DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 1, 2008
Electronic Journal,
reg. N P2375 at 07.03.97
ISSN 1817-2172

http://www.neva.ru/journal
http://www.math.spbu.ru/diffjournal/
e-mail: jodiff@mail.ru

Differential equations with randomness

EXTINCTION TIME OF SOME STOCHASTIC DIFFERENTIAL EQUATIONS

Diabate Nabongo
Universite d'Abobo-Adjame, UFR-SFA,
Departement de Mathmatiques et Informatiques,
16 BP 372 Abidjan 16, (Cote d'Ivoire)
E-mail:nabongo_diabate@yahoo.fr

Theodore K. Boni
Institut National Polytechnique
Houphouet-Boigny de Yamoussoukro,
BP 1093 Yamoussoukro, (Cote d'Ivoire)
E-mail:theokboni@yahoo.fr

Abstract. This paper concerns the study of the following stochastic differential equation:

$$dX = -f(X)dt + \sigma(X, \varepsilon)_0 dW,$$

$$X(0) = x_0 > 0,$$

where ε is a positive parameter, $f(s) \in C^1(\mathbb{R})$ is a positive and increasing function for the positive values of s, $\sigma \in C^1(\mathbb{R} \times \mathbb{R})$, W is a (one dimensional) Wiener process defined on a given probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration $\{\mathfrak{F}\}_{t \geq 0}$ satisfying the usual conditions. Under some conditions, we show that any solution of the above problem extincts in a finite time and its extinction

time goes to the one of the solution of a certain differential equation as ε goes to zero. We also extend the above results to other classes of extinction problems.

AMS subject classifications(2000): *60H10, 60G17, 34F05*

Key-words: *Stochastic differential equations, extinction, extinction time.*

1 Introduction

In this paper, we consider the following stochastic differential equation (SDE)

$$dX = -f(X)dt + \sigma(X,\varepsilon)_0 dW, \tag{1}$$

$$X(0) = x_0 > 0, \tag{2}$$

where ε is a positive parameter, $f(s) \in C^1(\mathbb{R})$ is a positive and increasing function for the positive values of s, $\sigma \in C^1(\mathbb{R}\times\mathbb{R})$, W is a (one dimensional) Wiener process defined on a given probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration $\{\mathfrak{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e it is right continuous and $\{\mathfrak{F}\}_0$ contains all $\mathbb{P}$-null sets (see [11]). Let us notice that our stochastic differential equation is given in Stratonovich form. It is well known that if a SDE is given in It form, it may be rewritten in Stratonovich form. In fact, if $X(t)$ solves $dX = -f(X)dt + g(X)dW$, where the SDE is given in It form, then $X(t)$ solves

$$dX = -f(X)dt + b(X)_0 dW$$

with

$$b(s) = f(s) + \frac{1}{2}g^{'}(s)g(s).$$

The first SDE dates back to 1930 and has been written by Uhlenbeck and Ornstein (see [17]) and has been used as a model for the Brownian motion (irregular motion of a particle suspended in a fluid first observed on the microscope by the botanist Brown in the XIX century). A mathematical study of SDE_s is due to It half a century age and they have extensively used in practically all branches of science and technology from physics to biology (see [1], [4], [9], [11]–[13], [15]-[17] and the references cited therein). We know that a solution $X(t)$ of the SDE in (1)–(2) may extinct in a finite time, namely there exists a finite time T such that $X(t) > 0$ for $t \in (0,T)$ but $X(t) = 0$ for $t \geq T$. The time T is called the extinction time of $X(t)$. In the case of ordinary differential equations (ODE), one may determine easily the extinction time in a lot of situations. In the case of SDE, the problem is more complicated because of the stochastic term. Our aim in this paper is to describe the extinction time when ε is small enough. Our work was motived by the paper of Friedman and Lacey in [8] and the one of Groisman and Rossi in [10], concerning the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). In [8], Friedman and Lacey have considered the following initial-boundary value problem

$$u_t(x,t) = \varepsilon \Delta u(x,t) + f(u(x,t)) \quad in \quad \Omega \times (0,T),$$

$$u(x,t) = 0 \quad on \quad \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) \quad in \quad \Omega,$$

where Δ is the Laplacian, ε is a positive parameter, $f(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^{+\infty} \frac{ds}{f(s)} < +\infty$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $u_0(x)$ is a continuous function in Ω. Under some additional conditions on the initial data, they have shown that if ε is small enough, the solution u of the above problem blows up in a finite time and its blow-up time goes to the one of the solution $\alpha(t)$ of the following differential equation

$$\alpha'(t) = f(\alpha(t)), \quad t > 0, \quad \alpha(0) = M,$$

as ε goes to zero where $M = \sup_{x\in\Omega} u_0(x)$.
Let us notice that the blow-up time of the solution $\alpha(t)$ of the differential equation is given by $T_0 = \int_M^{+\infty} \frac{ds}{f(s)}$. A similar result has been obtained by Groisman and Rossi in [10], where they have considered the SDE below

$$dX = f(X)dt + \sigma(X,\varepsilon)_0 dW,$$

$$X(0) = x_0.$$

In this problem, $\sigma(X,\varepsilon)$ which represents the diffusion of the SDE plays the same role as $\varepsilon\Delta u$ of the partial differential equation considered in [8]. In this paper, we obtain an analogous result for the problem of extinction.
Our paper is written in the following manner. In the next section, we show that when ε is small enough, any solution $X(t)$ of the SDE defined in (1)–(2) extincts in a finite time and its extinction time goes to the one of the solution of a certain differential equation. Finally, in the last section, we extend the results of section 2 to other classes of extinction problems.

