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Computer modeling in dynamical and control systems

ROUT TO CHAOS IN FOOD CHAIN DYNAMICS

George Osipenko
Laboratory of Nonlinear Analysis
St. Petersburg State Polytechnic University
math@math.tu.neva.ru

Abstract

The paper proves that a real discrete system with biological origin possesses a non-oriented invariant manifold — Möbius band. The obtained results demonstrate the existence of multiple attractors in food-chains models. Moreover, the parameter region with three coexisting and closely-spaced attractors was found. It should be noted that such a proximity does not exclude the possibility that a complicated situation may appear, which may lead to more intriguing biological consequences in the system under study or similar systems.

The route to chaos in the food-chain dynamics is investivated. The initial system (as a parameter $M_0 < 2.9$) has a single stable fixed point, when the parameter M_0 increases the systems passes through non-trivial cascad of the bifurcations which, when $M_0 = 3.65$, results in the appearance of a minimal chaotic attractor covering a Möbius band.

1 Analytical results

The 3-dimensional dynamical system describes a discrete food chain model. Lindström [7] proposed the model that displays a lot of properties commonly known for continuous food-chains [11, 2].

The discrete food chain model is defined by the mapping f of the form

$$\begin{aligned} X_{t+1} &= \frac{M_0 X_t \exp(-Y_t)}{1 + X_t \max(\exp(-Y_t), K(Z_T)K(Y_t))} \\ Y_{t+1} &= M_1 X_t Y_t \exp(-Z_t) K(Y_t) \cdot K(M_3 Y_t Z_t) \\ Z_{t+1} &= M_2 Y_t Z_t, \end{aligned} \tag{1}$$

where

$$K(\gamma) = \begin{cases} \frac{1-\exp(-\gamma)}{\gamma}, & \text{if } \ \gamma \neq 0. \\ 1, & \text{if } \ \gamma = 0 \end{cases}, \tag{2}$$

The detailed description of the model was given in [7]. The variables are related to the different trophic levels of the system, so X is proportional to vegetation abundance whereas Z is proportional to carnivore abundance. Since the relation between herbivores and Y is nonlinear, a more complicated relation describes the situation here. However, such relationships do not change the topological properties of the system under investigation. So, for convenience, we will refer to vegetation, herbivore, and carnivore levels in the sequel.

It should be noted that $M_2 = M_3$ in the original equation. The fourth parameter M_3 is introduced in order to generate additional cases and obtain the complete analysis of system characteristics. So, the result from [10], p. 207-209 shows the existence of an invariant Möbius band at the parameter position $M_0 = 4.0$, $M_1 = 1.0$, $M_2 = 3.0$, $M_3 = 4.0$. However in this paper we will consider the original model with $M_2 = M_3$.

It should be marked that the solutions of (1) remain positive and bounded. Repeating the arguments given in [7] we can show that all solutions starting in the positive cone enter the box $0 < X_t < M_0$, $0 < Y_t < M_0 M_1$, $0 < Z_t < M_0 M_1 M_2^2 / M_3$ within three iterations.

The system (1) has at most four equilibria [7] which are given by:

$$\begin{aligned} E_0 &= (0, 0, 0), \\ E_1 &= (M_0 - 1, 0, 0), \\ E_2 &= \left(\frac{M_0 \log\left(\frac{M_1 M_0}{1+M_1}\right)}{(M_0 - 1)M_1 - 1}, \log\left(\frac{M_1 M_0}{1 + M_1}\right), 0 \right), \end{aligned}$$

and $E_3(X, Y, Z)$ is given by

$$\left(\frac{M_0 \exp(-\frac{1}{M_2}) - 1}{K\left(\frac{1}{M_2}\right) K\left(\log M_1 \left(M_0 \exp(-\frac{1}{M_2}) - 1\right)\right)}, \frac{1}{M_2}, \log M_1 \left(M_0 \exp(-\frac{1}{M_2}) - 1\right) \right),$$

if $M_2 = M_3$ and $\max(\exp(-Y), K(Y)K(Z)) = K(Y)K(Z)$ at E_3.

Some general features of the system can be described by the Morse spectrum of the determinant $\det Df$ [9] which is the rate of change of phase volume.

If $\xi = \{v_0, ..., v_p = v_0\}$ is a p-periodic ε-orbit, then the determinant exponent is defined by

$$\lambda(\det Df, \xi) = \frac{1}{p}\sum_{i=0}^{p-1} \ln|\det Df(v_i)|.$$

The Morse spectrum of the determinant is defined as the following

$$\Sigma(\det Df) = \{\lambda \in R : \textit{ there are } \varepsilon_k \to 0 \textit{ and periodic } \varepsilon_k - \textit{orbits } \xi_k \textit{ with } \lambda(\det Df, \xi_k) \to \lambda \textit{ as } k \to \infty\}.$$

It is well known [17] that if λ_1, λ_2, and λ_3 are Lyapunov exponents of a periodic orbit ξ and $\lambda(\det Df, \xi)$ is its determinant exponent then

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda(\det Df, \xi).$$

Our computing experiments show that in the selected area the considered system has the negative Morse spectrum of $\det Df$. It follows from [9] that the volume tends to zero with negative exponent along each chain recurrent orbit, being its determinant exponent has at least one negative Lyapunov exponent.

2 Numerical results

We limit our discussion to the parameter range $M_0 \in [3.00; 3.65]$ and fix $M_1 = 1.0, M_2 = M_3 = 4.0$. We commence an overview about some general features and bifurcations. The selected area is located along the route to chaos.

