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On oscillatory behavior of neutral first order differential equations

Distribution of zeros of solutions of first order neutral differential equations

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Abstract

In this paper, the distribution of zeros of solutions of the first order neutral differential equation

$$[x(t)+p(t)x(g(t))]'+f(t,\ x(h(t)))=0$$

is discussed. New criteria are deduced . Illustrative example is given.

Keywords: Distribution of zeros, Neutral differential equations.

1. Introduction

The aim of this paper is to study the distribution of zeros of solutions of the first order neutral differential equations of the type

$$[x(t)+p(t)x(g(t))]'+f(t,\ x(h(t)))=0 \tag{1.1}$$

where $p,h\in C([t_0,\infty),[0,\infty))$, $g\in C([t_0,\infty),[0,\infty))$, $f\in C([t_0,\infty)\times R,R)$ and $g(t),h(t)$ are nondecreasing in t, and $f(t,x(h(t)))$ is nondecreasing in $x(t)$.

Further we assume that

(I) There exist $Q(t)$, $B(x(h(t)))$ such that for $t\ge t_0$, $\dfrac{f(t,x(h(t)))}{x(h(t))}\ge Q(t)B(x(h(t)))>0$

$Q\in C([t_0,\infty),[0,\infty))$, and $B\in C(R,\ R^+)$

(II) $\lim\limits_{t\to\infty} g(t)=\lim\limits_{t\to\infty} h(t)=\infty$.

The results of this paper improve and extend those of Wu et al ([2] and [3]).

Eq. (1.1) includes the differential equation

$$\left[x(t)+p(t)x(g(t))\right]'+Q(t)x(h(t))=0 \qquad (1)$$

Which recently discussed by Wu et al([1] and [2]). In Sec 2 we deduce some preliminaries about the first-order inequality

$$x'(t)+p(t)R(x(t))x(\tau(t))\le 0 \qquad (1.2)$$

Where $p,\tau\in C\left([t_0,\infty),(0,\infty)\right)$, $\tau(t)\le t$, $\tau(t)$ is nondecreasing with $\lim\limits_{t\to\infty}\tau(t)=\infty$, and $R\in\left([t_0,\infty),(1,\infty)\right), R(x(t))\ge 1$.

The inequality

$$x'(t)+p(t)x(\tau(t))\le 0 \qquad (2)$$

of [2], and [3] is a special case of (1.2). Sec.3 includes the main results for Eq.(1.1). Our results depend and improve those of [1-6]. At the end, we give an example to illustrate our results.

2. First-order differential inequalities

Following [2] , we use the following notation. Let $\{f_n(\rho)\}_{n=1}^{\infty}$ be a sequence of functions defined by

$$f_0(\rho)=1,\ f_1(\rho)=\frac{1}{1-\rho},\ f_{n+2}(\rho)=\frac{f_n(\rho)}{f_n(\rho)+1-e^{\rho f_n(\rho)}},\ n=0,1,2,3,........ \qquad (2.1)$$

where $\rho\in(0,1)$. It is easy to see that if $\rho>\frac{1}{e}$, then either $f_n(\rho)$ is nondecreasing and $\lim\limits_{n\to\infty}f_n(\rho)=\infty$ or $f_n(\rho)$ is negative or ∞ after a finite numbers of terms. However for $0<\rho\le 1$ we have $1\le f_n(\rho)\le f_{n+2}(\rho)\le e$, $n=0,1,2,........$

and $\lim\limits_{n\to\infty}f_n(\rho)=f(\rho)\in[1,e]$, where $f(\rho)$ satisfies

$$f(\rho)=e^{\rho f(\rho)} \qquad (2.2)$$

The authors in [1] defined a sequence $\{\varphi_m(\rho)\}_{m=1}^{\infty}$ for $0<\rho<1$ by

$$\varphi_1(\rho)=\frac{2(1-\rho)}{\rho^2},\ \varphi_{m+1}(\rho)=\frac{2\left(1-\rho-\frac{1}{\varphi_m(\rho)}\right)}{\rho^2},\ m=1,2,3,.......... \qquad (2.3)$$

It is easy to see that for $0<\rho<1$, we have $\varphi_{m+1}(\rho)<\varphi_m(\rho), m=1,2,3,.....$, We also observe that when $0\le\rho\le\frac{1}{e}$, then $\varphi_1(\rho)>\frac{2(1-\rho)}{\rho^2}$, and in general

$$\varphi_{m+1}(\rho)=\frac{2\left(1-\rho-\frac{1}{\varphi_m(\rho)}\right)}{\rho^2}>\frac{2(1-\rho)}{\rho^2},\ m=1,2,3,......$$