2 Extinction times

In this section, under some conditions, we show that if ε is small enough, any solution $X(t)$ of (1)–(2) extincts in a finite and its extinction time goes the one of the solution of a certain ordinary differential equation (ODE). For the sake of simplicity, let us start with an example concerning the ODEs.
Consider the following ODE

$$y^{'}(t) = -y^p(t), \ \ t > 0, \tag{3}$$

$$y(0) = M > 0, \tag{4}$$

where $p > 0$. An explicit solution of (3)–(4) is given by

$$y(t) = \frac{1}{(M^{1-p} + (p-1)t)^{\frac{1}{p-1}}} \quad if \quad p > 1,$$

$$y(t) = Me^{-t} \quad if \quad p = 1,$$

$$y(t) = (M^{1-p} - (1-p)t)_+^{\frac{1}{p-1}} \quad if \quad 0 < p < 1,$$

where $(x)_+ = \max\{x, 0\}$.
Thus, we see that if $p \geq 1$, $0 < y(t) < M$ for $t \geq 0$ and $\lim_{t \to +\infty} y(t) = 0$, but if $0 < p < 1$, $0 < y(t) < M$ for $t \in [0, \frac{M^{1-p}}{1-p})$ but $y(t) = 0$ for $t \geq \frac{M^{1-p}}{1-p}$. In this case, we say that the solution $y(t)$ of (3)–(4) extincts in a finite time and the time $T_0 = \frac{M^{1-p}}{1-p}$ is called the extinction time of the solution $y(t)$.
More generally, consider the following ODE

$$\alpha^{'}(t) = -f(\alpha(t)), \ \ t > 0, \tag{5}$$

$$\alpha(0) = M > 0 \tag{6}$$

where $f(s) \in C^1(\mathbb{R})$ is a positive and increasing function for the positive values of s. It is well known that if the integral $\int_0^M \frac{ds}{f(s)}$ diverges then the solution $\alpha(t)$ of (5)–(6) satisfies $0 < \alpha(t) < M$ and $\lim_{t \to +\infty} \alpha(t) = 0$ but if the integral $\int_0^M \frac{ds}{f(s)}$ converges then the solution of (5)–(6) extincts in a finite time and its extinction time T_0 is given by $T_0 = \int_0^M \frac{ds}{f(s)}$.
Thus, we see that for certain ODEs, the extinction times of solutions are given explicitly.
Now, let us consider the SDE_s. Our first result is the following.

Theorem 1 *Suppose that $\sigma(X,\varepsilon) = \varepsilon X$ and $\int_0 \frac{ds}{f(s)} < +\infty$. Then for almost every ω, any solution $X(t)$ of the SDE in (1)–(2) extincts in a finite time for every $\varepsilon > 0$ and its extinction time T_ε^ω goes to $\int_0^{x_0} \frac{ds}{f(s)}$ as ε tends to zero.*

Proof. Since $\sigma(X,\varepsilon) = \varepsilon X$, then we have

$$dX = -f(X)dt + \varepsilon X_0 dW,$$

$$X(0) = x_0.$$

Setting $Z = Xe^{-\varepsilon W}$, it is not hard to see that $dZ = e^{-\varepsilon W}dX - \varepsilon e^{-\varepsilon W}X_0 dW$, which implies that

$$dZ = -e^{-\varepsilon W} f(e^{\varepsilon W} Z)dt.$$

This gives a non-autonomous ODE for each ω such that $W(.,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = -e^{-\varepsilon W(t,\omega)} f(e^{\varepsilon W(t,\omega)} Z_\omega(t)), \quad t > 0,$$

$$Z_\omega(0) = x_0.$$

In the above problem ω is regarded as a parameter. Consider $M > 0$ and define

$$A_M = \{\omega : W(.,\omega) \quad \textit{is continuous and} \quad \max_{0 \le t \le T+1} |W(.,\omega)| \le M\},$$

where $T = \int_0^{x_0} \frac{ds}{f(s)}$. Let Z_1 be the solution of the following ODE

$$Z_1'(t) = -e^{-\varepsilon M} f(e^{-\varepsilon M} Z_1(t)), \quad t > 0,$$

$$Z_1(0) = x_0.$$

Similarly, let Z_2 be the solution of the ODE below

$$Z_2'(t) = -e^{\varepsilon M} f(e^{\varepsilon M} Z_2(t)), \quad t > 0,$$

$$Z_2(0) = x_0.$$

It is not hard to see that $Z_1(t)$ extincts at the time $T_1^\varepsilon = e^{2\varepsilon M} \int_0^{e^{-\varepsilon M}x_0} \frac{ds}{f(s)}$ and $Z_2(t)$ extincts at the time $T_2^\varepsilon = e^{-2\varepsilon M} \int_0^{e^{\varepsilon M}x_0} \frac{ds}{f(s)}$. By the maximum principle for ODE, we discover that

$$Z_2(t) \le Z_\omega(t) \le Z_1(t) \quad for \quad t \ge 0, \quad \omega \in A_M.$$

Therefore, if $\omega \in A_M$ then $Z_\omega(t)$ extincts in a finite time T_ε^ω such that $T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon$. Obviously, we have $\lim_{\varepsilon\to 0} T_2^\varepsilon = \lim_{\varepsilon\to 0} T_1^\varepsilon = \int_0^{x_0} \frac{ds}{f(s)}$. Consequently, we obtain $\lim_{\varepsilon\to 0} T_\varepsilon^\omega = \int_0^{x_0} \frac{ds}{f(s)}$. Use the fact that $\mathbb{P}(\bigcup_{M=1}^\infty A_M) = 1$ and $X = e^{\varepsilon W} Z$ to complete the rest of the proof. $\Box$
Consider now the SDE in to sense. It may be rewritten as follows

$$dX = -f(X)dt + \varepsilon X dW, \tag{7}$$

$$X(0) = x_0. \tag{8}$$

We have the following result.