At $M_0 \approx 2.93$ a Neimark-Sacker bifurcation occurs. The fixed point E_3 loses its stability and an invariant circle appears in its vicinity. This curve becomes the minimal attractor of the system. As the parameter M_0 increases, the attractor is alternatingly quasi-periodic and periodic, like the dynamics of circle maps [1, 16]. This holds as long as the parameter value stays moderately far from the bifurcation value.

2.1 $M_0 = 3.000$

The Morse spectrum of det(Df) is estimated as [-0.403970,-0.322668]. The system has the fixed point E_2 with the coordinates (1.2164, 0.4055, 0) and the Lyapunov exponents $L_1 = 0.4837$; $L_{2,3} = -0.1667$. The (xy)-plane is invariant for the differential at E_2 and the exponents $L_{2,3}$ correspond to a focus on this plane. Thus, the unstable manifold $W^u(E_2)$ is a curve transversal to the (xy)-plane. The system has the fixed point E_3 with the coordinates (1.7160, 0.2500, 0.2806) and the Lyapunov exponents $L_{1,2} = 0.0630$; $L_3 = -0.6430$. The unstable manifold $W^u(E_3)$ is a 2-dimensional surface with an unstable focus. Our computing results show that the closure of $W^u(E_3)$ is a global attractor in the positive corner $\{x > 0,\ y > 0,\ z > 0\}$. In particular, $W^u(E_2)$ tends to the closure of $W^u(E_3)$, see Fig. 1. Such a dynamics may be observed when the parameter M_0 changes from 3 to 4.

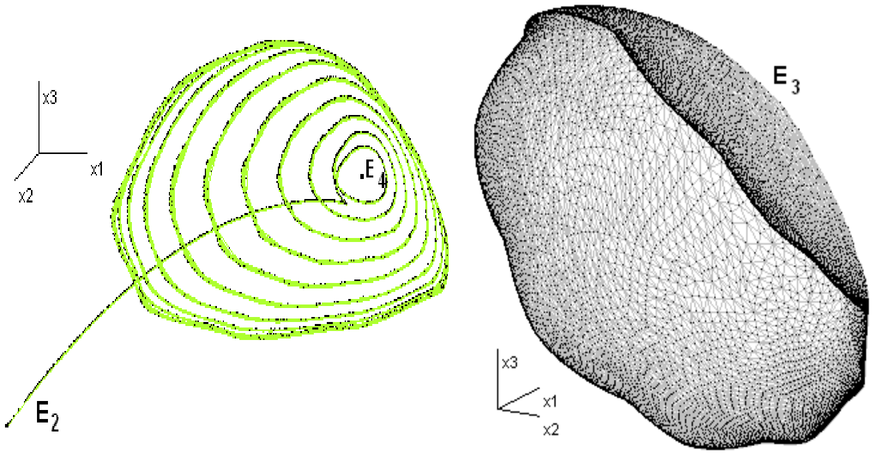

Figure 1: Unstable manifolds $W^u(E_2)$ and $W^u(E_3)$, $M_0 = 3.000$.

The closure of the unstable manifold $W^u(E_3)$ is diffeomorphic to a standard 2-dimensional closed disc, see Fig. 1. The boundary C of the unstable manifold $W^u(E_3)$ is homeomorphic to circle S^1. The stable invariant curve C appears at the Neimark-Sacker bifurcation at $M_0 \approx 2.93$ and looses its stability at $M_0 \approx 3.366$. As $M_0 = 3.0$ we observe a quasiperiodic behavior on C. The first approximation of this rotation looks like as 9-periodic. However, Danny Fundinger found the coordinates of a point X_0 on C whose 50,000 iterations form a line,being the iterations from 40,000 to 50,000 form a line as well. We conclude from this that the movement on the cycle is not periodic. The coordinates of the iterations are $X_0 = (1.336740, 0.379555, 0.253432)$, $X_8 = (1.471666, 0.417361, 0.145821)$, $X_9 = (1.372226, 0.385229, 0.243440)$, $X_{18} = (1.378605, 0.390052, 0.234376)$, $X_{36} = (1.392378, 0.397436, 0.218947)$. These results show that if we start from a point X_0, then X_9 is shifted a little bit from the position of X_0, X_{18} a little bit further and so on. The chain recurrent sets E_2, E_3 and C are localized by the symbolic method, the unstable manifolds $W^u(E_2)$ and $W^u(E_3)$ are constructed by the iterations of broken lines and polytopes, respectively.

When parameter M_0 changes from 3 to 3.3, both the topological structure of trajectories and the manifolds $W^u(E_2)$, $W^u(E_3)$ persist.

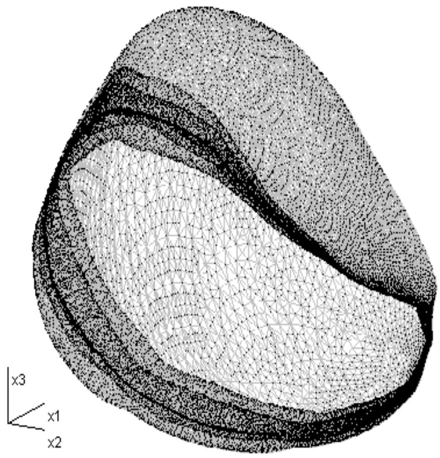

Figure 2: Unstable manifolds $W^u(E_3)$, $M_0 = 3.300$.