Hence, the sequence $\{\varphi_m(\rho)\}_{m=1}^{\infty}$ is decreasing and bounded from below. Thus there exists a function $\varphi(\rho)$ such that

$\lim\limits_{m\to\infty}\varphi_m(\rho)=\varphi(\rho)$, and $\varphi(\rho)=\dfrac{2\left(1-\rho-\frac{1}{\varphi(\rho)}\right)}{\rho^2}$. This implies that

$$\varphi(\rho)=\frac{1-\rho+\sqrt{1-2\rho-\rho^2}}{\rho^2}=\frac{2}{1-\rho-\sqrt{1-2\rho-\rho^2}},\ 0<\rho\le\frac{1}{e} \qquad (2.4)$$

We will need the iteration of the inverse of each of the functions τ, g and h , using the notation $\tau^0(t)=t$ and inductively define the iterates of τ^{-1} by

$$\tau^{-i}(t)=\tau^{-1}\left(\tau^{-(i-1)}(t)\right),\ i=1,2,......$$

Like wise for g and h .

Lemma 2.1 Let $x(t)$ be a solution of (1.2) on $[t_0,\infty)$. Further assume that there exist $t_1 \ge t_0$ and a positive constant ρ such that

$$\int_{\tau(t)}^{t} p(s)ds \ge \rho \ , \ t \ge t_1 \tag{2.5}$$

and that there exists $T_0 \ge t_1$ and $T \ge \tau^{-3}(T_0)$ such that $x(t)$ is positive on $[T_0,T]$. Then for some $n > 0$, we get

$$\frac{x(\tau(t))}{x(t)} \ge f_n(\rho) > 0 \ \text{ for } t \in \left[\tau^{-(2+n)}(T_0),\ T\right] \tag{2.6}$$

where $f_n(\rho)$ is defined by (2.1) .

Proof: From (1.2) , we obtain

$$x'(t) \le -p(t)R(x(t))x(\tau(t)) \le 0 \ \text{ for } t \in [\tau^{-1}(T_0),\ T] \tag{2.7}$$

which implies that $x(t)$ is nonincreasing on $t \in [\tau^{-1}(T_0),T]$. Thus it follows that

$$\int_{\tau(t)}^{t} x'(s)ds \le 0 \ \text{ then } \ x(t) - x(\tau(t)) \le 0$$

Then

$$\frac{x(\tau(t))}{x(t)} \ge 1 = f_0(\rho) \ \text{ for } t \in \left[\tau^{-2}(T_0),\ T\right] \tag{2.8}$$

If $\tau^{-3}(t) \le t \le T$, then integrating (1.2) from $\tau(t)$ to t we get

$$x(\tau(t)) \ge x(t) + \int_{\tau(t)}^{t} p(s)R(x(s))x(\tau(s))ds$$

Let $E(t,x(t)) = R(x(t))x(\tau(t))$. Then

$$x(\tau(t)) \ge x(t) + \int_{\tau(t)}^{t} p(s)E(t,x(s))ds$$

Thus

$$x(\tau(t)) \ge x(t) + E(t,x(t))\int_{\tau(t)}^{t} p(s)ds$$

Now from (2.5), we have

$$x(\tau(t)) \ge x(t) + \rho\, E(t,x(t))$$
$$x(\tau(t)) \ge x(t) + \rho\, R(x(t))x(\tau(t))$$

So we get

$$\frac{x(\tau(t))}{x(t)} \ge 1 + \rho\, R(x(t))\frac{x(\tau(t))}{x(t)}$$

and

$$\frac{x(\tau(t))}{x(t)}\left(1 - \rho R(x(t))\right) \ge 1$$

Then

$$\frac{x(\tau(t))}{x(t)} \ge \frac{1}{\left(1 - \rho R(x(t))\right)} \ge \frac{1}{1-\rho} = f_1(\rho) > 0 \ \text{ for } t \in \left[\tau^{-3}(T_0),\ T\right]$$

where $R(x(t)) \ge 1$.

Next, we show that

$$\frac{x(\tau(t))}{x(t)} \ge f_2(\rho) > 0 \ \text{ for } t \in \left[\tau^{-4}(T_0),T\right] \tag{2.9}$$

Integrating (1.2) from $\tau(t)$ to t, we get

$$x(\tau(t)) \geq x(t)+\int_{\tau(t)}^{t} p(s)\, R(x(s))\, x(\tau(s))ds \ , \ \tau(s) \leq s \leq t$$

Dividing (1.2) by $x(t)$ and integrating again from $\tau(s)$ to $\tau(t)$, we get

$$\int_{\tau(s)}^{\tau(t)} \frac{x'(\eta)}{x(\eta)}\, d\eta \leq -\int_{\tau(s)}^{\tau(t)} p(\eta)\, R(x(\eta))\, \frac{x(\tau(\eta))}{x(\eta)} d\eta$$