Theorem 2 *Theorem 1 remains valid if $X(t)$ solves (7)–(8).*

Proof. Applying the formula given in the introduction, the SDE in (7)–(8) may be rewritten in Stratonovich sense in the following manner

$$dX = -f(X) - \frac{\varepsilon^2 X}{2} + \varepsilon X_0 dW,$$

$$X(0) = x_0.$$

Setting $Z = Xe^{-\varepsilon W}$, it is not hard to see that

$$dZ = e^{-\varepsilon W} dX - \varepsilon e^{-\varepsilon W} X_0 dW$$

which implies that

$$dZ = -[e^{\varepsilon W} f(e^{\varepsilon W} Z) + \frac{\varepsilon^2}{2} e^{\varepsilon W} Z]dt.$$

This gives a non-autonomous ODE for each ω such that $W(.,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = -[e^{\varepsilon W(t,\omega)} f(e^{\varepsilon W(t,\omega)} Z_\omega(t)) + \frac{\varepsilon^2}{2} e^{\varepsilon W(t,\omega)} Z_\omega(t)], \quad t > 0,$$

$$Z_\omega(0) = x_0.$$

Consider $M > 0$ and define

$$A_M = \{\omega : W(.,\omega) \quad \textit{is continuous and} \quad \max_{0\leq t\leq T+1} |W(.,\omega)| \leq M\},$$

where $T = \int_0^{x_0} \frac{ds}{f(s)}$. Let Z_1 be the solution of the following ODE

$$\dot{Z}_1(t) = -[e^{-\varepsilon M} f(e^{-\varepsilon M} Z_1(t)) + \frac{\varepsilon^2}{2} e^{-\varepsilon M} Z_1(t)], \quad t > 0,$$

$$Z_1(0) = x_0.$$

Similarly, let Z_2 be the solution of the ODE below

$$\dot{Z}_2(t) = -[e^{\varepsilon M} f(e^{\varepsilon M} Z_2(t)) + \frac{\varepsilon^2}{2} e^{\varepsilon M} Z_2(t)], \quad t > 0,$$

$$Z_2(0) = x_0.$$

It is not difficult to see that $Z_1(t)$ extincts at the time

$$T_1^\varepsilon = e^{2\varepsilon M} \int_0^{e^{-\varepsilon M} x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2} e^{\varepsilon M} \sigma}$$

and $Z_2(t)$ extincts at the time

$$T_2^\varepsilon = e^{-2\varepsilon M} \int_0^{e^{\varepsilon M} x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2} e^{-\varepsilon M} \sigma}.$$

We observe that the above times are finite because $\int_0 \frac{ds}{f(s)}$ is finite. Owing to the maximum principle for ODE, we obtain

$$Z_2(t) \leq Z_\omega(t) \leq Z_1(t) \quad for \quad t > 0, \quad \omega \in A_M.$$

We deduce that if $\omega \in A_M$ then $Z_\omega(t)$ extincts at the time T_ε^ω which is estimated as follows

$$T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon.$$

Since $\int_0 \frac{ds}{f(s)}$ is finite, applying the dominated convergence theorem, it is not hard to see that

$$\lim_{\varepsilon \to 0} T_1^\varepsilon = \lim_{\varepsilon \to 0} T_2^\varepsilon = \int_0^{x_0} \frac{d\sigma}{f(\sigma)}.$$

Hence, we have

$$\lim_{\varepsilon \to 0} T_\varepsilon^\omega = \int_0^{x_0} \frac{ds}{f(s)}.$$

Use the fact that $X = e^{\varepsilon W} Z$ and $\mathbb{P}(\cup_{M=1}^\infty A_M) = 1$ to complete the rest of the proof. $\square$

Now, let us consider the general case. We have the following result.

Theorem 3 *Let $\phi_\varepsilon(s, x)$ be the flux associated to the ODE $\dot{y} = \sigma(y, \varepsilon)$, $y(0) = x$ and let $H(s, x, \varepsilon) = \frac{f(\phi_\varepsilon(s,x))\sigma(x,\varepsilon)}{\sigma(\phi_\varepsilon(s,x),\varepsilon)}$. Suppose that*

$$\lim_{\varepsilon \to 0} H(s, x, \varepsilon) = f(x), \tag{9}$$

$$H(s,x,\varepsilon) \geq H(t,x,\varepsilon) \quad if \quad s \geq t \tag{10}$$

and there exists a function $k_s(x)$ such that

$$\frac{1}{H(s,x,\varepsilon)} \leq k_s(x) \in L^1([0,x_0]). \tag{11}$$

Then for almost every ω, any solution of (1)–(2) extincts in a finite time and its extinction time T_ε^ω satisfies the following relation $\lim_{\varepsilon\to 0} T_\varepsilon^\omega = T_0$ where $T_0 = \int_0^{x_0} \frac{ds}{f(s)}$. In addition if for every $s \in \mathbb{R}$, there exists l_s such that

$$\frac{\partial}{\partial\varepsilon}\frac{1}{H(s,x,\varepsilon)} \leq l_s(x) \in L^1([0,x_0]), \tag{12}$$

then there exists a random variable $K = K(\omega)$ such that with total probability $|T_\varepsilon - T_0| \leq \varepsilon K$.