2.2 $M_0 = 3.300$

The Morse spectrum of det(Df) is estimated as [-0.431418,-0.289534]. The fixed points E_2 and E_3 have the coordinates (1.27119, 0.50077, 0) and (2.01597, 0.25, 0.39002) respectively. As in previous case they have the same type of stability. Moreover, there is a minimal attractor C with quasiperiodic motion. However, as $M_0 = 3.3$ the manifold $W^u(E_2)$ tends to the invariant curve C by winding around C, see Fig. 2. Hence C should not be considered as a boundary of the smooth manifold $W^u(E_2)$, but as its limit set.

2.3 $M_0 = 3.3701$

The Morse spectrum of det(Df) is estimated as [-0.526124,-0.194662]. The stable invariant curve C looses its stability at $M_0 \approx 3.366$. When the parameter M_0 becomes 3.3701, this bifurcation results in the appearance of a Möbius band MB(3.3701). Later we prove that the invariant manifold MB(3.3701) is non-oriented. The bifurcation is like period doubling bifurcation which is usually observed in continuous dynamical systems. However, we can not speak of period doubling bifurcations here in the same sense as is usually meant for discrete systems. More precisely, in the discrete system we observe a pattern which is typical for continuous systems [4, 3]. So we can say about a "Feigenbaum-like bifurcation". As we will see later, several Feigenbaum-like bifurcations of the same kind happen close to the transition to chaos. The manifold MB(3.3701) is the limit set of the unstable manifold $W^u(E_3)$. Construct the unstable manifold $W^u(E_2)$ of the fixed point E_2 as above. The boundary $L = \partial MB(3.3701)$ is a limit set of the unstable manifold $W^u(E_2)$, see Fig. 3. Center line C of

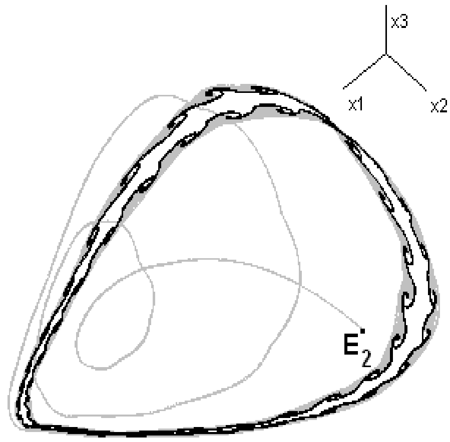

Figure 3: Unstable manifold $W^u(E_2)$ and the Möbius band MB(3.3701).

$MB(3.3701)$ is an unstable invariant curve (on $MB(3.3701)$) with quasiperiodic motion, see Fig. 4. Note that L is a stable invariant curve homeomorphic to circle which is two times longer than the center line C. On L there are the stable 55-periodic orbit $\{P, \dots\}$ and the hyperbolic 55-perodic orbit $\{H, \dots\}$ which is stable on L. The rotation number [3] of the system on L is 3/55. The points R(1.631969, 0.105806, 0.778837), Q(1.456810, 0.157710, 0.718353) lies on the orbit of P(1.519275, 0.140199, 0.847081), and H has the coordinates (1.582335, 0.118405, 0.815238), see Fig. 4. The eigenvalues of the differential Df^{55} at P are approximately equal to $\lambda_{1,2} = 0.8584 \pm 0.2398i$ and $\lambda_3 = 5 \cdot 10^{-8}$. Hence the point P is focus for f^{55}. To prove non-orientability of the band MB(3.3701) we consider the direction of the rotation on the orbit of the point P. Comparing the rotation at points along the center line, i.e. at P, R, etc. (see Fig. 4), we see that the direction of the rotation persists. But the directions of the rotation at the points P and Q are opposites. So, if we go from P to Q along the center line, we save the orientation and if we go transversal to C we get opposite orientation. This is possible if and only if the band is non-oriented. There is only single 2-dimensional non-oriented strip - Möbius band.

Thus, the global dynamics of the system in the positive corner $\{x > 0,\ y > 0,\ z > 0\}$ is the following. The stable 55-periodic orbit of the point P is a single

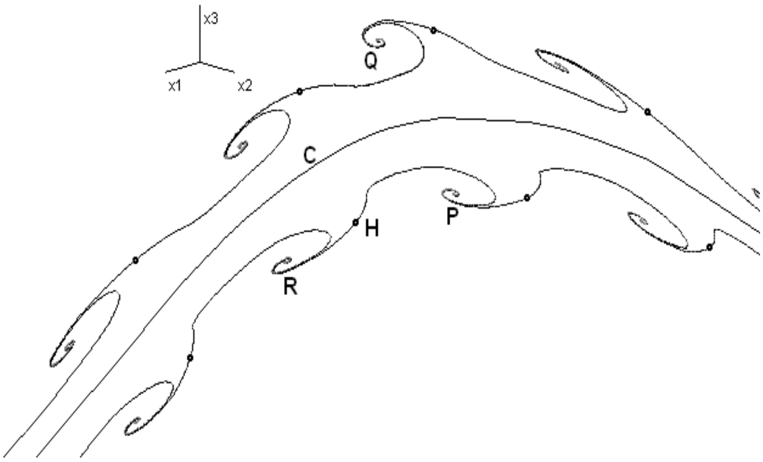

Figure 4: Detail of the Möbius band MB(3.3701).

attractor A minimal by inclusion. Other trajectories, except the fixed points E_3, the center line C, and the hyperbolic 55-periodic orbit of the point H, tend to A.

