max_{t\in\mathbb{R}} u(t), \quad u(\bar{t_2}) = \min_{t\in\mathbb{R}} u(t), \quad \text{where } t_1, t_2 \in \mathbb{R}.$$

Therefore we find

$$\dot{u}(\bar{t_1}) = \dot{x}_1(\bar{t_1}) - \dot{x}_2(\bar{t_1}) = 0, \quad \ddot{u}(\bar{t_1}) \le 0, \quad \dddot{u}(\bar{t_1}) \le 0, \tag{2.21}$$

and the following

$$\dot{u}(\bar{t_2}) = \dot{x}_1(\bar{t_2}) - \dot{x}_2(\bar{t_2}) = 0, \quad \ddot{u}(\bar{t_2}) \ge 0, \quad \dddot{u}(\bar{t_2}) \ge 0. \tag{2.22}$$

Now we shall prove that there exists a constant $\bar{\eta} \in \mathbb{R}$ such that

$$u(\bar{\eta}) = 0. \tag{2.23}$$

Contrarily, one of the following cases satisfies:

(i) $u(t) = x_1(t) - x_2(t) > 0$, for all $t \in \mathbb{R}$.

(ii) $u(t) = x_1(t) - x_2(t) < 0$, for all $t \in \mathbb{R}$.

If (i) holds, from (2.20) together with condition (A_3), $\dot{x}_1(\bar{t_1}) = \dot{x}_2(\bar{t_1})$, we get

$$\begin{aligned} \dddot{u}(\bar{t_1}) &= -f(\dot{x}_1(\bar{t_1}))(\ddot{x}_1(\bar{t_1}))^2 + f(\dot{x}_2(\bar{t_1}))(\ddot{x}_2(\bar{t_1}))^2 - g(\bar{t_1})\{\dot{x}_1(\bar{t_1}) - \dot{x}_2(\bar{t_1})\} \\ &\quad - h(t, x_1(\bar{t_1} - \tau(\bar{t_1}))) + h(t, x_2(\bar{t_1} - \tau(\bar{t_1}))) \\ &= -(\ddot{x}_1(\bar{t_1}))^2\{f(\dot{x}_1(\bar{t_1})) - f(\dot{x}_2(\bar{t_1}))\} \\ &\quad - \{h(\bar{t_1}, x_1(\bar{t_1} - \tau(\bar{t_1}))) - h(\bar{t_1}, x_2(\bar{t_1} - \tau(\bar{t_1})))\} \\ &> 0, \end{aligned}$$

which contradicts (2.21), so we have that (2.23) is true.
If (ii) holds, in view of (2.20) and condition (A_3), and $\dot{x}_1(\bar{t_2}) = \dot{x}_2(\bar{t_2})$, we obtain

$$\begin{aligned} \dddot{u}(\bar{t_2}) &= -f(\dot{x}_1(\bar{t_2}))(\ddot{x}_1(\bar{t_2}))^2 + f(\dot{x}_2(\bar{t_2}))(\ddot{x}_2(\bar{t_2}))^2 - g(\bar{t_2})\{\dot{x}_1(\bar{t_2}) - \dot{x}_2(\bar{t_2})\} \\ &\quad - h(t, x_1(\bar{t_2} - \tau(\bar{t_2}))) + h(t, x_2(\bar{t_2} - \tau(\bar{t_2}))) \\ &= -(\ddot{x}_1(\bar{t_2}))^2\{f(\dot{x}_1(\bar{t_2})) - f(\dot{x}_2(\bar{t_2}))\} \\ &\quad - \{h(\bar{t_2}, x_1(\bar{t_2} - \tau(\bar{t_2}))) - h(\bar{t_2}, x_2(\bar{t_2} - \tau(\bar{t_2})))\} \\ &< 0, \end{aligned}$$

which contradicts of (2.22), thus (2.23) is true.
Let $\bar{\eta} = nT + \bar{\gamma}$, where $\bar{\gamma} \in [0, T]$ and n is an integer. Then $u(\bar{\gamma}) = 0$.