Then

$$\ln x(\eta)|_{\tau(s)}^{\tau(t)} \leq -\int_{\tau(s)}^{\tau(t)} p(\eta)\, R(x(\eta))\, \frac{x(\tau(\eta))}{x(\eta)} d\eta$$

i.e.

$$\ln x(\tau(t)) - \ln x(\tau(s)) \leq -\int_{\tau(s)}^{\tau(t)} p(\eta)\, R(x(\eta))\, \frac{x(\tau(\eta))}{x(\eta)} d\eta$$

Thus, we have

$$\frac{x(\tau(s))}{x(\tau(t))} \geq \exp \int_{\tau(s)}^{\tau(t)} p(\eta)\, R(x(\eta))\, \frac{x(\tau(\eta))}{x(\eta)} d\eta$$

From the condition $R(x(t)) \geq 1$ and (2.8), we have

$$\frac{x(\tau(s))}{x(\tau(t))} \geq \exp \int_{\tau(s)}^{\tau(t)} p(\eta) \frac{x(\tau(\eta))}{x(\eta)} d\eta \geq \exp\left(f_0(\rho) \int_{\tau(s)}^{\tau(t)} p(\eta)\right) d\eta$$

Moreover from (2.9) and the above inequality we get

$$x(\tau(t)) \geq x(t)+\int_{\tau(t)}^{t} p(s) R(x(s)) x(\tau(s))ds \geq x(t)+\int_{\tau(t)}^{t} p(s) x(\tau(s))ds$$

So we have

$$x(\tau(t)) \geq x(t)+ x(\tau(t)) \int_{\tau(t)}^{t} p(s) \exp\left(f_0(\rho) \int_{\tau(s)}^{\tau(t)} p(\eta)\right) d\eta$$

Now as in [2], we get

$$x(\tau(t)) \geq x(t)+ x(\tau(t)) \frac{\left(e^{\rho f(\rho)} - 1\right)}{f_0(\rho)}$$

So

$$\frac{x(\tau(t))}{x(t)} \geq \frac{f_0(\rho)}{f_0(\rho)+1-e^{\rho f_0(\rho)}} > 0 \ \text{ for } t \in \left[\tau^{-4}(T_0), T\right]$$

Repeating the above procedures, we get

$$\frac{x(\tau(t))}{x(t)} \geq f_n(\rho) \ \text{ for } \ t \in \left[\tau^{-(2+n)}(T_0),\, T\right] \tag{2.10}$$

Lemma 2.2: Assume that there exist $t_1 \geq t_0$, and a positive constant $\rho < 1$ such that (2.5) be satisfied and $R(x(t)) \geq 1$. Suppose that there exist $T_0 \geq t_1$ and a positive solution $x(t)$ of (1.2) on $\left[T_0, \tau^{-N}(T_0)\right]$ .Then for some $m \leq N-3$,we have

$$\frac{x(\tau(t))}{x(t)} < \varphi_m(\rho) \ \text{ for } \ t \in \left[\tau^{-3}(T_0),\, \tau^{-(N-3)}(T_0)\right] \tag{2.11}$$

where $\varphi_{m+1}(\rho)$ be as defined in (2.3) .

Proof: From (2.11) , we know that

$$\int_{\tau(t)}^{t} p(s)ds \geq \rho \quad \text{and} \quad \int_{t}^{\tau^{-1}(t)} p(s)ds \geq \rho \ \ , \ t \geq t_1 \tag{2.12}$$

Now since $F(\lambda) = \int_t^{\lambda} p(s)ds$ is a continuous function, $F(\tau^{-1}(t)) \geq \rho$ and $F(t)=0$ .Thus, there exists a λ_1 such that $\int_t^{\lambda} p(s)ds \ = \rho$, where $t \leq \lambda_1 \leq \tau^{-1}(t)$.

Consider the case $\tau^{-3}(T_0) \le t \le \tau^{-(N-1)}(T_0)$. Integrating both sides of (1.2) from t to λ_1, we obtain

$$x(t) - x(\lambda_1) \ge \int_t^{\lambda_1} p(s)\, R(x(s))\, x(\tau(s)) ds \qquad (2.13)$$

Since $t \le s \le \lambda_1 \le \tau^{-1}(t)$, it follows that $\tau^{-2}(T_0) \le \tau(t) \le \tau(s) \le \tau(\lambda_1) \le t$. Integrating both sides of (1.2) again but from $\tau(s)$ to t, we get

$$x(\tau(s)) - x(t) \ge \int_{\tau(s)}^{t} p(u)\, R(x(u)) x(\tau(u)) du$$

From (2.7), $x(\tau(u))$ is nonincreasing on $\tau^{-2}(T_0) \le \tau(s) \le u \le t$. Thus, we have

$$x(\tau(s)) \ge x(t) + R(x(t))\, x(\tau(t)) \left[\rho - \int_t^s p(u) du\right]. \qquad (2.14)$$