2.4 $M_0 = 3.4001$

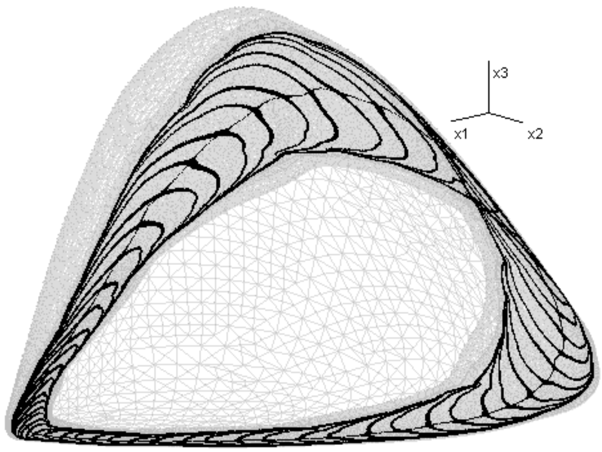

Figure 5: The unstable manifold $W^u(E_3)$ and the Möbius band $MB(3.4001)$.

The Morse spectrum of det(Df) is estimated as [-0.580550,-0.186926]. By using symbolic methods we determined the coordinates of the fixed points E_2 (1.288505, 0.5309395, 0.0), E_3 (2.11595, 0.24995, 0.42300) and two chain recurrent curves C and L. The eigenvalues at E_2 and E_3 are $\lambda_1(E_2) = 2.124$, $\lambda_{2,3}(E_2) = 0.628869 \pm 0.627i$, and $\lambda_{1,2}(E_3) = 0.806993 \pm 0.812688i$, $\lambda_3(E_3) = 0.447195$. Both C and L are homeomorphic to a circle, being L is two times longer than C. The curve L becomes the minimal attractor of the system. These two curves belong to a 2-dimensional invariant manifold $MB(3.4001)$ which is homeomorphic to an Möbius band, Fig. 5. On the picture you can see a collection of curves being ustable manifold of an orbit on the line C. Geometrically, the curves resemble the shape of a Möbius strip. The stable invariant curve can be imagined as the edges of the strip, and the unstable curve as its center line. Numerical studies of forward and backward iterations so far indicate that the curve at the center line has saddle type in the space and unstable type on $MB(3.4001)$. We constructed the unstable manifold $W^u(E_3)$ and the Möbius band $MB(3.4001)$, see Fig. 5.

2.5 $M_0 = 3.480$

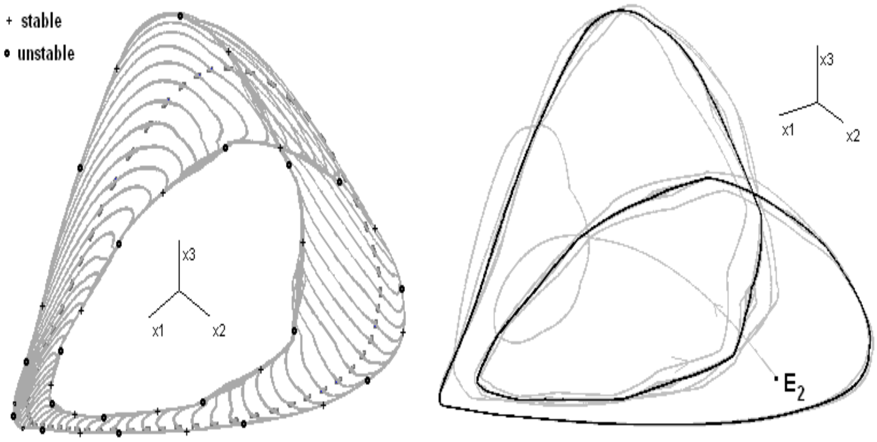

Figure 6: The boundary L of $MB(3.48)$ is a limit set of $W^u(E_2)$.

The Morse spectrum of det(Df) is estimated as [-0.552695,-0.181412]. The Möbius band $MB(3.48)$ persists. It is a global attractor in the positive corner $\{x > 0,\ y > 0,\ z > 0\}$. The center line C of $MB(3.48)$ is an unstable invariant curve on $MB(3.48)$. The dynamics on C is periodic with the rotation number 7/64. There is a stable 64-perodic orbit of the point $P \approx (1.231941, 0.800882, 0.137928)$ on C. The limit set of $MB(3.48)$ is a stable invariant curve L with periodic dynamics of the rotation number 1/18. There is a hyperbolic 18-periodic orbit of the point $U \approx (2.194811, 0.356991, 0.065345)$ on L. The differential $Df^{18}(U)$ has the eigenvalues $\lambda_1 \approx 1.21$, $\lambda_2 \approx -0.28$, $\lambda_3 \approx -0.002$. The first eigenvalue corresponds to L so the orbit of U is unstable on L. There is a stable 18-periodic orbit of the point $S \approx (1.652079, 0.131587, 0.502229)$ on L. The differential $Df^{18}(S)$ has the eigenvalues $\lambda_1 \approx 0.81$, $\lambda_2 \approx -0.40$, $\lambda_3 \approx -0.003$. Since the pair of the eigenvalues at U and S are negative, the Möbius band $MB(3.48)$ tends to L by winding around one. The periodic orbit of S is a single attractor minimal by inclusion. Thus, almost all trajectories from positive corner tend to this orbit.