Proof. Since $\phi_\varepsilon(t,x)$ is the flux of the following ODE

$$\dot{y} = \sigma(y,\varepsilon),$$

$$y(0) = x.$$

we have

$$(\phi_\varepsilon)_t(t,x) = \sigma(\phi_\varepsilon(t,x),\varepsilon), \tag{13}$$

$$\phi_\varepsilon(0,x) = x. \tag{14}$$

Let $Z_\omega(t)$ be the solution of the Random differential equation

$$dZ_\omega = \frac{-f(\phi_\varepsilon(W(t,\omega), Z_\omega(t)))}{\phi_x(W(t,\omega), Z_\omega(t))}, \quad t > 0, \tag{15}$$

$$Z_\omega(0) = x_0. \tag{16}$$

Setting $X(t,\omega) = \phi_\varepsilon(W(t,\omega), Z_\omega(t))$, we observe that

$$\begin{aligned} dX &= (\phi_\varepsilon)_t(W,Z_\omega)dW + (\phi_\varepsilon)_x(W,Z_\omega)dZ_\omega \\ &= \sigma(\phi_\varepsilon(W,Z_\omega),\varepsilon)dW + (\phi_\varepsilon)_x(W,Z_\omega)dZ_\omega \\ &= \sigma(X,\varepsilon)dW - f(X)dX. \end{aligned}$$

Therefore, X is a solution of the SDE defined in (1)–(2). On the other hand, from (13)–(14), we get

$$\frac{d\phi_\varepsilon}{\sigma(\phi_\varepsilon(t,x),\varepsilon)} = dt,$$

which implies that

$$\int_x^{\phi_\varepsilon(t,x)} \frac{ds}{\sigma(s,\varepsilon)} = t. \tag{17}$$

Take the derivative in x of (17) to obtain

$$\frac{(\phi_\varepsilon)_x(t,x)}{\sigma(\phi_\varepsilon(t,x),\varepsilon)} - \frac{1}{\sigma(x,\varepsilon)} = 0,$$

which implies that

$$(\phi_\varepsilon)_x(t,x) = \frac{\sigma(\phi_\varepsilon(t,x),\varepsilon)}{\sigma(x,\varepsilon)}.$$

It follows from (13) that

$$\dot{Z}_\omega = \frac{-f(\phi_\varepsilon(W(t,\omega),Z_\omega(t)))\sigma(Z_\omega(t),\varepsilon)}{\sigma(\phi_\varepsilon(W(t,\omega),Z_\omega(t)),\varepsilon)}.$$

Take the expression of H to arrive at

$$\dot{Z}_\omega(t) = -H(W(t,\omega),Z_\omega(t),\varepsilon), \quad t>0,$$

$$Z_\omega(0) = x_0.$$

Consider $M>0$ and define

$$A_M = \{\omega : W(.,\omega) \quad \textit{is continuous and} \quad \max_{0\le t\le T_0+1} |W(.,\omega)| \le M\},$$

where $T_0 = \int_0^{x_0} \frac{ds}{f(s)}$. Let $Z_1(t)$ be the solution of the following ODE

$$Z_1'(t) = -H(-M,Z_1(t),\varepsilon), \quad t>0,$$

$$Z_1(0) = x_0$$

and let $Z_2(t)$ be the one of the ODE below

$$Z_2'(t) = -H(M,Z_2(t),\varepsilon), \quad t>0,$$

$$Z_2(0) = x_0.$$

From (11), we observe that the integrals $\int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)}$ and $\int_0^{x_0} \frac{ds}{H(M,s,\varepsilon)}$ are finite. We deduce that the solution $Z_1(t)$ extincts in a finite time $T_1^\varepsilon = \int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)}$ and the solution $Z_2(t)$ extincts in a finite time $T_2^\varepsilon = \int_0^{x_0} \frac{ds}{H(M,s,\varepsilon)}$. The maximum principle for ODE implies that

$$Z_2(t) \leq Z_\omega(t) \leq Z_1(t) \quad for \quad t \geq 0, \quad \omega \in A_M.$$

Therefore, if $\omega \in A_M$ then $Z_\omega(t)$ extincts in a finite time T_ε^ω such that

$$T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon.$$

Due to (11) and the dominated convergence theorem, it is not hard to see that

$$\lim_{\varepsilon \to 0} T_1^\varepsilon = \lim_{\varepsilon \to 0} T_2^\varepsilon = T_0 = \int_0^{x_0} \frac{ds}{f(s)}.$$

Therefore, $\lim_{\varepsilon \to 0} T_\varepsilon^\omega = T_0$. We observe that $\mathbb{P}(\bigcup_{M=1}^{+\infty} A_M) = 1$. Use Taylor's expansion to obtain

$$\frac{1}{H(-M,s,\varepsilon)} = \frac{1}{H(-M,s,0)} + \varepsilon \frac{\partial}{\partial \varepsilon} \frac{1}{H(-M,s,\tilde{\varepsilon})},$$

where $\tilde{\varepsilon}$ is an intermediate value between 0 and ε. We deduce from (12) that

$$\int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)} \leq \int_0^{x_0} \frac{ds}{f(s)} + \varepsilon \int_0^{x_0} l_{-M}(s) ds.$$