Now from (2.13) and (2.14), we have

$$x(t) \ge x(\lambda_1) + R(x(\lambda_1)) \int_t^{\lambda_1} p(s) \left[x(t) + R(x(t)) x(\tau(t)) \left[\rho - \int_t^s p(u) du\right]\right] ds$$

Thus

$$x(t) \ge x(\lambda_1) + \rho\, x(t) R(x(\lambda_1)) + \rho^2 R(x(t)) R(x(\lambda_1)) x(\tau(t)) - R(x(t)) R(x(\lambda_1)) x(\tau(t)) \int_t^{\lambda_1} p(s) \int_t^s p(u) du\, ds \qquad (2.15)$$

By changing the variables, we get

$$\int_t^{\lambda_1} p(s) \int_t^s p(u) du\, ds = \int_t^{\lambda_1} \int_u^{\lambda_1} p(s)\, p(u) ds\, du$$

Thus

$$\int_t^{\lambda_1} ds \int_t^s p(s)\, p(u) du = \int_t^{\lambda_1} ds \int_u^{\lambda_1} p(u)\, p(s)\, du$$

This implies that

$$\int_t^{\lambda_1} ds \int_t^s p(s)\, p(u) du = \frac{1}{2} \int_t^{\lambda_1} \int_t^{\lambda_1} p(u)\, p(s)\, du\, ds = \frac{1}{2} \left(\int_t^{\lambda_1} p(u) du\right)^2 = \frac{\rho^2}{2}$$

Substituting into (2.15) , we have

$$x(t) \ge x(\lambda_1) + \rho\, x(t)\, R(x(\lambda_1)) + \frac{\rho^2}{2} R(x(t)) R(x(\lambda_1))\, x(\tau(t))$$

Since $t \le s < \lambda_1$ so $x(t) \le x(s) \le x(\lambda_1)$ and $R(x(t)) \le R(x(s)) \le R(x(\lambda_1))$, then

$$x(t) \ge x(\lambda_1) + \rho\, x(t) R(x(\lambda_1)) + \frac{\rho^2}{2} x(\tau(t)) R^2(x(\lambda_1)) . \qquad (2.16)$$

Thus

$$\frac{x(\tau(t))}{x(t)} < \frac{2(1-\rho)}{\rho^2 R(x(\lambda_1))} , \qquad (2.17)$$

Now since $\dfrac{2(1-\rho)}{\rho^2 R(x(\lambda_1))} < \dfrac{2(1-\rho)}{\rho^2}$ where $R(x(\lambda_1)) \ge 1$, then

$$\frac{x(\tau(t))}{x(t)} < \frac{2(1-\rho)}{\rho^2} = \varphi_1(\rho) \text{ for } t \in \left[\tau^{-3}(T_0),\, \tau^{-(N-1)}(T_0)\right] \qquad (2.18)$$

If $\tau^{-3}(T_0) \le t \le \tau^{-(N-2)}(T_0)$, we have $\tau^{-3}(T_0) \le t \le \lambda_1 \le \tau^{-(N-1)}(T_0)$. Thus, by (2.18)

$$x(\lambda_1) > \frac{1}{\varphi_1(\rho)} x(\tau(t)) \text{ for } t \in \left[\tau^{-3}(T_0),\, \tau^{-(N-2)}(T_0)\right] , \qquad (2.19)$$

Since $x(t)$ is nonincreasing on $[\tau^{-1}(T_0), \tau^{-N}(T_0)]$ and $\tau^{-2}(T_0) \le \tau(\lambda_1) < t < \lambda_1 \le \tau^{-(N-1)}(T_0)$, we obtain

$$x(\lambda_1) > \frac{1}{\varphi_1(\rho)}\, x(t) \ .$$

Substituting into (2.16) , we have

$$x(t) \ge \frac{1}{\varphi_1(\rho)} x(t) + \rho\, x(t) R(x(\lambda_1)) + \frac{\rho^2}{2} x(\tau(t)) R^2(x(\lambda_1)) \text{ for } t \in \left[\tau^{-3}(T_0), \tau^{-(N-2)}(T_0)\right]$$

Thus

$$1 > \frac{1}{\varphi_1(\rho)} + \rho R(x(\lambda_1)) + \frac{\rho^2}{2} \frac{x(\tau(t))}{x(t)} R^2(x(\lambda_1))$$