2.6 $M_0 = 3.532$

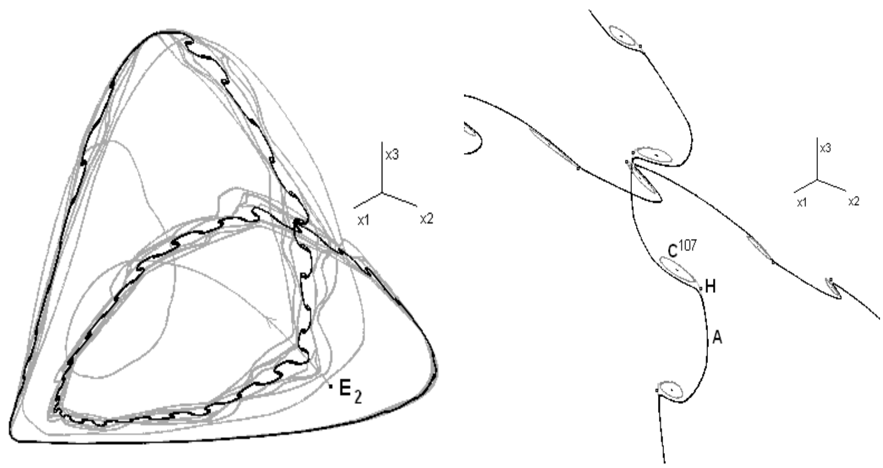

Figure 7: Unstable manifold $W^u(E_2)$ and its limit set $LimW^u(E_2)$, $M_0 = 3.532$.

The Morse spectrum of det(Df) is estimated as [-0.572940,-0.162688]. The coordinates of the fixed point E_2 are (1.305196,0.561747,0). We constructed the unstable manifold $W^u(E_2)$ and its limit set $LimW^u(E_2)$. It turns out that $LimW^u(E_2)$ has nontrivial structure, see Fig. 7. The limit set consists of the 107 circles (C^{107}), the hyperbolic 107-periodic orbit H, its unstable manifold $W^u(H)$, and the attractor A, see Fig. 8. The unstable 107-periodic orbit U is inside of the set C^{107}. The limit set of the Möbius band $MB(3.532)$ coincides with $LimW^u(E_2)$. The attractors A and C^{107} are minimal by inclusion, so we have a non-ordinary phenomenon —the existence of two minimal attractors in a biology system. It should be noted that the set C^{107} exists in a short interval for M_0. The circles, the orbits U and H disappear when $M_0 \approx 3.536$, whereas the attractor A persists. The paper [6] deals with the appearance of multiple attractors in the chain food dynamics and contains detailed information about the considered system.

2.7 $M_0 = 3.540$

The Morse spectrum of det(Df) is estimated as [-0.578320,-0.173867]. The attractor A is the limit set of the Möbius band MB from $M_0 \approx 3.536$ to $M_0 \approx 3.538$, but for all that the invariant curve A looses its stability for the last parameter value. When the parameter M_0 becomes 3.54, the bifurcation results in the appearance of the second Möbius band $MB_2(3.54)$, see Fig. 9. Hence we have

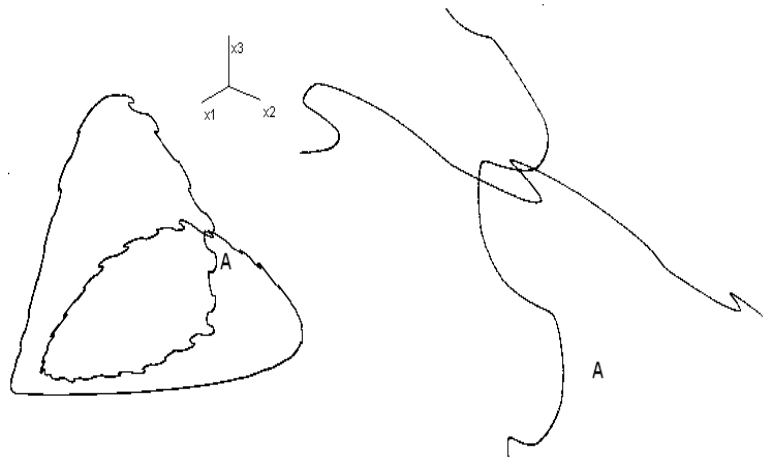

Figure 8: The attractor A and its details, $M_0 = 3.532$.

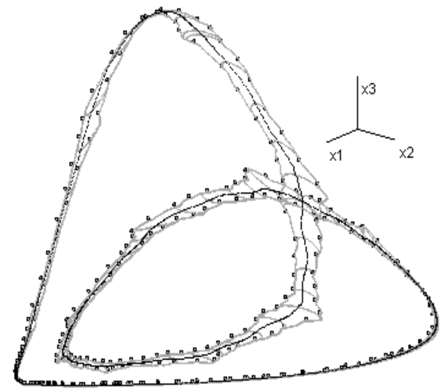

Figure 9: The second Möbius band $MB_2(3.54)$, its unstable center line C_2, and the stable 283-periodic orbit S on L_2.

"Feigenbaum-like bifurcation".

The second Möbius band $MB_2(3.54)$ appears as a limit set of the first Möbius band $MB_1(3.54)$. The center line of the second Möbius band $MB_2(3.54)$ is unstable invariant curve C_2 with quasiperiodic motion. The boundary L_2 of $MB_2(3.54)$ is a stable invariant circle which is two times longer than the center line C_2. On the boundary L_2 there are two 283-periodic orbits: stable $S \approx \{(1.354245, 0.467325, 0.679182), \dots\}$ and unstable $U \approx \{(1.354245, 0.467325, 0.679182), \dots\}$. The points of these orbits alternate.