It follows that there exists a random variable $K = K(\omega)$ such that

$$|T_\varepsilon - T_0| \leq \varepsilon K$$

and the proof is complete. □

Remark 1 *If $\sigma(x,\varepsilon) = \varepsilon x$ and $f(s) = s^q$ with $0 < q < 1$ then*

$$\phi_\varepsilon(s,x) = x e^{\varepsilon s} \quad and \quad H(s,x,\varepsilon) = x^q e^{-\varepsilon(1-q)s}.$$

If $\sigma(x,\varepsilon) = \varepsilon x^p$ and $f(s) = s^q$ with $0 < p \leq q < 1$, then

$$\phi_\varepsilon(s,x) = (x^{1-p} + (1-p)\varepsilon s)^{\frac{1}{1-p}} \quad and \quad H(s,x,\varepsilon) = x^p (x^{1-p} + (1-p)\varepsilon s)^{\frac{q-p}{1-p}}.$$

3 Other extinctions times

In this section, we show the possibility to extend the results of the previous section to another problem of extinction which is called problem of quenching. To illustrate our analysis, let us consider the following ODE

$$y^{'}(t) = -y^{-p}(t), \ \ t > 0, \tag{18}$$

$$y(0) = M > 0, \tag{19}$$

where $p > 0$. An explicit solution of (18)–(19) is given by

$$y(t) = (M^{p+1} - (p+1)t)^{\frac{1}{p+1}} \quad for \quad t \in [0, \frac{M^{p+1}}{p+1}).$$

Hence we see that if $t = \frac{M^{p+1}}{p+1}$ then $y(t)$ reaches the value zero which implies that $y^{'}(t)$ explodes at the same time. In this case, we say that the solution $y(t)$ quenches in a finite time.

More generally, consider the following ODE

$$y^{'}(t) = -f(y(t)), \ \ t > 0, \tag{20}$$

$$y(0) = M > 0, \tag{21}$$

where $f(s)$ is a positive, decreasing function for the positive values of s, $\lim_{s \to 0^+} f(s) = +\infty$, $\int_0^M \frac{ds}{f(s)} < +\infty$. It is not hard to see that $M > y(t) > 0$ for $t \in [0, T_0)$ but $\lim_{t \to T_0} y(t) = 0$ where $T_0 = \int_0^M \frac{ds}{f(s)}$. Therefore, we discover that $y(t)$ quenches in a finite time and the time T_0 is called the quenching time of $y(t)$. Let us also notice that the derivative in t of $y(t)$ explodes at the time T_0.

Now, let us consider the following SDE

$$dX = -f(X)dt + \sigma(X, \varepsilon)_0 dW, \tag{22}$$

$$X(0) = x_0 > 0, \tag{23}$$

where $f(s)$ is positive, decreasing function for the positive values of s, $\lim_{s \to 0^+} f(s) = +\infty$. We have the following result.

Theorem 4 *Suppose that $\sigma(X, \varepsilon) = \varepsilon X$ and $\int_0 \frac{ds}{f(s)} < +\infty$. Then for almost every ω, any solution $X(t)$ of the SDE in (22)–(23) quenches in a finite time for every $\varepsilon > 0$ and its quenching time T_ε^ω goes to $\int_0^{x_0} \frac{ds}{f(s)}$ as ε tends to zero.*

Proof. Since $\sigma(X,\varepsilon)=\varepsilon X$, then we have

$$dX=-f(X)dt+\varepsilon X_0 dW,$$

$$X(0)=x_0.$$

Take $Z=Xe^{-\varepsilon W}$. A straightforward computation yields

$$dZ=dXe^{-\varepsilon W}-\varepsilon Xe^{-\varepsilon W}dW,$$

which implies that

$$dZ=-e^{-\varepsilon W}f(X)dt.$$

Use the expression of X to obtain

$$dZ=-e^{-\varepsilon W}f(e^{\varepsilon W}Z).$$

Introduce the following Random differential equation

$$\dot{Z}_\omega(t)=-e^{-\varepsilon W(t,\omega)}f(e^{\varepsilon W(t,\omega)}Z_\omega(t)),$$

$$Z_\omega(0)=x_0.$$

Consider $M>0$ and define

$$A_M=\{\omega : W(.,\omega)\quad \textit{is continuous and}\quad \max_{0\le t\le T_0+1}|W(.,\omega)|\le M\},$$

where $T_0=\int_0^{x_0}\frac{ds}{f(s)}$. Let $Z_1(t)$ be the solution of the following ODE

$$\dot{Z}_1(t)=-e^{-\varepsilon M}f(e^{-\varepsilon M}Z_1(t)),\quad t>0,$$

$$Z_1(0)=x_0,$$

and let $Z_2(t)$ be the one of the ODE below

$$\dot{Z}_2(t)=-e^{\varepsilon M}f(e^{\varepsilon M}Z_2(t)),\quad t>0,$$

$$Z_2(0)=x_0.$$

Setting $g_1(s)=e^{-\varepsilon M}f(e^{-\varepsilon M}s)$ and $g_2(s)=e^{\varepsilon M}f(e^{\varepsilon M}s)$, one easily sees that $\lim_{s\to 0^+}g_1(s)=+\infty$, $\lim_{s\to 0^+}g_2(s)=+\infty$,

$$\int_0^{x_0}\frac{ds}{g_1(s)}=e^{2\varepsilon M}\int_0^{e^{-\varepsilon M}x_0}\frac{d\sigma}{f(\sigma)}<+\infty,$$

$$\int_0^{x_0} \frac{ds}{g_2(s)} = e^{-2\varepsilon M} \int_0^{e^{\varepsilon M} x_0} \frac{d\sigma}{f(\sigma)} < +\infty.$$