Thus
$$|u(t)| = |u(\bar{\gamma}) + \int_{\bar{\gamma}}^{t} \dot{u}(s)ds| \leq \int_{\bar{\gamma}}^{t} |\dot{u}(s)|ds.$$
Again
$$|u(t)| = |u(\bar{\gamma}+T) + \int_{\bar{\gamma}+T}^{t} \dot{u}(s)ds| \leq \int_{t}^{\bar{\gamma}+T} |\dot{u}(s)|ds.$$
Hence by using Cauchy-Schwarz inequality we have
$$2|u(t)| \leq \int_{\bar{\gamma}}^{\bar{\gamma}+T} |\dot{u}(s)|ds = \int_{0}^{T} |\dot{u}(s)|ds \leq \sqrt{T}(\int_{0}^{T} |\dot{u}(s)|^2 ds)^{\frac{1}{2}} = \sqrt{T}|\dot{u}|_2.$$
Therefore
$$|u|_\infty \leq \frac{1}{2}\sqrt{T}|\dot{u}|_2. \tag{2.24}$$
Multiplying (2.20) by $\dddot{u}(t)$ and then integrating it over $[0, T]$ it follows
$$\begin{aligned}|\dddot{u}(t)|_2^2 = &- \int_0^T \{f(y_1)(\ddot{x}_1(t))^2 - f(y_2)(\ddot{x}_2(t))^2\}\dddot{u}(t)dt \\ &- \int_0^T g(t)\{\dot{x}_1(t) - \dot{x}_2(t)\}\dddot{u}(t)dt \\ &- \int_0^T \{h(t, x_1(t-\tau(t))) - h(t, x_2(t-\tau(t)))\}\dddot{u}(t)dt.\end{aligned}$$
From (A_3) we get
$$\begin{aligned}|\dddot{u}(t)|_2^2 \leq &\int_0^T |f(y_1) - f(y_2)||(\ddot{x}_2(t))^2||\dddot{u}(t)|dt + \int_0^T |g(t)||\dot{x}_1(t) - \dot{x}_2(t))||\dddot{u}(t)|dt \\ &+ b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||\dddot{u}(t)|dt \\ &+ \int_0^T |f(y_1)||(\ddot{x}_1(t))^2 - (\ddot{x}_2(t))^2||\dddot{u}(t)|dt.\end{aligned}$$
It follows from (A_0) and (A_3) that
$$\begin{aligned}|\dddot{u}(t)|_2^2 \leq c_1 &\int_0^T |y_1 - y_2||\ddot{x}_2(t)|^2|\dddot{u}(t)|dt + c_2\int_0^T |\dot{u}(t)||\dddot{u}(t)|dt \\ &+ b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||\dddot{u}(t)|dt \\ &+ c_1|\dot{x}_1|_\infty \int_0^T |\dot{x}_1 + \dot{x}_2||\dot{x}_1 - \dot{x}_2||\dddot{u}(t)|dt.\end{aligned}$$

Therefore from (2.18) and (2.19) and by using Cauchy-Schwarz inequality, we have

$$|\dddot{u}|_2^2 \leq c_1 D^2 |\dot{u}|_2 |\dddot{u}|_2 + c_2 |\dot{u}|_2 |\dddot{u}|_2 + b|u|_\infty \sqrt{T} |\dddot{u}|_2 + \frac{1}{2} c_1 T^2 D^2 |\dot{u}|_2 |\dddot{u}|_2.$$

From (2.13) and (2.24) we obtain

$$|\dddot{u}|_2^2 \leq \frac{1}{4} c_2 T^2 |\dddot{u}|_2^2 + \frac{1}{8} c_1 D^2 T^4 |\dddot{u}|_2^2 + \frac{1}{4} c_1 D^2 T^2 |\dddot{u}|_2^2 + \frac{1}{8} b T^3 |\dddot{u}|_2^2.$$