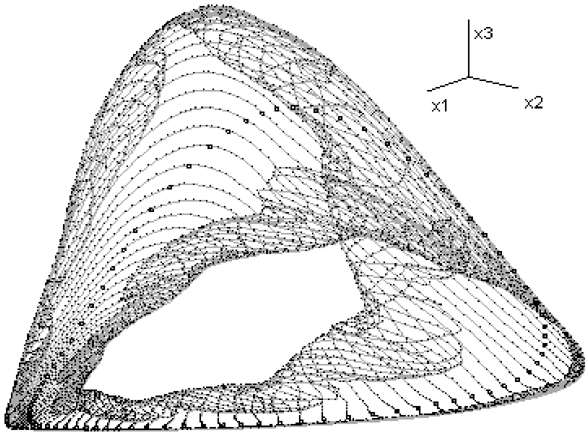

Figure 10: The trajectories of the first Möbius band tend to the second Möbius band by winding around it, $M_0 = 3.570$.

2.8 $M_0 = 3.570$

The Morse Spectrum of det(Df) is estimated as [-0.565295,-0.161637]. When the parameter M_0 is greater than 3.538, the second Möbius band becomes a limit set of the first Möbius band. More precisely, the trajectories on MB_1 tend to MB_2 by winding around it. The set $MB_1(3.57)$, the unstable malifold of an orbit on the center line C_1 and its behavior near the limit set MB_2 are shown on the Fig. 10.

Now we observe the next "Feigenbaum-like bifurcation" near the boundary of the second Möbius band MB_2. Here the bifurcation is the following. The invariant curve $L_2 = \partial MB_2$ loses its stability and a strange invariant set MB_3 appears, see Fig. 11. The set $MB_3(3.570)$ consists of 71 pieces,

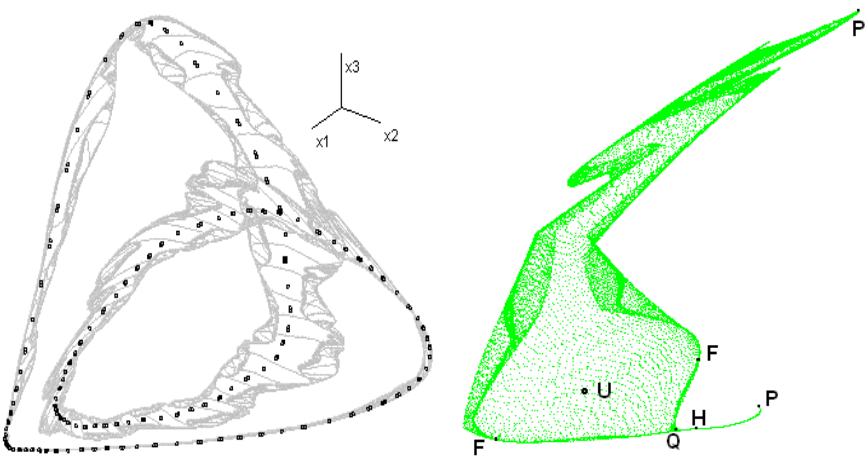

Figure 11: The second Möbius band and the detail of its limit set, $M_0 = 3.570$.

each piece has both 2- and 1-dimensional parts. This decomposition is invariant, i.e. an image of a piece part is a piece part. The 2-dimensional part is the closure of the unstable manifold $W^u(U)$ of the hyperbolic 71-periodic orbit $U \approx \{(1.323374, 0.186463, 0.646940), \dots\}$. The eigenvalues of the differential $Df^{71}(U)$ are $\lambda_1 \approx 1.374292$, $\lambda_2 \approx -1.501502$, $\lambda_3 \approx -0.794848$. The eigenvalues λ_1 and λ_2 correspond to the unstable manifold $W^u(U)$. The system behavior on $W^u(U)$ is similar to its dynamics on a Möbius band. As $\lambda_1\lambda_2 < 0$, the 71-th iteration f^{71} inverts the orientation on $W^u(U)$. The 1-dimensional part is formed by the unstable manifold $W^u(H)$ of the hyperbolic 71-periodic orbit $H \approx \{(1.576668, 0.44846, 0.72410), \dots\}$. The manifold $W^u(H)$ ends at the stable 71-periodic cycles $Q \approx \{(1.913398, 0.677286, 0.093128), \dots\}$ and $P \approx \{(2.489430, 0.086803, 0.027202), \dots\}$, see Fig. 11. The limit set of the 2-dimensional manifold $W^u(U)$ is a stable invariant curve which forms by the 1-dimensional unstable invariant manifold $W^u(F)$ of the hyperbolic 142-periodic orbit F. The structure of the both Möbius bands persists near $M_0 \approx 3.5708$. Moreover, the structure of the invariant set MB_3 persists as well. It should be noted that in this case we detect a new phenomenon. When $M_0 = 3.5708$ the set MB_3 contains at least three stable periodic orbits: the known 71-periodic orbits $Q \approx \{(1.560941, 0.447309, 0.731145), \dots\}$ and $P \approx \{(1.601819, 0.436388, 0.74072), \dots\}$ and a new stable 142-periodic orbit $S \approx \{(1.547730, 0.450719, 0.727846), \dots\}$. Thus, we obtain three minimal attractors in the biology system.