Using the conditions $R(x(\lambda_1)) > 1$, $0 < \rho < 1$ we have

$$1 > \frac{1}{\varphi_1(\rho)} + \rho + \frac{\rho^2}{2} \frac{x(\tau(t))}{x(t)} \quad \text{for } t \in \left[\tau^{-3}(T_0), \tau^{-(N-2)}(T_0)\right]$$

Thus

$$\frac{x(\tau(t))}{x(t)} < \frac{2\left(1 - \rho - \frac{1}{\varphi_1(\rho)}\right)}{\rho^2} = \varphi_2(\rho) \ \text{ for } t \in \left[\tau^{-3}(T_0), \tau^{-(N-2)}(T_0)\right]$$

Repeating the procedures, we have

$$\frac{x(\tau(t))}{x(t)} < \frac{2\left(1 - \rho - \frac{1}{\varphi_{m+1}(\rho)}\right)}{\rho^2} = \varphi_m(\rho) \text{ for } t \in \left[\tau^{-3}(T_0), \tau^{-(N-m)}(T_0)\right]$$

Remark 2.1.The above result depends and improves Lemma 2.2 of [2] and Lemma 2 of [4].

Theorem 2.1. Assume that there exists $t_1 \ge t_0$ and a positive constant ρ , $\rho > \frac{1}{e}$, such that Eq.(1.2) holds. Then, for any $T \ge t_1$, every solution of Eq. (1.2) has at least one zero on $[T, \tau^{-k}(T)]$, where

$$k = \begin{cases} 3 & \rho \ge 1 \\ \min\{\alpha, \beta\} & \frac{1}{e} < \rho < 1 \end{cases} \tag{2.21}$$

$\alpha = 2 + \min_{n \ge 1,\, m \ge 1} \{n + m \,/\, f_n(\rho) \ge \varphi_m(\rho)\}$ and

$\beta = 3 + \min_{n \ge 1} \{n \,/\, f_{n+1}(\rho) < 0 \text{ or } f_{n+1}(\rho) = \infty\}$.

Proof: Suppose that $x(t)$ is a solution of Eq. (1.2) for $t \in [T, \tau^{-k}(T)]$. If $x(t) > 0$ for $T \le t \le \tau^{-2}(T)$, then from Eq. (1.2) we obtain

$$x'(t) \le -p(t) R(x(t)) x(\tau(t)) \le 0 \text{ for } t \in [\tau^{-1}(T), \tau^{-3}(T)]$$

This implies that $x(t)$ is nonincreasing on $t \in [\tau^{-1}(T_0), \tau^{-3}(T_0)]$ and

$$x(t) \ge x(\tau^{-2}(T)) \quad \text{for} \quad t \in [\tau^{-1}(T), \tau^{-2}(T)]$$

Integrating both sides of Eq. (1.2) from $\tau^{-2}(T)$ to $\tau^{-3}(T)$, we obtain

$$\begin{aligned} x(\tau^{-3}(T)) &\le x(\tau^{-2}(T)) - \int_{\tau^{-2}(T)}^{\tau^{-3}(T)} p(s) R(x(s)) x(\tau(s))\, ds \\ &\le x(\tau^{-2}(T)) - \int_{\tau^{-2}(T)}^{\tau^{-3}(T)} p(s) x(\tau(s))\, ds \\ &\le x(\tau^{-2}(T)) \left\{1 - \int_{\tau^{-2}(T)}^{\tau^{-3}(T)} p(s)\, ds\right\} . \end{aligned}$$

In view of (2.7) and $\rho \ge 1$,we have $x(\tau^{-3}(T)) \le 0$ .This is a contradiction and so it is easy to see that $k = 3$.

In the case $\frac{1}{e} < \rho < 1$, assume that $x(t)$ is a solution of Eq. (1.2) satisfying $x(t) > 0$ for

$t \in [T, \tau^{-k}(T)]$. Let $k = 2 + n^* + m^*$

$$f_{n^*}(\rho) \geq \varphi_{m^*}(\rho) \text{ for } t \in \left[T, \tau^{-k}(T)\right]. \tag{2.22}$$

From the definitions of φ and f , then n^* and m^* must exist. By Lemma 2.1, we have

$$\frac{x(\tau(t))}{x(t)} \geq f_{n^*}(\rho) \text{ for } t \in \left[\tau^{-(2+n^*)}(T_0), T\right]. \tag{2.23}$$

On the other hand, by Lemma 2.2, we obtain

$$\frac{x(\tau(t))}{x(t)} < \varphi_{m^*}(\rho) \text{ for } t \in \left[\tau^{-3}(T_0), \tau^{-(N-m^*)}(T_0)\right] . \tag{2.24}$$

Setting $t = \tau^{-(2+n^*)}(T) = \tau^{-(N-m^*)}(T)$ in (2.23) and (2.24), we get

$$f_{n^*}(\rho) \leq \frac{x\left(\tau^{-(1+n^*)}(T)\right)}{x\left(\tau^{-(2+n^*)}(T)\right)} < \varphi_{m^*}(\rho) \tag{2.25}$$

This contradicts (2.22). It is easy to see that $k \leq 3 + \min_{k \geq 1}\{n \,/\, f_{n+1}(\rho) < 0 \text{ or } f_{n+1}(\rho) = \infty\}$,

Since $\varphi_m(\rho) > \dfrac{(1-\rho)}{\rho^2}$, $m = 1, 2, \ldots..$ .The proof is complete.