It follows that

$$\left\{1 - \left(c_2 \frac{T^2}{4} + c_1 D^2 \frac{T^4}{8} + c_1 D^2 \frac{T^2}{4} + b \frac{T^3}{8}\right)\right\} |\dddot{u}|_2^2 \leq 0. \qquad (2.25)$$

Since $u(t), \dot{u}(t), \ddot{u}(t)$ and $\dddot{u}(t)$ are T-periodic and continuous functions, in view of (A_3), (2.10), (2.23) and (2.25) we have

$$u(t) \equiv \dot{u}(t) \equiv \ddot{u}(t) \equiv \dddot{u}(t) = 0, \text{for all } \ t \in \mathbb{R}.$$

Thus

$$x_1(t) \equiv x_2(t), \ \text{for all } \ t \in \mathbb{R}.$$

Therefore (1.1) has at most one T-periodic solution.
This completes the proof of Theorem 2.2.

3 Main Result

The following theorem is the main result of the uniqueness of a T-periodic solution of (1.1).

Theorem 3.1 *Suppose that* $(A_0)-(A_3)$ *hold, then* (1.1) *has a unique T-periodic solution.*

Proof. By Theorem 2.2 states that (1.1) has at most one T-periodic solution. Thus to prove Theorem 3.1 it suffices to show that (1.1) has at least one T-periodic solution. To do this, we shall apply Theorem 2.1.
First we shall claim that the set of all possible T-periodic solutions of (2.3) is bounded.
Let $x(t)$ be a T-periodic solution of (2.3). Multiplying (2.3) by $\dddot{x}(t)$ and then integrating it from 0 to T in view of $(A_1) - (A_3)$ we obtain

$$\begin{aligned} \int_0^T |\dddot{x}(t)|^2 dt = & -\lambda \int_0^T f(\dot{x}(t))(\ddot{x}(t))^2 \dddot{x}(t) dt - \lambda \int_0^T g(t) \dot{x}(t) \dddot{x}(t) dt \\ & - \lambda \int_0^T h(t, x(t-\tau(t))) \dddot{x}(t) dt + \lambda \int_0^T p(t) \dddot{x}(t) dt. \end{aligned}$$

In view of (A_0) we have

$$\begin{aligned}\int_0^T |\dddot{x}(t)|^2 dt &= \frac{1}{3}\lambda \int_0^T f'(\dot{x}(t))(\ddot{x}(t))^4 dt - \lambda \int_0^T g(t)\dot{x}(t)\dddot{x}(t)dt \\ &\quad - \lambda \int_0^T h(t, x(t-\tau(t)))\dddot{x}(t)dt + \lambda \int_0^T p(t)\dddot{x}(t)dt \\ &\le \int_0^T \{|h(t, x(t-\tau(t))) - h(t,0)| + |h(t,0)|\}|\dddot{x}(t)|dt \\ &\quad + c_2 \int_0^T |\dot{x}(t)|\dddot{x}(t)|dt + \int_0^T |p(t)||\dddot{x}(t)|dt.\end{aligned}$$

Therefore from (A_2), (2.4), (2.5) and the inequality of Cauchy-Schwarz, we obtain

$$|\dddot{x}|_2^2 \le c_2|\dot{x}|_2|\dddot{x}|_2 + \frac{1}{2}b|\dot{x}|_2|\dddot{x}|_2 + [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2.$$

Which together with (A_2), implies that there exist positive constants D_1, D_2 and D_3 such that

$$|\ddot{x}|_\infty \le \frac{1}{2}\sqrt{T}|\dddot{x}|_2 := D_1, \quad |\dot{x}|_\infty \le \frac{T}{2}D_1 := D_2, \quad |x|_\infty \le d + \frac{T^2}{4}D_1 := D_3.$$