2.9 $M_0 = 3.571$

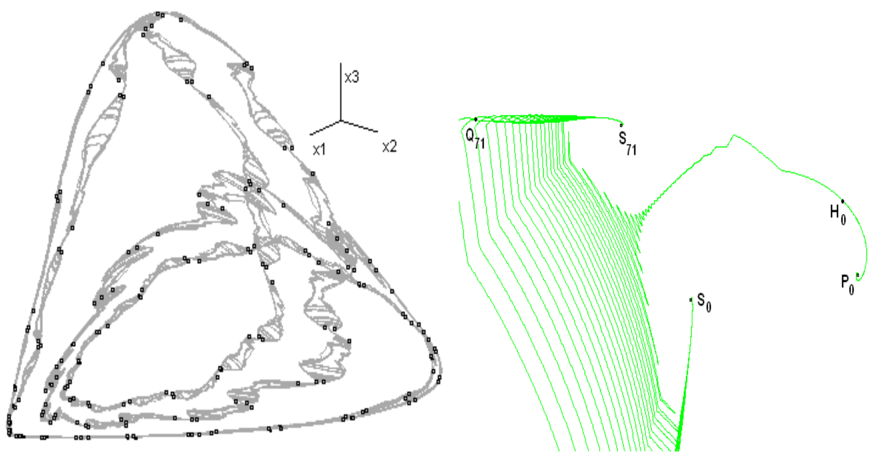

Figure 12: The invariant set of $MB_3(3.571)$ and the unstable manifold $W^u(H_0)$ on it.

In this case the Morse spectrum of the det(Df) is estimated as [-0.596287,-0.163302]. The structure of $MB_3(3.571)$ is the following. There are two attractors minimal by inclusion. One of them — P — is a stable 71-periodic cycle generated by the point $P_0 \approx (1.6014, 0.4362, 0.7415)$, see Fig. 12. The second attractor is a stable cycle S formed by 142-periodic orbit of the point $S_0 \approx (1.93364, 0.60068, 0.0002)$. The eigenvalues of these cycles are following:

$$\lambda_{1,2}(P) \approx 0.6477 \pm 0.3656i,\ |\lambda_{1,2}(P)| = 0.7438 < 1,\ \lambda_3(S^{71}) \approx 0$$

and

$$\lambda_{1,2}(S) \approx 0.5079 \pm 0.3626i,\ |\lambda_{1,2}(S)| = 0.6241 < 1,\ \lambda_3(S) \approx 0.$$

Thus, the both cycles are of focus type and obviously, they are attractors minimal by inclusion. Additionally, there are two unstable periodic cycles. One of them H is 71-periodic orbit of the point $H_0 \approx (1.5889, 0.4438, 0.7305)$ and the other Q is 142-periodic orbit of the point $Q_0 \approx (1.578620, 0, 410556, 0.798312)$, $f^{71}(Q_0) = Q_{71} \approx (1.649636, 0.279537, 1.066632)$. Again, we analyze the eigenvalues of these cycles:

$$\lambda_1(H) \approx 1.4817,\ \lambda_2(H) \approx 0.5603,\ \lambda_3(H) \approx 0;$$

and

$$\lambda_1(Q) \approx 1.8779,\ \lambda_2(Q) \approx -0.0669,\ \lambda_3(Q) \approx -0.$$

Moreover, we estimate the corresponding eigenvectors and eigenspaces. So the periodic cycles H and Q are of hyperbolic type with 1-dimensional unstable manifolds. The unstable manifold $W^u(H)$ are constructed and we see (Fig. 12) that the right part of $W^u(H_0)$ has a simple behavior and ends at the point P_0 of the orbit P. The left part of $W^u(H_0)$ has more complex oscillated behavior. The limit set of $W^u(H_0)$ contains the points S_0 and $S_{71} = f^{71}(S_0)$ of the stable orbit S. The limit set of $W^u(H_0)$ contains the points Q_0 and Q_{71} of the hyperbolic orbit Q, see Fig. 13. We estimate the angle between the stable subspace of the orbit Q at Q_0 and $W^u(H_0)$ and verify that their intersection is transversal near Q_0. Thus, we have the heteroclinic transversal connection $H \to Q$.

The next step is the construction of the unstable manifold $W^u(Q_0)$ at the point Q_0. It turn out that here we have a similar behavior: one part of $W^u(Q_0)$ ends at S_0 of the stable orbit S. The other part of $W^u(Q_0)$ has more complex oscillated behavior such that its limit set contains the point H_{36} of the hyperbolic orbit H and the point P_{36} of the stable orbit P. Here we also obtain the heteroclinic transversal connection $Q \to H$. From this it follows that there is the homoclinic connections $Q \to Q$ and $H \to H$. To check this conclusion we construct the global unstable manifold $W^u(H)$ of the hyperbolic orbit H. The result of our

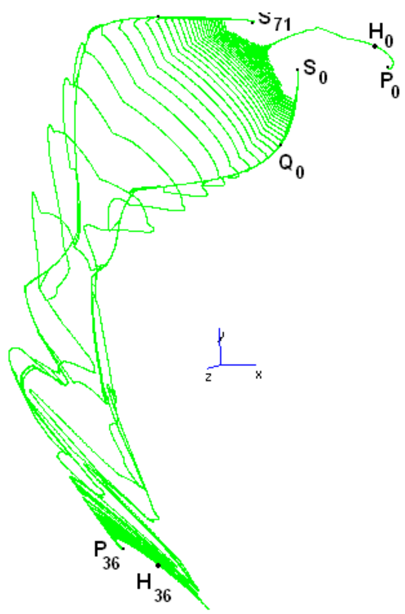

Figure 13: The closure of the unstable manifold $W^u(H)$, the stable orbits S and F.