3. Main Results

In this section, we discuss upper bound on the distance between zeros of solutions of Eq. (1.1), we consider the function $H(t) = p(h(t))Q(t) / Q(G(t))$ where $G(t) = h^{-1}(g(h(t)))$, We assume the following conditions.

(H_1) $h(t) \leq g(t) \leq t$, $H(t) \in C([t_0, \infty), [0, \infty)]$ and $G'(t) \geq 1$, when $H'(t) \leq 0$, or $H'(t) - (G'(t) - 1)\, Q(t) \leq 0$, when $H'(t) > 0$.

(H_2) $\displaystyle\int_{g^{-1}(h(t))}^{t} \frac{Q(s)\; B(x(h(s)))}{1 + H\left(g^{-1}(h(s))\right)}\, ds \geq \rho$, $\rho > \dfrac{1}{e}$, t ¤ t_1

(H_3) $\displaystyle\int_{g^{-1}(h(t))}^{t} \frac{Q(s)\; B(x(h(s)))}{1 + H\left(g^{-1}(h(s))\right)}\, ds \geq \rho$, $0 \leq \rho \leq \dfrac{1}{e}$, $t \geq t_1$

Theorem 3.1: Suppose that (H_1), and (H_2) hold. Then for any $T \geq h^{-2}(t_1)$ every solution of Eq. (1.1) has at least one zero in the interval $[T, (g^{-1}h)^{-k}(T)]$, where k is given by (2.20) .

Proof: Suppose that $x(t)$ is a solution of Eq. (1.1) with $x(t) > 0$ for all $t \in [T, T_1]$, where $T_1 = (g^{-1}h)^{-k}(T)$. Let

$$z(t) = x(t) + p(t)x(g(t)) \text{ for } t \in g^{-1}(T, T_1) \tag{3.1}$$

Then

$$z(t) > 0 \text{ for } t \in g^{-1}(T, T_1), \tag{3.2}$$

and

$$z'(t) = -f(t, x(h(t))) < 0 \text{ for } t \in h^{-1}(T, T_1) . \tag{3.3}$$

From (1.1), (3.3) and (I) , with $h(t) \in [g^{-1}(T), T]$, we get

$$z'(t) \leq -Q(t)B(x(h(t)))x(h(t)) \tag{3.4}$$

$$z'(t) \leq -Q(t)B(x(h(t)))\,[z(h(t)) - p(h(t))x(g(h(t)))].$$

So

$$z'(t) \leq -Q(t)B(x(h(t)))z(h(t)) + Q(t)B(x(h(t)))\,p(h(t))x(g(h(t))) . \tag{3.5}$$

But since by (3.4)

$$z'\left(h^{-1}(t)\right) \le -Q\left(h^{-1}(t)\right) B(x(t))\, x(t),$$

then ,

$$B(x(t))\, x(t) \le -\frac{z'\left(h^{-1}(t)\right)}{Q\left(h^{-1}(t)\right)} \ \text{ for } \ t \in (T, T_1),$$

and

$$x(g(h(t))) \le -\frac{z'\left(h^{-1}(g(h(t)))\right)}{Q\left(h^{-1}(g(h(t)))\right) B(x(g(h(t))))} .$$

Since $G(t) = h^{-1}(g(h(t)))$. Then $G(t) \ge g^{-1}(g(h(t))) = h(t)$. By substituting into (3.5) we obtain

$$z'(t) \le -Q(t)\, B(x(h(t)))z(h(t)) + Q(t)B(x(h(t)))\, p(h(t)) \left[-\frac{z'\left(h^{-1}(g(h(t)))\right)}{Q\left(h^{-1}(g(h(t)))\right) B(x(g(h(t))))} \right]$$

$$\le -Q(t)B(x(h(t)))z(h(t)) - Q(t)\, p(h(t)) \frac{B(x(h(t)))}{B(x(g(h(t))))} \left(\frac{z'(G(t))}{Q(G(t))} \right) \ \text{ for } t \in \left[h^{-1}(g(T)), T_1\right]$$