Let $D_0 = \max\{D_1, D_2, D_3\}$ and take $\Omega = \{x | x \in X, \ \ \|x\| < D_0\}$.
If $x \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}$, then x is a constant with $x(t) = D_0$ or $x(t) = -D_0$. Then

$$\begin{aligned}QNx &= \frac{1}{T}\int_0^T \{-f(\dot{x}(t))(\ddot{x}(t))^2 - g(t)\ddot{x}(t) - h(t, x(t-\tau(t))) + p(t)\}dt \\ &= \frac{1}{T}\int_0^T \{-h(t, \pm D_0) + p(t)\}dt \ne 0.\end{aligned}$$

So the conditions (1) and (2) in Theorem 2.1 hold.
Furthermore define a continuous function $H(x,\mu)$ by setting

$$H(x,\mu) = (1-\mu)x - \mu.\frac{1}{T}\int_0^T \{h(t,x) - p(t)\}dt, \ \ \mu \in [0,1].$$

It follows from (A_1) that

$$xH(x,\mu) \ne 0, \ \text{ for all } x \in \partial\Omega \cap KerL.$$

Thus $H(x,\mu)$ is a homotopy.
Hence by using the homotopy invariance theorem we have

$$\begin{aligned}deg\{QN, \Omega \cap KerL, 0\} &= deg\{-\frac{1}{T}\int_0^T [h(t,x) - p(t)]dt, \Omega \cap KerL, 0\} \\ &= deg\{x, \Omega \cap KerL, 0\} \ne 0.\end{aligned}$$

So condition (3) of Theorem 2.1 is satisfied.
In view of all the discussions above, we conclude from Theorem 2.1 that the main Theorem 3.1 is proved.

4 Example

In this section we shall provide an example to validate the main results.

Example 4.1. Let $h(t,x) = -\frac{1}{6\pi}x$, for all $t, x \in \mathbb{R}$. Then we get the third-order (FDE) with a time-delay as the following form

$$\dddot{x}(t) - \frac{1}{8}\left(\arctan \dot{x}(t)\right)(\ddot{x}(t))^2 + \frac{1}{8}(\sin 4t)\dot{x}(t) + \frac{x(t-\sin^2 t)}{6\pi} = \frac{1}{6\pi}e^{-\cos^2 t}, \quad (4.1)$$

has a unique π-periodic solution.
Proof. By (4.1) we obtain

$$g(t,x) = \frac{1}{8}\sin 4t, \quad f(y) = \arctan y, \quad h(t, x(t-\tau(t))) = -\frac{x(t-sin^2 t)}{6\pi},$$
$$\tau(t) = \sin^2 t, \ \ T = \pi \quad \text{and} \quad p(t) = \frac{1}{6\pi}e^{-\cos^2 t}.$$

Then from the condition (A_0), we find

$$c_1 = c_2 = \frac{1}{8}, \quad b = \frac{1}{6\pi}, \quad (d = \frac{1}{3} \text{ is an arbitrary small positive constant}).$$

Therefore we get

$$\begin{aligned} D := & \frac{1}{2}\frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]T}{1 - (c_2\frac{T^2}{4} + b\frac{T^3}{8})} \\ = & \frac{1}{2}\,\frac{(\frac{1}{6\pi}\frac{1}{3} + \frac{1}{6\pi})\pi}{1 - (\frac{1}{8}\frac{\pi^2}{4} - \frac{1}{6\pi}\frac{\pi^3}{8})} \cong 0.23, \end{aligned}$$

and

$$\begin{aligned} & c_2\frac{T^2}{4} + c_1 D^2\frac{T^4}{8} + c_1 D^2\frac{T^2}{4} + b\frac{T^3}{8} \\ & = \frac{1}{8}\,\frac{\pi^2}{4} + \frac{1}{8}\,(\frac{23}{100})^2\,\frac{\pi^4}{8} + \frac{1}{8}\,(\frac{23}{100})^2\,\frac{\pi^2}{4} + \frac{1}{6\pi}\,\frac{\pi^3}{8} \cong 0.61 < 1. \end{aligned}$$

It is obvious that the assumptions $(A_0) - (A_3)$ hold.
Hence by Theorem 3.1, equation (4.1) has a unique π-periodic solution.