On the other hand, the maximum principle for ODE implies that

$$Z_2(t) \leq Z_\omega(t) \leq Z_1(t),$$

as long as all of them are defined. Hence, it is not hard to see that $Z_1(t)$ quenches at the time

$$T_1^\varepsilon = e^{2\varepsilon M} \int_0^{e^{-\varepsilon M} x_0} \frac{ds}{f(s)}$$

and $Z_2(t)$ quenches at the time

$$T_2^\varepsilon = e^{-2\varepsilon M} \int_0^{e^{\varepsilon M} x_0} \frac{ds}{f(s)}.$$

We deduce that for $\omega \in A_M$, $Z_\omega(t)$ quenches at the time T_ε^ω which obeys the following estimates

$$T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon.$$

Let us notice that $\lim_{\varepsilon\to 0} T_2^\varepsilon = \lim_{\varepsilon\to 0} T_1^\varepsilon = \int_0^{x_0} \frac{ds}{f(s)}$. Therefore, we conclude that $\lim_{\varepsilon\to 0} T_\varepsilon^\omega = \int_0^{x_0} \frac{ds}{f(s)}$. Since $\mathbb{P}(\bigcup_{M=1}^\infty A_M) = 1$, using the fact that $X = e^{\varepsilon W} Z$, we see that the solution $X(t)$ of the SDE quenches in a finite time with total probability and its quenching time T_ε^ω goes to $\int_0^{x_0} \frac{ds}{f(s)}$ as ε tends to zero. This ends the proof. □

When $\sigma(X, \varepsilon) = \varepsilon X$, the above theorem reveals that any solution of (22)–(23) quenches in a finite time. In the following, we want to know what happens if we consider the SDE in to sense. In this case, our problem can be rewritten as follows

$$dX = -f(X)dt + \varepsilon X dW, \tag{24}$$

$$X(0) = x_0. \tag{25}$$

The result below gives an answer when the SDE is taken in to sense.

Theorem 5 *Theorem 4 remains valid if $X(t)$ solves (24)–(25).*

Proof. Taking into account the formula given in the introduction, the SDE in (24)–(25) may be rewritten in Stratonovich sense in the following manner

$$dX = -f(X) - \frac{\varepsilon^2 X}{2} + \varepsilon X_0 dW,$$

$$X(0) = x_0.$$

Setting $Z = Xe^{-\varepsilon W}$, we easily see that

$$dZ = e^{-\varepsilon W} dX - \varepsilon e^{-\varepsilon W} X_0 dW,$$

which implies that

$$dZ = -[e^{\varepsilon W} f(e^{\varepsilon W} Z) + \frac{\varepsilon^2}{2} e^{\varepsilon W} Z] dt.$$

This gives a non-autonomous ODE for each ω such that $W(.,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = -[e^{\varepsilon W(t,\omega)} f(e^{\varepsilon W(t,\omega)} Z_\omega(t)) + \frac{\varepsilon^2}{2} e^{\varepsilon W(t,\omega)} Z_\omega(t)], \quad t > 0,$$

$$Z_\omega(0) = x_0.$$

Consider $M > 0$ and define

$$A_M = \{\omega \ : \ W(.,\omega) \quad \textit{is continuous and} \quad \max_{0 \le t \le T+1} |W(.,\omega)| \le M\},$$

where $T = \int_0^{x_0} \frac{ds}{f(s)}$. Let Z_1 be the solution of the following ODE

$$\dot{Z}_1(t) = -[e^{-\varepsilon M} f(e^{-\varepsilon M} Z_1(t)) + \frac{\varepsilon^2}{2} e^{-\varepsilon M} Z_1(t)], \quad t > 0,$$

$$Z_1(0) = x_0.$$

Similarly, let Z_2 be the solution of the ODE below

$$\dot{Z}_2(t) = -[e^{\varepsilon M} f(e^{\varepsilon M} Z_2(t)) + \frac{\varepsilon^2}{2} e^{\varepsilon M} Z_2(t)], \quad t > 0,$$

$$Z_2(0) = x_0.$$

Setting

$$g_1(s) = e^{-\varepsilon M} f(e^{-\varepsilon M} s) + \frac{\varepsilon^2}{2} e^{-\varepsilon M} s,$$

and

$$g_2(s) = e^{\varepsilon M} f(e^{\varepsilon M} s) + \frac{\varepsilon^2}{2} e^{\varepsilon M} s,$$

we observe that

$$\lim_{s\to 0^+} g_1(s) = +\infty \quad and \quad \lim_{s\to 0^+} g_2(s) = +\infty.$$

Taking into account g_1 and g_2, the above ODE_s may be rewritten as follows

$$Z_1^{'}(t) = -g_1(Z_1(t)), \quad t > 0, \quad Z_1(0) = x_0,$$

$$Z_2^{'}(t) = -g_1(Z_2(t)), \quad t > 0, \quad Z_2(0) = x_0.$$

On the other hand, a routine calculation yields

$$\int_0^{x_0} \frac{ds}{g_1(s)} = \int_0^{e^{-\varepsilon M}x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2}e^{\varepsilon M}\sigma} \leq \int_0^{e^{-\varepsilon M}x_0} \frac{d\sigma}{f(\sigma)} < +\infty$$

and

$$\int_0^{x_0} \frac{ds}{g_2(s)} = \int_0^{e^{\varepsilon M}x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2}e^{-\varepsilon M}\sigma} \leq \int_0^{e^{\varepsilon M}x_0} \frac{d\sigma}{f(\sigma)} < +\infty.$$