computing displayed on Fig. 13 shows that the limit set of $W^u(H_0)$ contains $H_{36} = f^{36}(H_0)$. Thus, there exists the homoclinic orbit $H \to H$, which usually leads to chaotic dynamics near this orbit [12, 13, 14, 15, 8, 3]. Our numerical investigation shows that a chaotic dynamics is located inside of the closure of $W^u(H)$. This closure is not minimal by inclusion because it contains the stable orbits S and P, see Fig. 12.

2.10 Chaos

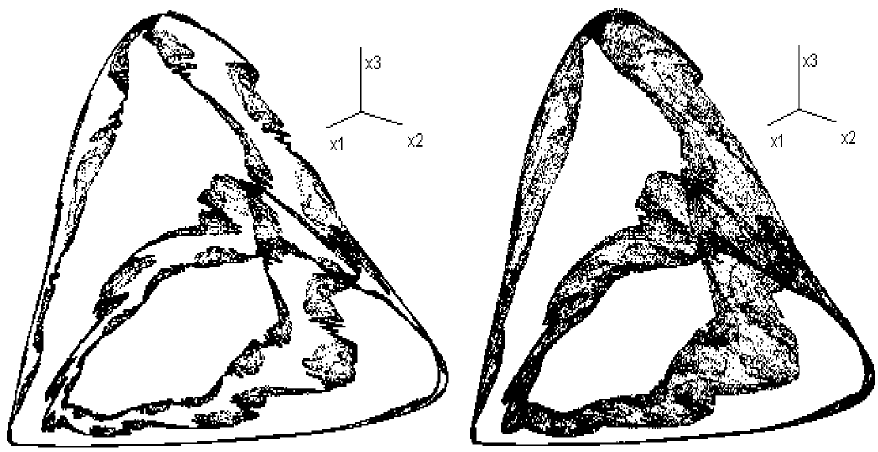

Figure 14: The chaotic attractors for $M_0 = 3.573$ and $M_0 = 3.58$.

Consider the system behaviour for $M_0 = 3.573,\ 3.580$ and 3.650. The Morse spectrum of the differential was obtained by the symbolic analysis methods. For $M_0 = 3.573$ the Morse spectrum of det(Df) is estimated as [-0.601288,-0.167536]. In the case $M_0 = 3.58$ the spectrum is estimated as [-0.582492,-0.156852] and for $M_0 = 3.65$ the estimate is [-0.601972,-0.136850]. Thus, we have negative spectrum in all cases. It results in zero volume of the chain recurrent set and, as a consequence, in zero volume of the chaotic attractor. It was mentioned above that the chaotic dynamics appears when $M_0 \approx 3.571$ and is located in very small domain. When $M_0 \in [3.573, 3.650]$ several local and global bifurcations lead to a number of subsequent changes in the dynamics. This involves the appearance of chaotic attractor which grows as M_0 increases. Our construction of a chaotic attractor is trivial: we pick up a point from its domain of attraction and consider its iterations. If the iteration number is huge, we obtain the desired attractor. In our cases the chaotic attractor is minimal by inclusion, so almost all points from the positive corner are in the basin of the desired attractor. When $M_0 = 3.573$ we can observe (see Fig. 14) that the chaotic attractor coincides with the invariant set $MB_3(3.573)$. If the parameter M_0 increases and reaches value $M_0 = 3.58$, the chaos occupies the second Möbius band $MB_2(3.58)$, see Fig. 14. After all, as $M_0 = 3.650$ the chaos covers the first Möbius band $MB_1(3.65)$, see Fig. 15.

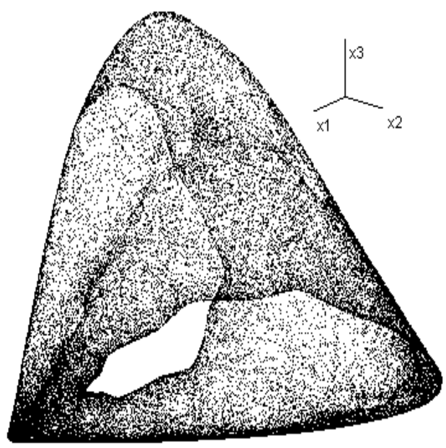

Figure 15: The chaotic attractor for $M_0 = 3.65$.

3 Conclusion

In this paper we have proved that a real discrete system with biological origin possesses a non-oriented invariant manifold — Möbius band. The obtained results demonstrate the existence of multiple attractors in food-chains models. We have found several parameter regions where attractors of this kind exist. Moreover, the parameter region with three coexisting and closely-spaced attractors was found. It should be noted that such a proximity does not exclude the possibility that a complicated situation may appear, which may lead to more intriguing biological consequences in the system under study or similar systems.

We have shown the route to chaos in the food-chain dynamics. The initial system ($M_0 < 2.9$) has a single stable fixed point, when the parameter M_0 increases the systems passes through non-trivial cascad of the bifurcations which, when $M_0 = 3.65$, results in the appearance of a minimal chaotic attractor covering a Möbius band.

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