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Control Theory

CHARACTERIZING A CONTROL VIA AN INITIAL VALUE PROBLEM

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Abstract

For a controllable autonomous linear system, an initial value problem is presented, which depends only on (a) the matrices in the system (b) the initial state and (c) duration of control, such that its (unique) solution provides one possible control function. The dependence need not determine the initial value uniquely.

Keywords: Control, linear autonomous, initial value problem.

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1. Introduction

Let B be an $n\times m$ matrix and A and $n\times n$ matrix such that the pair (A,B) is *controllable* in the usual sense that the autonomous initial value problem

$$x' = Ax + Bu, \qquad x(0) = x_0$$

on $[0,\infty)$ has the property that, for any $n\times 1$ vector x_0 and any positive T, there exists an $m\times 1$ vector function u, called *control*, such that the solution vanishes at T. We shall show that one possible control is the unique solution of an n^{th} order differential equation that depends on A, B, x_0 as well as T. In fact, the coefficients in the differential equation are the same as in the characteristic polynomial of A but alternate in sign.

2. Notation

Throughout this article, A will denote a real $n\times n$ matrix with characteristic polynomial

$$\det(\lambda I - A) = \lambda^n - b_{n-1}\lambda^{n-1} - b_{n-2}\lambda^{n-2} - \cdots - b_0,$$

where I is the identity matrix. The transpose of a matrix will be indicated by a superscript *tr*. By $p_0, p_1, \ldots, p_{n-1}$, we shall mean the functions whose existence is asserted by Theorem 3.1.

3. Background results

We begin with a result about $\exp(tA)$ which is essentially proved in [3].

3.1. Theorem. *There exist real valued functions* $p_0, p_1, \ldots, p_{n-1}$ *on* $\mathbf{R}$ *such that*

$$\exp(tA) = p_0(t)I + p_1(t)A + \cdots + p_{n-1}(t)A^{n-1}, \ldots\ldots(1)$$

and

$$p_0, p_1, \ldots, p_{n-1} \text{ have Wronskian matrix } I \text{ at } t = 0. \ldots\ldots(2)$$

Moreover

$$p_j^{(n)} - b_{n-1}p_j^{(n-1)} - \cdots - b_0 p_j = 0 \text{ for } 0 \le j \le n-1 \ldots\ldots(3)$$

and

$$p_0, p_1, \ldots, p_{n-1} \text{ are linearly independent}\ldots\ldots(4)$$

PROOF: The initial value problem

$$\begin{aligned}
p'_0(t) &= & & b_0 p_{n-1}(t)\\
p'_1(t) &= p_0(t) + & & b_1 p_{n-1}(t)\\
p'_2(t) &= \quad p_1(t) + & & b_2 p_{n-1}(t)\\
& \qquad \cdots\ \cdots\ \cdots\ \cdots & & \\
p'_{n-1}(t) &= & p_{n-2}(t) + & b_{n-1}p_{n-1}(t),
\end{aligned}$$

$$p_0(0) = 1,\ p_1(0) = 0,\ \ldots,\ p_{n-1}(0) = 0,$$

has a (unique) solution. A computation from the differential equations and the initial condition shows that the Wronskian matrix of the solution at $t = 0$ is the identity matrix I. Thus (2) holds.

Now, by the Cayley-Hamilton Theorem,

$$A^n = b_0 I + b_1 A + \cdots + b_{n-1}A^{n-1}.$$

Therefore the $n\times n$ matrix valued function

$$u(t) = p_0(t)I + \cdots + p_{n-1}(t)A^{n-1}$$

satisfies

$$\begin{aligned}
u'(t) &= p'_0(t)I + \cdots + p'_{n-1}(t)A^{n-1}\\
&= b_0 p_{n-1}(t)I + (p_0(t) + b_1 p_{n-1}(t))A + \cdots + (p_{n-2}(t) + b_{n-1}p_{n-1}(t))A^{n-1}\\
&= p_0(t)A + \cdots + p_{n-2}(t)A^{n-1} + p_{n-1}(t)(b_0 I + b_1 A + \cdots + b_{n-1}A^{n-1})\\
&= p_0(t)A + \cdots + p_{n-1}(t)A^{n-1} + p_{n-1}(t)A^n\\
&= Au(t).
\end{aligned}$$

Since $u(0) = I$, it follows that (1) and (4) hold.