5 Conclusion

Based on the coincidence degree theory and analysis techniques, new results on the existence and uniqueness of a T-periodic solution for the third-order functional differential equation, with a time-delay have been established. The obtained results extend existing results in the literature on deterministic systems. In addition an example is given to illustrate the new main results, which we obtain in this paper.

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References

[1] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, Periodic solutions for a kind of third-order delay differential equations with a deviating argument, *J. Math. Sci. Univ. Tokyo*, **18**, 35-49, 2011.

[2] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, Existence and uniqueness of a periodic solution for third-order delay differential equation with two deviating arguments, *IAENG International Journal of Applied Mathematics*, **42**(1), 7-12, 2012.

[3] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, 1985.

[4] R. E. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math., Springer-Verlag, Berlin, **568**, 1977.

[5] Z. Gui, Existence of positive periodic solutions to third-order delay differential equations, *Electron. J. Differential Equations*, **91**, 1-7, 2006.

[6] S. A. Iyase and O. J. Adeleke, On the existence and uniqueness of periodic solution for a third-order neutral functional differential equation, *Int. J. Math. Anal.*, **10**(17), 817-831, 2016.

[7] V. Kolmanovskii and A. Myshkis, *Introduction to the theory and applications of functional differential equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[8] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamical system*, Academic Press, New York, 1993.

[9] B. Liu, L. Huang and Y. Li, Existence of periodic solutions of a third-order functional differential equation, *Acta Math. Appl. Sin.*, **29**(2), 226-233, 2006.

[10] H. P. Long and X. H. Hou, Existence of periodic solutions of a third-order functional differential equation, *Math. Theory Appl.*, **28**(1), 82-85, 2008.

[11] G. Makay, Periodic solutions of linear functional differential and integral equations, *Differential Integral Equations*, **8**(8), 2177-2187, 1995.

[12] G. Makay, Periodic solutions of dissipative functional differential equations with infinite delay, *Tohoku Math. J.*, **48**, 355-362, 1996.

[13] H. O. Tẹjumọla and B. Tchegnani, Stability, boundedness and existence of periodic solutions of some third and fourth-order nonlinear delay differential equations, *J. Nigerian Math. Soc.*, **19**, 9-19, 2000.

[14] C. Tunç, On existence of periodic solution to certain nonlinear third-order differential equations, *Proyecciones*, **28**(2), 125-132, 2009.

[15] C. Tunç, On the existence of periodic solutions to nonlinear third-order ordinary differential equations with delay, *J. Comput. Anal. Appl.*, **12**(1-B), 191-201, 2010.

[16] C. Tunç, Existence of periodic solutions to nonlinear differential equations of third-order with multiple deviating arguments, *International Journal of Differential Equations*, **2012**, 1-13, 2012.

[17] C. Tunç, On existence of periodic solutions of functional differential equations of the third-order, *Appl. Comput. Math.*, **15**(2), 189-199, 2016.

[18] Z. Zhang and Z. Wang, Periodic solutions of third-order functional differential equations with variable coefficients, *Appl. Math. J. Chinese Univ. Ser. B*, **17**(2), 145-154, 2002.

[19] Z. Zhang and Z. Wang, Periodic solutions of the third-order functional differential equations, *J. Math. Anal. Appl.*, **292**(1), 115-134, 2004.

[20] Z. Zhang, Z. Wang and J. Yu, On the existence of periodic solutions of third-order functional differential equations, *Funkcial. Ekvac.*, **43**(3), 461-469, 2000.

[21] Y. Zhu, On stability, boundedness and existence of periodic solution of a kind of third-order nonlinear delay differential system, *Ann. Differential Equations*, **8**(2), 249-259, 1992.