Hence

$$z'(t) + Q(t)\, p(h(t)) \frac{B(x(h(t)))}{B(x(g(h(t))))} \left(\frac{z'(G(t))}{Q(G(t))} \right) + Q(t)B(x(h(t)))z(h(t)) \le 0 \ \text{ for } t \in \left[h^{-1}(g(T)), T_1\right]$$

$$z'(t) + H(t) \frac{B(x(h(t)))}{B(x(g(h(t))))} z'(G(t)) + Q(t)B(x(h(t)))z(h(t)) \le 0 \ \text{ for } t \in \left[h^{-1}(g(T)), T_1\right] . \qquad (3.6)$$

Now let

$$\omega(t) = z(t) + H(t) \frac{B(x(h(t)))}{B(x(g(h(t))))}\, z(G(t)) \ \text{for}\ t \in \left[G^{-1}(T), T_1\right] \qquad (3.7)$$

Then from (3.2) and (3.7), we have

$$\omega(t) > 0 \ \text{ for } \ t \in [G^{-1}(T), T_1] \ , \qquad (3.8)$$

and

$$\omega'(t) = z'(t) + H'(t) \frac{B(x(h(t)))}{B(x(g(h(t))))} z(G(t)) + H(t) \left(\frac{B(x(h(t)))}{B(x(g(h(t))))} \right)' z(G(t))$$

$$+ H(t) \frac{B(x(h(t)))}{B(x(g(h(t))))}\, z'(G(t))G'(t) \qquad (3.9)$$

where $Y(t) = z'(t) < 0$.

Now from (3.6) and (3.9) , we obtain

$$\omega'(t) \le H'(t) \frac{B(x(h(t)))}{B(x(g(h(t))))} z(G(t)) + H(t) \left(\frac{B(x(h(t)))}{B(x(g(h(t))))} \right)' z(G(t)) -$$
$$Q(t)B(x(h(t)))z(h(t)) + H(t) \frac{B(x(h(t)))}{B(x(g(h(t))))}\, Y(G(t))(G'(t) - 1) \ , \qquad (3.10)$$

Let

$$\left(\frac{B(x(h(t)))}{B(x(g(h(t))))} \right)' \le 0 \ . \qquad (3.11)$$

If $H'(t) \le 0$ and $G'(t) - 1 > 0$, then from (3.11) , we have

$$\omega'(t) + Q(t)B(x(h(t)))\, z(h(t)) \le 0 \ \text{ for } \ t \in \left[G^{-1}(T), T_1\right] \qquad (3.12)$$

If $H'(t) > 0$ and $H'(t) - (G'(t) - 1)Q(t) < 0$, then from (3.11) , then we have

$$H'(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(G(t))+H(t)\left(\frac{B(x(h(t)))}{B(x(g(h(t))))}\right)' z(G(t))+H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}Y(G(t))(G'(t)-1)$$
$$\le H'(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(G(t))+H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}Y(G(t))(G'(t)-1)$$
$$= H'(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(G(t))-H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}Y(G(t))(G'(t)-1)$$
$$\le H'(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(h(t))-H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}Y(G(t))(G'(t)-1)$$
$$\le H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(h(t))\left[\frac{H'(t)}{H(t)}-\frac{Y(G(t))(G'(t)-1)}{z(h(t))}\right]$$
$$\le H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(h(t))\left[\frac{H'(t)}{H(t)}-\frac{(G'(t)-1)Q(G(t))x(h(G(t)))B(x(h(G(t))))}{x(h(t))+p(h(t))x(g(h(t)))}\right]$$

$$\le H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(h(t))\left[\frac{H'(t)}{H(t)}-\frac{(G'(t)-1)Q(G(t))x(h(G(t)))B(x(h(G(t))))}{p(h(t))x(g(h(t)))}\right]$$
$$\le H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(h(t))\left[\frac{H'(t)}{H(t)}-\frac{(G'(t)-1)\,Q(G(t))}{p(h(t))}\right]$$

If we have $0<\dfrac{B(x(h(t)))}{B(x(g(h(t))))}<1$, then we have

$$\le H(t)z(h(t))\left[\frac{H'(t)}{H(t)}-\frac{(G'(t)-1)\,Q(G(t))}{p(h(t))}\right]$$
$$\le H'(t)z(h(t))-(G'(t)-1)\,Q(t)z(h(t))<0$$
$$= z(h(t))[H'(t)-(G'(t)-1)\,Q(t)]<0\ .$$

Also (3.12) holds.