Hence, we easily see that $Z_1(t)$ quenches at the time

$$T_1^\varepsilon = \int_0^{e^{-\varepsilon M}x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2}e^{\varepsilon M}\sigma}$$

and $Z_2(t)$ quenches at the time

$$T_2^\varepsilon = \int_0^{e^{\varepsilon M}x_0} \frac{d\sigma}{f(\sigma) + \frac{\varepsilon^2}{2}e^{-\varepsilon M}\sigma}.$$

According to the maximum principle for ODE, we obtain

$$Z_2(t) \leq Z_\omega(t) \leq Z_1(t) \quad for \quad \omega \in A_M$$

as long as all of them are defined. We deduce that if $\omega \in A_M$ then $Z_\omega(t)$ quenches at the time T_ε^ω which satisfies the following estimates

$$T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon.$$

Since $\int_0 \frac{ds}{f(s)}$ is finite, using the dominated convergence theorem, we easily derive the following equalities

$$\lim_{\varepsilon\to 0} T_1^\varepsilon = \lim_{\varepsilon\to 0} T_2^\varepsilon = \int_0^{x_0} \frac{ds}{f(s)}.$$

Hence, we have

$$\lim_{\varepsilon\to 0} T_\varepsilon^\omega = \int_0^{x_0} \frac{ds}{f(s)}.$$

Use the fact that $X = e^{\varepsilon W} Z$ and $\mathbb{P}(\cup_{M=1}^{\infty} A_M) = 1$ to complete the rest of the proof. $\square$

Now, let us consider the general case concerning the phenomenon of quenching. We can derive the following important result.

Theorem 6 *Let $\phi_\varepsilon(s, x)$ be the flux associated to the ODE $\dot{y} = \sigma(y, \varepsilon)$, $y(0) = x$ and let $H(s, x, \varepsilon) = \frac{f(\phi_\varepsilon(s,x))\sigma(x,\varepsilon)}{\sigma(\phi_\varepsilon(s,x),\varepsilon)}$. Suppose that*

$$\lim_{x\to 0^+} H(s, x, \varepsilon) = +\infty, \quad \lim_{\varepsilon\to 0} H(s, x, \varepsilon) = f(x), \tag{26}$$

$$H(s, x, \varepsilon) \geq H(t, x, \varepsilon) \quad if \quad s \geq t \tag{27}$$

and there exists a function $k_s(x)$ such that

$$\frac{1}{H(s, x, \varepsilon)} \leq k_s(x) \in L^1([0, x_0]). \tag{28}$$

Then for almost every ω, any solution of (22)–(23) quenches in a finite time and its quenching time T_ε^ω satisfies the following relation $\lim_{\varepsilon\to 0} T_\varepsilon^\omega = T_0$ where $T_0 = \int_0^{x_0} \frac{ds}{f(s)}$. In addition if for every $s \in \mathbb{R}$, there exists l_s such that

$$\frac{\partial}{\partial\varepsilon}\frac{1}{H(s, x, \varepsilon)} \leq l_s(x) \in L^1([0, x_0]), \tag{29}$$

then there exists a random variable $K = K(\omega)$ such that with total probability $|T_\varepsilon - T_0| \leq \varepsilon K$.

Proof. Since $\phi_\varepsilon(t, x)$ is the flux of the following ODE

$$\dot{y} = \sigma(y, \varepsilon),$$

$$y(0) = x,$$

we easily see that

$$(\phi_\varepsilon)_t(t, x) = \sigma(\phi_\varepsilon(t, x), \varepsilon),$$

$$\phi_\varepsilon(0, x) = x.$$

Let $Z_\omega(t)$ be the solution of the Random differential equation

$$dZ_\omega(t) = \frac{-f(\phi_\varepsilon(W(t,\omega), Z_\omega(t))}{\phi_\varepsilon(W(t,\omega), Z_\omega(t))}, \quad t > 0,$$

$$Z_\omega(0) = x_0.$$

As in the proof of Theorem 3, we find that

$$\dot{Z}_\omega(t) = -H(W(t,\omega), Z_\omega(t), \varepsilon), \quad t > 0,$$

$$Z_\omega(0) = x_0$$

and $X(t,\omega) = \phi_\varepsilon(W(t,\omega), Z_\omega(t))$ is a solution of (22)–(23).
Consider $M > 0$ and define

$$A_M = \{\omega : W(.,\omega) \quad \textit{is continuous and} \quad \max_{0 \le t \le T_0+1} |W(.,\omega)| \le M\},$$

where $T_0 = \int_0^{x_0} \frac{ds}{f(s)}$. Let $Z_1(t)$ be the solution of the following ODE

$$Z_1^{'}(t) = -H(-M, Z_1(t), \varepsilon), \quad t > 0$$

$$Z_1(0) = x_0$$

and let $Z_2(t)$ be the one of the ODE below

$$Z_2^{'}(t) = -H(M, Z_2(t), \varepsilon), \quad t > 0,$$

$$Z_2(0) = x_0.$$

Setting

$$g_1(s) = H(-M, s, \varepsilon) \quad \textit{and} \quad g_2(s) = H(M, s, \varepsilon),$$

we easily see that

$$\lim_{s \to 0^+} g_1(s) = +\infty \quad \textit{and} \quad \lim_{s \to 0^+} g_2(s) = +\infty.$$