Finally, the characteristic polynomial of the constant matrix on the right side of the linear system, through which the functions $p_0, p_1, \ldots, p_{n-1}$ have been set up, is the same at that of A and the secular equation [1; see definition on p.115] of the system is

$$y^{(n)} - b_{n-1}y^{(n-1)} - \cdots - b_0 y = 0.$$

However, each solution of the linear system must satisfy the secular equation [1, Theorem 3, p.116]. Thus (3) must hold. ■

3.2. Proposition. *Any element of* $\mathbf{R}^{mn}$ *can be regarded as* $\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$, *or equivalently as* $[c_0^{tr}, c_1^{tr}, \ldots, c_{n-1}^{tr}]^{tr}$, *where* $c_0, \ldots, c_{n-1}$ *are columns in* $\mathbf{R}^m$. *Therefore there is an associated* $\mathbf{R}^m$ *valued function*

$$u(t) = \sum_{j=0}^{n-1} p_j(-t)c_j \ldots\ldots (5).$$

The linear transformation $\Lambda: \mathbf{R}^{mn} \to \mathbf{R}^{mn}$ *defined by*

$$\Lambda([c_0^{tr}, c_1^{tr}, \ldots, c_{n-1}^{tr}]^{tr}) = \begin{bmatrix} \int_0^T p_0(-t)u(t)dt \\ \int_0^T p_1(-t)u(t)dt \\ \vdots \\ \int_0^T p_{n-1}(-t)u(t)dt \end{bmatrix},$$

where T is a fixed positive number, is onto and hence one-to-one.

PROOF: If Λ is not onto, then there exist n columns $\gamma_0, \ldots, \gamma_{n-1}$ in $\mathbf{R}^m$ such that the mn-vector $[\gamma_0^{tr}, \gamma_1^{tr}, \ldots, \gamma_{n-1}^{tr}]^{tr}$ is nonzero and is orthogonal to every vector in $\Lambda(\mathbf{R}^{mn})$. Thus for every $[c_0^{tr}, c_1^{tr}, \ldots, c_{n-1}^{tr}]^{tr} \in \mathbf{R}^{mn}$ and u given by (5),

$$\gamma_0 \cdot \int_0^T p_0(-t)u(t)dt + \cdots + \gamma_{n-1} \cdot \int_0^T p_{n-1}(-t)u(t)dt = 0,$$

that is to say,

$$\int_0^T \sum_{j=0}^{n-1} p_j(-t)\gamma_j \cdot u(t)dt = 0.$$

Choosing $c_j = \gamma_j$ for each j, we get $\int_0^T (u(t) \cdot u(t))dt = 0$, i.e. $u(t) = 0$ on $[0, T]$. Reading this componentwise and applying (4) of Theorem 3.1, we conclude that $\gamma_j = 0$ for every j. This is a contradiction, showing that Λ must be onto and hence also one-to-one. ■

3.3. Remark. By (3) of Theorem 3.1, any function u given by (5) satisfies the n^{th} order differential equation

$$(-1)^n u^{(n)} - (-1)^{n-1} b_{n-1} u^{(n-1)} - \cdots - b_0 u = 0, \ldots\ldots (6)$$

and by (2), it also satisfies

$$u(0) = c_0, u'(0) = -c_1, \ldots, u^{(n-1)} = (-1)^{n-1} c_{n-1}, \ldots\ldots (7)$$

where $c_0, \ldots, c_{n-1}$ are the vectors in the representation (5). However, this initial value problem defines the function u uniquely. Consequently, the function u of (5) has an alternative description as the solution of this initial value problem. Thus fact will turn out to be significant in Theorem 4.1.

4. Main result

Now we come to the result that describes an initial value problem, the unique solution of which steers the controllable system $x' = Ax + Bu$ from initial state x_0 to terminal state 0 at time *T*. In order for such a control to exist for every initial value x_0, it is necessary and sufficient that the rank of the $n \times mn$ matrix

$$[B, AB, \ldots, AB^{n-1}]$$

be n (Kalman controllability criterion). When this is the case, the matrix has a *right inverse*, that is, an $mn \times n$ matrix C such that $[B, AB, \ldots, AB^{n-1}]C$ is the $n \times n$ identity matrix.

Although the hypothesis in Theorem 4.1 takes all this for granted, parts of the argument can be harvested for justifying the Kalman controllability criterion; see Remark 4.2 below.

4.1. Theorem. *Let B be an* $n \times m$ *matrix such that the pair* (A,B) *is controllable, so that the rank of the matrix* $[B, AB, \ldots, AB^{n-1}]$ *is n. Denote by C any right inverse of* $[B, AB, \ldots, AB^{n-1}]$ and *let L be the* $mn \times mn$ *matrix of the linear map* $\Lambda: \mathbf{R}^{mn} \to \mathbf{R}^{mn}$ *of Proposition 3.2. Then the unique solution of the differential equation* (6) *and the initial condition*

$$\begin{bmatrix} u(0) \\ -u'(0) \\ \vdots \\ (-1)^{n-1}u^{(n-1)}(0) \end{bmatrix} = -L^{-1}Cx_0 \quad \text{(8)}$$

is a control that steers the system $x' = Ax + Bu$ *from initial state* x_0 *to terminal state* 0 *at time T. In other words, the solution* $\begin{bmatrix} x \\ u \end{bmatrix}$ *of*

$$x' = Ax + Bu, \quad x(0) = x_0 \quad \text{(9)}$$

has the property that $x(T) = 0$.