Since $z'(t)<0$ and (3.7) we have

$$\omega(t)<\left[1+H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}\right]z(g(t))\ \text{ for } t\in\left[h^{-1}\left(g^{-1}(T)\right),T_1\right]$$

From $0<\dfrac{B(x(h(t)))}{B(x(g(h(t))))}<1$, then

$$\omega(t)<[1+H(t)]\,z(g(t))\ \text{ for } t\in[h^{-1}\left(g^{-1}(T)\right),T_1]$$

So

$$z(h(t))>\frac{\omega\left(g^{-1}(h(t))\right)}{1+H\left(g^{-1}(h(t))\right)}\ \text{ for } t\in\left[h^{-2}(T),T_1\right] \tag{3.13}$$

Substituting (3.13) into (3.10), we have

$$\omega'(t)\le H'(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}z(G(t))+H(t)\left(\frac{B(x(h(t)))}{B(x(g(h(t))))}\right)' z(G(t))-$$
$$Q(t)B(x(h(t)))\frac{\omega\left(g^{-1}(h(t))\right)}{1+H\left(g^{-1}(h(t))\right)}+H(t)\frac{B(x(h(t)))}{B(x(g(h(t))))}Y(G(t))(G'(t)-1)$$

$$\omega'(t)+\frac{Q(t)}{1+H\left(g^{-1}(h(t))\right)}B(x(h(t)))\omega\left(g^{-1}(h(t))\right)<0\ \text{ for } t\in\left[h^{-2}(T),T_1\right] \tag{3.14}$$

Then from Theorem 2.1, the proof is completed.
Remark 3.1. Theorem 3.1 depends and extends those in [2] and [3].

4. Example

Consider the delay differential equation

$$x'(t)+4\,(1+2t)\,x(t-1)=0 \tag{4.1}$$

Where $p(t)=0$, $B(t)=t$, $g(t)=t$, $x(t-1)=(t-1+2)=(t+1)$, $h(t)=t-1$. Here

$$Q(t)=4(2t+1),\ H(t)=\frac{p(h(t))\,Q(t)}{Q(G(t))}=0$$

$h(t)=t-1 \Rightarrow x(h(t))=x(t-1)=t+1$, then

$$B(x(h(t)))=B(t+1)=t+1 \tag{4.2}$$

$h(t)=t-1 \Rightarrow g^{-1}(h(t))=g^{-1}(t-1)=t-1$, then

$$H\left(g^{-1}(h(t))\right)=H(t-1)=0 \tag{4.3}$$

Then from $(4.2),(4.3)$ we have

$$\int_{g^{-1}(h(t))}^{t}\frac{Q(s)B(x(h(s)))}{1+H\left(g^{-1}(h(s))\right)}ds=8t^2+4t+\frac{2}{3}\geq\frac{2}{3}$$

$t\geq t_1=\max\left\{t_0,\frac{2}{3}\right\}$ .Hence

$$\varphi_1(\rho)=\frac{2\,(1-\rho)}{\rho^2}=\frac{3}{2}$$

$$\varphi_2(\rho)=\frac{2\left(1-\rho-\frac{1}{\varphi_1(\rho)}\right)}{\rho^2}=-\frac{3}{2},$$

And

$$f_0(\rho)=1$$

$$f_1(\rho)=\frac{1}{1-\rho}=3$$

$$f_2(\rho)=19.13291213$$

$$f_3(\rho)=-0.885202225$$

$$f_3(\rho)\leq 0,\beta=3+n=3+2=5$$

$$f_2(\rho)\geq\varphi_1(\rho)\Rightarrow\alpha=2+n+m=2+1+2=5$$

Thus $k=\min\{\alpha,\beta\}=\min\{5,5\}=5$ .Thus the hypotheses of Theorem 3.1 be satisfied. Then every solution of Eq. (1.1) has at least one zero in $[T,\left(g^{-1}h\right)^{-5}(T)]$.

Remark4.1.The above example may show that the conclusions do not follow the known oscillation criteria in the literature ([1],[2],[4],[5],and [6]).

References

F.X.Liang, The distribution of zeros of solutions of first order delay differential equations, J.Math.Anal.Appl.186(1994) 383-392.
[2] H.W.Wu, Y.T.Xu, The distribution of zeros of solutions of neutral differential equations, Appl.Math.Comput. 156(2004)665-677.

[3] H.W.Wu, S.S.Cheng, Q.R.Wang, Distribution of zeros of solutions of functional equations, Appl.Math.Comput. 193(2007)154-161.
[4]B.G.Zhang , Y.Zhou The distribution of zeros of solutions of differential equations with a ,variable delay , J.Math.Anal.Appl.256(2001) 216-228.
[5] Y.Zhou, Z.R. Liu, and Y.H.Yu, An estimate for distance between adjacent zeros of solutions of neutral delay differential equations, Acta . Math. Appl. Sin. 21 (4) (1998) ,505-512.
[6] Y.Zhou, The distribution of zeros of solutions of first order functional differential equations, Bull. Aust. Math.Soc.59(1999)305-314.