We also observe that the integrals $\int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)}$ and $\int_0^{x_0} \frac{ds}{H(M,s,\varepsilon)}$ are finite because of (28). Hence, we deduce that the solution $Z_1(t)$ quenches in a finite time $T_1^\varepsilon = \int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)}$ and the solution $Z_2(t)$ quenches in a finite time $T_2^\varepsilon = \int_0^{x_0} \frac{ds}{H(M,s,\varepsilon)}$.
On the other hand, the maximum principle for ODE implies that

$$Z_2(t) \le Z_\omega(t) \le Z_1(t) \quad \textit{for} \quad \omega \in A_M,$$

as long as all of them are defined. Therefore, if $\omega \in A_M$ then $Z_\omega(t)$ quenches in a finite time T_ε^ω such that

$$T_2^\varepsilon \leq T_\varepsilon^\omega \leq T_1^\varepsilon.$$

Due to (26) and the dominated convergence theorem, it is not hard to see that

$$\lim_{\varepsilon\to 0} T_1^\varepsilon = \lim_{\varepsilon\to 0} T_2^\varepsilon = T_0 = \int_0^{x_0} \frac{ds}{f(s)}.$$

Therefore, $\lim_{\varepsilon\to 0} T_\varepsilon^\omega = T_0$. Obviously $\mathbb{P}(\bigcup_{M=1}^{+\infty} A_M) = 1$. Using Taylor's expansion, we find that

$$\frac{1}{H(-M,s,\varepsilon)} = \frac{1}{H(-M,s,0)} + \varepsilon\frac{\partial}{\partial\varepsilon}\frac{1}{H(-M,s,\tilde{\varepsilon})},$$

where $\tilde{\varepsilon}$ is an intermediate value between 0 and ε. We deduce from (29) that

$$\int_0^{x_0} \frac{ds}{H(-M,s,\varepsilon)} \leq \int_0^{x_0} \frac{ds}{f(s)} + \varepsilon \int_0^{x_0} l_{-M}(s)ds.$$

It follows that there exists a random variable $K = K(\omega)$ such that

$$|T_\varepsilon - T_0| \leq \varepsilon K$$

and the proof is complete. $\square$

Remark 2 *If $\sigma(x,\varepsilon) = \varepsilon x$ and $f(s) = s^{-q}$ with $q > 0$, then*

$$\phi_\varepsilon(s,x) = xe^{\varepsilon s} \quad \textit{and} \quad H(s,x,\varepsilon) = x^{-q}e^{-\varepsilon(q+1)s}.$$

If $\sigma(x,\varepsilon) = \varepsilon x^p$ and $f(s) = s^{-q}$ with $q > 0$, $p > 1$ then

$$\phi_\varepsilon(s,x) = (x^{1-p} + (1-p)\varepsilon s)^{\frac{1}{1-p}} \quad \textit{and} \quad H(s,x,\varepsilon) = x^p(x^{1-p} + (1-p)\varepsilon s)^{\frac{p+q}{p-1}}.$$

References

[1] **L. Arnold,** Stochastic differential equations, Theory and applications, *Wiley, New-York*, (1974).

[2] **T. K. Boni,** Extinction for discretizations of some semilinear parabolic equations, *C.R.A.S*, Serie I, **333** (2001), 795-800.

[3] **T. K. Boni,** On the extinction of the solution for an elliptic equation in a cylindrical domain, *Ann. Math. Blaise Pascal*, **6** (1999), 13-20.

[4] **C. A. Braumann,** Introduction to stochastic differential equations, Stochastic Finance 2004.

[5] **J. Davila, J. Fernandez Bonder, J. D. Rossi, P. Groisman and M. Sued,** Numerical analysis to stochastic differential equations with explosions, *Stoch. Anal. Appl.*, 23**4** (2005), 809-825.

[6] **H. Doss,** Liens entre equations differentielles stochastiques et ordinaires, *Ann. Inst. Henri Poincar*, 13**2** (1977), 99-125.

[7] **J. Fernandez Bonder, P. Groisman, J. D. Rossi,** Continuity of the explosion time in stochastic differential equations, Submitted.

[8] **A. Friedman and A. A. Lacey,** The blow-up time for solutions of nonlinear heat equations with small diffusion, *SIAM J. Math. Anal.*, **18** (1987), 711-721.

[9] **I. I. Gikhman and A. V. Skorohod,** Introduction to the theory of Random processes, *W. B. Sounders, Philadelphia.*

[10] **P. Groisman, J. D. Rossi,** Elect. *Jour. Diff. Equat.*, Vol. 2007 (2007), n140, 1-9.

[11] **I. Karatzas and S. E. Shreve,** Brownian motion and stochastic calculus, Volume 113 of Graduate Texts in Mathematics, *Springer-Verlag, New-York*, second edition , (1991).

[12] **H. P. Mckean,** Jr. Stochastic integrals, Probability and Mathematical statistics, n5. *Academie Press New-York*, 1969.

[13] **B. ϕksendal,** Stochastic differential equations, An introduction with applications, *(fifth edition), Springer-Verlag, Berlin.*

[14] **M. H. Protter and H. F. Weinberger,** Maximum principles in differential equations, *Prentice Hall, Englewood Cliffs, NJ*, (1967).

[15] **K. Sobczyk and B. F. Spencer,** Jr. Random fatigue, *Academie Press Inc., Boston, MA*, 1992.

[16] **H. J. Sussmann,** On the gap between deterministic and stochastic ordinary differential equations, *Ann. Probability*, 6**1** 1978, 1941.

[17] **G. E. Uhlenbeck and L. S. Ornstein,** On the theory of Brownian motion, *Physical Review*, **36** (1930), 823–841.

[18] **W. Walter,** Differential-und Integral-Ungleichungen, *Springer, Berlin*,(1964).