PROOF: By Remark 3.3, u is of the form (5) with c_j given by (7). Recall that the scalar functions p_j of (5) are as in Theorem 3.1 and therefore satisfy (1). Now, the solution of (9) is given by the variation of parameters formula [2, p.269, eq.(8)] or [3, p.15] as

$$x(t) = \exp(tA)x_0 + \exp(tA)\int_0^t \exp(-sA)Bu(s)\,ds. \quad \text{(10)}$$

Using (1) of Theorem 3.1, we find that this solution satisfies $x(T) = 0$ if and only if

$$\begin{aligned} -x_0 &= \int_0^T \exp(-sA)Bu(s)\,ds \\ &= \int_0^T [p_0(-s)I + \cdots + p_{n-1}(-s)A^{n-1}]Bu(s)\,ds \\ &= \int_0^T [p_0(-s)B + p_1(-s)AB + \cdots + p_{n-1}(-s)A^{n-1}B]\,ds \\ &= [B,\ AB, \dots,\ AB^{n-1}] \begin{bmatrix} \int_0^T p_0(-s)u(s)\,ds \\ \int_0^T p_1(-s)u(s)\,ds \\ \vdots \\ \int_0^T p_{n-1}(-s)u(s)\,ds \end{bmatrix}. \quad \text{(11)} \end{aligned}$$

By definition of L, this is equivalent to

$$-x_0 = [B,\ AB, \dots,\ AB^{n-1}]\,L \begin{bmatrix} u(0) \\ -u'(0) \\ \vdots \\ (-1)^{n-1}u^{(n-1)}(0) \end{bmatrix},$$

and this is true whenever (8) holds. ■

4.2. Remark. The rank condition was not used until after (11). Once this equation is derived, one can easily prove the equivalence of the rank condition and the existence of a suitable control. In fact, it is clear from equation (11) that if a suitable control exists, then every x_0 is in the range of the linear transformation given by the matrix in the equation. The converse also follows from equation (11) because the column vector on the extreme right in the equation can be chosen to be any vector in $\mathbf{R}^{mn}$ by virtue of Proposition 3.2.

5. Illustration

We illustrate Theorem 4.1 in a simple instance.

5.1. Example. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that $m = n = 2$. Take $T = 2\pi$ and $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The characteristic polynomial of A is λ^2+1 and therefore $b_0 = -1$ and $b_1 = 0$. The familiar linearly independent solutions of $u''+u=0$ have Wronskian matrix equal to the identity matrix at $t=0$. Therefore $\cos t = p_0(t)$ and $\sin t = p_1(t)$. By Theorem 3.1 (or otherwise),

$$\exp(tA) = p_0(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p_1(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Now

$$[B,\ AB] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

which has rank 2. So the system is controllable. Moreover, the matrices

$$C_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

are both right inverses for $[B,\ AB]$. Next, Λ maps $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$ into

$$\begin{bmatrix} c_0 \int_0^{2\pi} \cos(-t)(\cos(-t)+\sin(-t))\,dt \\ c_1 \int_0^{2\pi} \cos(-t)(\cos(-t)+\sin(-t))\,dt \\ c_2 \int_0^{2\pi} \sin(-t)(\cos(-t)+\sin(-t))\,dt \\ c_3 \int_0^{2\pi} \sin(-t)(\cos(-t)+\sin(-t))\,dt \end{bmatrix},$$

which works out to be $\pi\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Therefore the matrix L of Λ is π times the 4×4 identity matrix. So, L^{-1} is $\frac{1}{\pi}$ times the 4×4 identity matrix.

Now $C_1 x_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that

$$L^{-1} C_1 x_0 = \frac{1}{\pi}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore according to Theorem 4.1, one possible control that makes $x(2\pi) = 0$ is the solution of

$$\begin{bmatrix} u''_1 \\ u''_2 \end{bmatrix} = -\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \frac{1}{\pi}\begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u'_1(0) \\ u'_2(0) \end{bmatrix} = \frac{1}{\pi}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The unique solution of this IVP is

$$u(t) = \begin{bmatrix} -\cos t + \sin t \\ -\cos t \end{bmatrix}.$$

This u satisfies

$$\exp(-sA)u(s) = \frac{1}{\pi}\begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}\begin{bmatrix} -\cos s + \sin s \\ -\cos s \end{bmatrix}$$

$$= \frac{1}{\pi}\begin{bmatrix} \frac{1}{2}(\cos 2s + 1) + \sin 2s \\ -\cos 2s - \frac{1}{2}\sin 2s \end{bmatrix}.$$

Therefore

$$\int_0^{2\pi} \exp(-sA)u(s)\,ds = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -x_0.$$

The variation of parameters formula (10) now shows that $x(2\pi) = 0$, as required.

Similar computations using the other right inverse lead to the control

$$u(t) = \begin{bmatrix} \sin t \\ -\cos t + \sin t \end{bmatrix}.$$

6. Concluding Remarks

6.1. Remark. Both controls computed in Example 5.1 satisfy $\int_0^{2\pi} u(t)\cdot u(t)\,dt = \frac{3}{\pi}$.

6.2. Remark. If in Example 5.1, we change B to the 2×1 matrix $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the control system becomes the same as [4, p.109]. The solution computed there agrees with what Theorem 4.1 here yields.

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